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ASYMPTOTIC BEHAVIOR OF TRANSFORMS OF DISTRIBUTIONS(1)

BY

E. O. MILTON(2)

ABSTRACT. In this paper final and initial value type Abelian theorems for Laplace and Fourier transforms of certain types of distributions are obtained. The class of distributions under consideration contains the singular distributions. Thus we generalize the results previously obtained by A. H. Zemanian in two ways: we add Fourier transforms to those considered, and we also deal with a larger class of distributions.

1. **Introduction.** Some classical Abelian theorems provide the initial motivation. Such theorems relate a singularity of a function at zero to the rate at which its transform tends to zero at infinity (initial value type) or relate the rate at which a function tends to zero at infinity to a singularity of its transform at zero (final value type). It turns out that, roughly speaking, transforms of distributions are better behaved at zero and worse behaved at infinity than are the transforms of functions.

Let T be a Laplace transformable distribution with support in $[0, \infty)$ such that T is regular (defined by a function f) in $[0, a]$ or $[a, \infty)$ for some $a > 0$. Such distributions will be called semiregular. Also, let $\mathfrak{L}[\ ]$ denote the appropriate Laplace transform of the function or distribution within the brackets. A. H. Zemanian [7, Chapter 8] showed that Abelian theorems relating the behavior of f to the behavior of $\mathfrak{L}[f]$ imply Abelian theorems relating the behavior of f to that of $\mathfrak{L}[T]$. We will now obtain results of the final and initial value type for both the Laplace transform and the Fourier transform of distributions which are not necessarily semiregular.

The final value type Abelian theorem for a semiregular distribution T is obtained in the following way. T can be written as $T = U + V$ where U is of compact support and $V = T_f$. Then $\mathfrak{L}[T] = \mathfrak{L}[U] + \mathfrak{L}[V]$ and $\mathfrak{L}[T]$ is a function of one variable, say σ. $\mathfrak{L}[U]$ is a function which can be extended to an entire

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function, and $\mathfrak{L}[V]$ is identical to $\mathfrak{L}[f]$. In Zemanian's example, $f(x) \sim K_1 x^\alpha$ $(\alpha < 1)$ as $x \to \infty$ implies $\mathfrak{L}[f](\sigma) \sim K_2 \sigma^{\alpha-1}$ as $\sigma \to 0^+$, and since $\mathfrak{L}[U]$ is bounded at the origin this contribution from $\mathfrak{L}[V]$ is the dominant one in the behavior of $\mathfrak{L}[T]$ as $\sigma \to 0^+$, giving an Abelian theorem for distributions.

The method of investigating the behavior of the transforms of singular distributions is based on representation theorems for the distributions. For the Laplace transform the basic idea is to take a Laplace transformable distribution T with support in $[0, \infty)$ and write it as $T = U + V$ where U is of compact support and $e^{-cx}V$ is tempered, for some real c. For the Fourier transform the decomposition is similar except the support is not necessarily restricted to $[0, \infty)$: Take a tempered distribution T and write it as $T = U + S + V$ where S is of compact support $[-a-\epsilon, b+\delta]$ and U and V are tempered, having support in $(-\infty, -a]$ and $[b, \infty)$, respectively, for positive numbers a and b. Then for both transforms representations of these components of the original distributions are used. Tempered distributions are represented as distributional derivatives of tempered functions, that is, functions of the form $(1+x^2)^k g(x)$ for any x where g is bounded and continuous. Distributions of compact support are represented as sums of distributional derivatives of Radon measures. Then the assumed behavior of the tempered functions or of the Radon measures is reflected in the asymptotic behavior of the transforms. A distributional derivative of order l, for some nonnegative integer l, contributes the factor $K\sigma^l$ ($K = 1$ or i^l depending on the transform) to the asymptotic behavior of the transform.

2. **Notation and definitions.** The evaluation of a distribution T at a test function ϕ will be denoted by $\langle T, \phi \rangle$. All integrals are Lebesgue integrals and $f \in BV(\Omega)$ will mean the function f is of bounded variation over the set Ω. D_x^i will denote the ith derivative with respect to the variable x, and the subscript may not appear when the variable is clear from the context. $x \to \pm a$ is shorthand for the two statements $x \to a^-$ (approach from the left only) and $x \to a^+$ (approach from the right only). As usual, $f \sim Kg(x \to a)$ for $K \neq 0$ will mean $f/g \to K$ as $x \to a$.

A distribution is said to be regular if it is defined by a locally integrable function f, that is, if $\langle T, \phi \rangle = \langle T_f, \phi \rangle = \int_{-\infty}^{\infty} f(x)\phi(x)\,dx$ for each test function ϕ. Then a distribution which is regular over a subset of its support will be called semiregular. A distribution which is not semiregular is said to be singular. The class of distributions denoted by $\mathcal{S}'$ will be the class of all tempered distributions and $\mathcal{S}$ is the corresponding test function space. $\mathcal{S}'_+$ will denote those elements of $\mathcal{S}'$ whose supports are bounded on the left.

If f is a complex valued, locally integrable function such that $f(x) = 0$ for $-\infty < x < a$ and there exists a real number c such that $e^{-cx}f(x)$ is absolutely

integrable over $-\infty < x < \infty$, then the Laplace transform of f is the function of the complex variable $s = \sigma + i\omega$ defined by

$$G(s) = \mathfrak{L}[f] = \int_{-\infty}^{\infty} e^{-sx} f(x)\,dx, \qquad \sigma > c.$$

Also, if f is a complex valued function absolutely integrable over $(-\infty, \infty)$, then the Fourier transform of f is the function of the real variable σ defined by

$$\hat{f}(\sigma) = \mathfrak{F}[f] = (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(x) e^{-i\sigma x}\,dx.$$

The analogous definitions for the transforms of distributions are as follows: if T is a distribution with support in $[a, \infty)$ for some real number a such that $e^{-cx}T \in \mathcal{S}'_+$ for some real number c, then T is said to be Laplace-transformable and its Laplace transform is the function defined by

$$G(s) = \mathfrak{L}[T] = \langle e^{-cx}T, \alpha(x)e^{-(s-c)x}\rangle$$

for $\mathfrak{Re}(s) > c$, where α is infinitely differentiable with support bounded on the left and $\alpha = 1$ on a neighborhood the support of T; and if $T \in \mathcal{S}'$, then the Fourier transform of T is the distribution $\mathfrak{F}[T]$ or $\hat{T}$ defined by $\langle \hat{T}, \phi\rangle = \langle T, \hat{\phi}\rangle$ for any test function $\phi \in \mathcal{S}$. Note that $\alpha(x)e^{-(s-c)x} \in \mathcal{S}$, so the right-hand side of the definition of the Laplace transform has meaning. It can be shown to be independent of any particular α, thus we commonly use the abbreviation $G(s) = \langle T, e^{-sx}\rangle$ instead of the actual definition. Also, the equation defining $\hat{T}$ can be shown to define a distribution by the use of properties of the classical Fourier transform.

3. **Fourier transforms of semiregular distributions.** The modification of a theorem of Titchmarsh [5, Theorem 126, Chapter VI, p. 172] is the starting point for the Fourier transform. If an L^1 function f is of the form $x^{-\alpha}g(x)$ for each x over appropriate parts of the interval $(0, \infty)$, we can get the asymptotic behavior of the Fourier sine $(\hat{f}_s)$ or Fourier cosine $(\hat{f}_c)$ transform at either 0 or ∞. Then to get information about $\hat{f}$, we write it as an integral from $-\infty$ to 0 $(\hat{f}^-)$ plus an integral from 0 to ∞ $(\hat{f}^+)$. Both $\hat{f}^-$ and $\hat{f}^+$ can be written in terms of $\hat{f}_s$ and $\hat{f}_c$, hence the desired behavior can be deduced from that of $\hat{f}_s$ and $\hat{f}_c$.

The generalization is then carried over to Fourier transforms of distributions by considering a tempered distribution T $(T \in \mathcal{S}')$ which is regular in both an interval neighborhood of $-\infty$ (where it is defined by g) and an interval neighborhood of ∞ (where it is defined by f). The Fourier transform of T is then regular and the behavior of the defining function as x approaches 0 from the right or left $(x \to \pm 0)$ can be determined from the classical results concerning f and g. It is interesting to note that we cannot get an analogue of the Laplace transform initial value theorem by these methods. That is, if $T \in \mathcal{S}'$ and T is regular in

an interval neighborhood of the origin, we cannot deduce similar information about the behavior of the defining function of $\mathfrak{F}[T]$ as $x \to \pm\infty$. This is because the contribution from the nonregular part does not necessarily approach 0 at ∞ and so may swamp the small contribution from the regular part.

If f is a complex valued function which is absolutely integrable over $-\infty < x < \infty$, then the Fourier sine transform of f is

$$\hat{f}_s(x) = \mathfrak{F}_s[f(t)] = (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(t) \sin xt\, dt,$$

and the Fourier cosine $\hat{f}_c(x)$ or $\mathfrak{F}_c[f(t)]$ is the similar expression with sine replaced by cosine. Then the one sided sine $(\hat{f}_s^+)$ and one sided cosine $(\hat{f}_c^+)$ transforms are defined in the same way, but the interval of integration is $[0, \infty)$ and the constant is changed to $(2/\pi)^{1/2}$.

Instead of considering one function $x^{-\alpha}g(x)$ for x over the whole interval $(0, \infty)$ we generalize Titchmarsh's result as follows.

Theorem 3.1. *Let $f \in L^1(0, \infty)$ such that*

$$f(x) = \begin{cases} x^{-\alpha}g(x), & 0 < x < a, \\ x^{-\beta}h(x), & a < x < \infty, \end{cases}$$

where $a \geq 0$, $g \in BV(0, a)$ and $h \in BV(a, \infty)$; then

(i) *if $0 < \alpha < 1$ and $\beta \geq 0$,*

$$\hat{f}_c^+(x) \sim g(0^+)(2/\pi)^{1/2}\Gamma(1-\alpha)\sin(\pi\alpha/2)\,|x|^{\alpha-1} \qquad (x \to \pm\infty),$$

and

$$\hat{f}_s^+(x) \sim \operatorname{sgn}(x)\, g(0^+)(2/\pi)^{1/2}\Gamma(1-\alpha)\cos(\pi\alpha/2)\,|x|^{\alpha-1} \qquad (x \to \pm\infty);$$

(ii) *if $\alpha < 1$ and $0 < \beta < 1$,*

$$\hat{f}_c^+(x) \sim h(\infty)(2/\pi)^{1/2}\Gamma(1-\beta)\sin(\pi\beta/2)\,|x|^{\beta-1} \qquad (x \to 0),$$

and

$$\hat{f}_s^+(x) \sim \operatorname{sgn}(x)\, h(\infty)(2/\pi)^{1/2}\Gamma(1-\beta)\cos(\pi\beta/2)\,|x|^{\beta-1} \qquad (x \to \pm 0).$$

Proof. The proof is similar to that of Titchmarsh. If $x > 0$ and $a > 0$,

$$(\pi/2)^{1/2}\hat{f}_c^+(x) = \int_0^a t^{-\alpha}g(t)\cos xt\, dt + \int_a^{\infty} t^{-\beta}h(t)\cos xt\, dt$$

(i)

$$= I_g + I_h, \quad \text{say.}$$

Consider I_h and let $t = u/x$, then

$$|I_h| = \left| x^{\beta-1} \int_{ax}^{\infty} u^{-\beta}h(u/x)\cos u\, du \right| = x^{\beta-1}|h(a)| \left| \int_{ax}^{\delta} u^{-\beta}\cos u\, du \right|$$

by Bonnet's form of the second mean value theorem, where for any fixed number ax, $ax \leq \delta < \infty$. But

$$\int_{ax}^{\delta} u^{-\beta} \cos u \, du = O([ax]^{-\beta}) \quad \text{as } x \to \infty$$

for all $\beta \geq 0$, again by Bonnet's form of the second mean value theorem. This implies that

$$x^{\beta-1} \int_{ax}^{\delta} u^{-\beta} \cos u \, du \leq Kx^{\beta-1}(ax)^{-\beta} = Cx^{-1},$$

or

$$x^{\beta-1} \int_{ax}^{\delta} u^{-\beta} \cos u \, du = O(1/x) \quad \text{as } x \to \infty.$$

Then as in the original theorem

$$I_g(x) \sim (2/\pi)^{1/2} g(0^+) \Gamma(1-\alpha) \sin(\pi\alpha/2) x^{\alpha-1} \qquad (x \to \infty).$$

But $I_b = O(1/x)$ and hence is negligible compared to I_g as $x \to \infty$. Hence we have the first part of (i). The proofs of the second part of (i) and of both parts of (ii) are similar. To deal with the case $x < 0$, we initially perform the change of variable $t = u/(-x)$ and (i) and (ii) follow for $x \to -\infty$ and $x \to 0^-$, respectively.

This theorem is a valuable tool for deducing the behavior of $\hat{f}(x)$ at $\pm\infty$ and 0. To do this we need two new definitions: the right-sided Fourier transform of a function f is

$$\hat{f}^+(x) = (2\pi)^{-1/2} \int_0^{\infty} f(t) e^{-ixt} \, dt,$$

and the left-sided Fourier transform is

$$\hat{f}^-(x) = (2\pi)^{-1/2} \int_{-\infty}^{0} f(t) e^{-ixt} \, dt.$$

Theorem 3.2. *Let $f \in L^1(-\infty, \infty)$ such that*

$$f(x) = \begin{cases} x^{-\beta} b(x), & x > a, \\ x^{-\alpha} g(x), & 0 < x < a, \\ (-x)^{-\lambda} p(-x), & -b < x < 0, \\ (-x)^{-\nu} q(-x), & x < -b, \end{cases}$$

where $a > 0$, $b > 0$, $g \in BV(0, a)$, $b \in BV(a, \infty)$, $p \in BV(0, b)$, and $q \in BV(b, \infty)$; then

(i) *if $0 < \alpha < 1$ and $\beta \geq 0$,*

$$\hat{f}^+(x) \sim (2\pi)^{-1/2} g(0^+)\Gamma(1-\alpha)e^{\pi i \operatorname{sgn}(x)(\alpha-1)/2}|x|^{\alpha-1} \qquad (x \to \pm\infty);$$

(ii) *if* $\alpha < 1$ *and* $0 < \beta < 1$,

$$\hat{f}^+(x) \sim (2\pi)^{-1/2} h(\infty)\Gamma(1-\beta)e^{\pi i \operatorname{sgn}(x)(\beta-1)/2}|x|^{\beta-1} \qquad (x \to \pm 0);$$

(iii) *if* $0 < \lambda < 1$ *and* $\nu \geq 0$,

$$\hat{f}^-(x) \sim (2\pi)^{-1/2} p(0^+)\Gamma(1-\lambda)e^{\pi i \operatorname{sgn}(x)(1-\lambda)/2}|x|^{\lambda-1} \qquad (x \to \pm\infty);$$

(iv) *if* $\lambda < 1$ *and* $0 < \nu < 1$,

$$\hat{f}^-(x) \sim (2\pi)^{-1/2} q(\infty)\Gamma(1-\nu)e^{\pi i \operatorname{sgn}(x)(1-\nu)/2}|x|^{\nu-1} \qquad (x \to \pm 0).$$

Proof. (i) and (ii).

$$\hat{f}^+(x) = (2\pi)^{-1/2}\int_0^\infty f(t)[\cos xt - i \sin xt]\,dt = \tfrac{1}{2}[\hat{f}^+_c - i\hat{f}^+_s](x),$$

hence the results by Theorem 3.1.

(iii) and (iv).

$$\hat{f}^-(x) = (2\pi)^{-1/2}\int_{-\infty}^0 f(t)e^{-ixt}\,dt = (2\pi)^{-1/2}\int_0^\infty f(-t)e^{ixt}\,dt$$
$$= \tfrac{1}{2}[\mathcal{F}^+_c[f(-t)] + i\mathcal{F}^+_s[f(-t)]],$$

and, for $0 < t < \infty$,

$$f(-t) = \begin{cases} t^{-\lambda}p(t), & 0 < t < b, \\ t^{-\nu}q(t), & t > b, \end{cases}$$

and again the results by Theorem 3.1.

Corollary 3.3. *If f is as in the preceding theorem, then*

(i) *if* $0 < \alpha < 1$, $0 < \lambda < 1$, $\beta \geq 0$, *and* $\nu \geq 0$,

$$\hat{f}(x) \sim (2\pi)^{-1/2} g(0^+)\Gamma(1-\alpha)e^{\pi i \operatorname{sgn}(x)(\alpha-1)/2}|x|^{\alpha-1}$$
$$+ (2\pi)^{-1/2} p(0^+)\Gamma(1-\lambda)e^{\pi i \operatorname{sgn}(x)(1-\lambda)/2}|x|^{\lambda-1} \qquad (x \to \pm\infty);$$

(ii) *if* $\alpha < 1$, $\lambda < 1$, $0 < \beta < 1$, *and* $0 < \nu < 1$,

$$\hat{f}(x) \sim (2\pi)^{-1/2} h(\infty)\Gamma(1-\beta)e^{\pi i \operatorname{sgn}(x)(\beta-1)/2}|x|^{\beta-1}$$
$$+ (2\pi)^{-1/2} q(\infty)\Gamma(1-\nu)e^{\pi i \operatorname{sgn}(x)(1-\nu)/2}|x|^{\nu-1} \qquad (x \to \pm 0).$$

In each part of this corollary one of the two terms in the sum will dominate, depending upon the relative sizes of the power of $|x|$ in each term. Similar sums will be written throughout the rest of this paper, with the understanding that one of the terms in any given sum will dominate.

This corollary completes the classical preliminaries and we are ready to prove a result for the Fourier transform of semiregular distributions.

Theorem 3.4. *Let T be a tempered distribution with support in $(-\infty, \infty)$ such that T equals the distribution corresponding to a function g over $(-\infty, b]$ and corresponding to a function f over $[a, \infty)$ for any two real numbers $b \leq a$. If $g \in L^1(-\infty, b]$ and $g(x) = (-x)^{-\beta}q(-x)$ for each $x \in (-\infty, d]$, and if $f \in L^1[a, \infty)$ and $f(x) = x^{-\alpha}p(x)$ for each $x \in [c, \infty)$, where $c \geq a$, $c \geq 0$, $d \leq b$, $d \leq 0$, $0 < \alpha < 1$, $0 < \beta < 1$, $p \in BV[c, \infty)$, and $q \in BV[-d, \infty)$, then $\hat{T}$ is a regular distribution defined by a function Φ and*

$$\Phi(x) \sim (2\pi)^{-1/2}q(\infty)\Gamma(1-\beta)e^{\pi i \operatorname{sgn}(x)(1-\beta)/2}|x|^{\beta-1}$$
$$+ (2\pi)^{-1/2}p(\infty)\Gamma(1-\alpha)e^{\pi i \operatorname{sgn}(x)(\alpha-1)/2}|x|^{\alpha-1} \qquad (x \to \pm 0).$$

Proof. T can be decomposed into $T = T_g + S + T_f$ where the supports of T_g, S, and T_f are $(-\infty, d]$, $[d, c]$, and $[c, \infty)$, respectively. S has compact support, hence $\hat{S}$ is regular and defined by the function u such that $u(x) = (2\pi)^{-1/2}\langle S_t, e^{-ixt}\rangle$ and u can be extended to the complex plane as an entire function [6, p. 307].

Then for T_g and T_f we have $\hat{T}_g = T_{\hat{g}}$ and $\hat{T}_f = T_{\hat{f}}$. Also, $\mathcal{F}[T] = \mathcal{F}[T_g + S + T_f] = \hat{T}_g + \hat{S} + \hat{T}_f = T_{\hat{g}} + T_u + T_{\hat{f}} = T_\Phi$ where $\Phi = \hat{g} + u + \hat{f}$. Then u is bounded as $x \to \pm 0$, and the behavior of $\hat{g}$ and $\hat{f}$ is given by Theorem 3.2 with the functions g and p appearing in that theorem being identically 0.

It is now easy to see why we do not have a similar result for $x \to \pm\infty$. The dominant term in the behavior of Φ as $x \to \pm\infty$ may be u, the restriction to the real axis of an entire function, and it does not necessarily approach 0 at ∞ and may swamp the contributions from $\hat{g}$ and $\hat{f}$.

This suggests the question of what happens if we begin with $T \in \mathcal{S}'$ such that T is regular in an interval neighborhood of the origin, say $[-b, a]$. T could then be decomposed into $T = U + T_f + V$ where f is the locally integrable function defining T over $[-b, a]$. We would then have $\hat{T}_f = T_{\hat{f}}$, where $\hat{f}$ is bounded. However, at present we are not prepared to cope with U and V, both arbitrary elements of $\mathcal{S}'$. A more general form of this question will be answered in the next section.

4. **Laplace and Fourier transforms of singular distributions.** This section deals with arbitrary elements of $\mathcal{S}'$, and results of the final and initial value type are obtained for both the Laplace and Fourier transforms.

The decomposition of an arbitrary distribution is not quite as straightforward as in the semiregular case, but there is no great difficulty. Given any distribution $T \in \mathcal{D}'$, any point y in the support of T and an arbitrary positive number δ,

T can be written as $S + V$ where the supports of S and V are contained in $(-\infty, y]$ and $[y - \delta, \infty)$, respectively. This is because the two sets $(-\infty, y)$ and $(y - \delta, \infty)$ certainly form a locally finite open covering of the support of T, and the assertion follows from [3, Theorem 25, p. 66]. Zemanian's result for semiregular distributions can now be generalized by performing such a decomposition on a distribution $T \in \mathcal{S}'$ with support bounded on the left at zero.

Theorem 4.1. *Let $e^{-cx}T \in \mathcal{S}'$ with support in $[0, \infty)$ for some real number c, and decompose T into $S + V$ where S has support in $[0, a]$ and V has support in $[a - \delta, \infty)$ for positive numbers a and δ such that $\delta < a$. Then there exists a positive integer m such that $S = \sum_{i=0}^{m} D^i T_{\mu_i}$ where each μ_i is a Radon measure whose support is contained in $[0, a]$. If each μ_i is absolutely continuous so that $d\mu_i = f_i\,dx$ where $f_i \in L^1[0, a]$, and if $f_i(x) = x^{-\alpha_i} g_i(x)$ for each x in $(0, a)$ where $0 < \alpha_i < 1$, and $g_i(0^+)$ exist for all i between 0 and m, then we have*

$$G(\sigma) \sim \sum_{i=0}^{m} \Gamma(1 - \alpha_i) g_i(0^+) \sigma^{i+\alpha_i - 1} \qquad (\sigma \to \infty).$$

Proof. $T = S + V$, hence $G(\sigma) = \mathcal{L}[T] = \mathcal{L}[S] + \mathcal{L}[V] = G_1(\sigma) + G_2(\sigma)$. Since $|G_2(\sigma)| \leq Me^{-\sigma b}$ for some positive b [7, p. 246], $G_2(\sigma) \to 0$ faster than any negative power of σ. For S, the compact support is a closed interval and since any bounded convex set is regular, the given representation for S is justified [4, p. 99]. The Laplace transform converts differentiation into multiplication by powers [1, Corollary 2.4, p. 49], so $\mathcal{L}[S] = \sum_{i=0}^{m} \sigma^i \mathcal{L}[T_{\mu_i}]$. If each μ_i is absolutely continuous, the Radon-Nikodym theorem assures us of the existence of the unique $f_i(x) \in L^1[0, a]$ and if each $f_i(x) = x^{-\alpha_i} g_i(x)$, we have

$$\mathcal{L}[S] = \sum_{i=0}^{m} \sigma^i \mathcal{L}\left[T_{x^{-\alpha_i} g_i(x)}\right].$$

But the Laplace transform of a regular distribution is the same as that of the defining function, hence

$$\mathcal{L}\left[T_{x^{-\alpha_i} g_i(x)}\right] = \int_0^a e^{-\sigma x} x^{-\alpha_i} g_i(x)\,dx.$$

Then by the classical theorem for the Laplace transform [7, Theorem 8.6-1, p. 243],

$$G_1(\sigma) \sim \sum_{i=0}^{m} \Gamma(1 - \alpha_i) g_i(0^+) \sigma^{i+\alpha_i - 1} \qquad (\sigma \to \infty).$$

But $G_2(\sigma) \to 0$ faster than any negative power of σ, so its contribution is negligible compared to that of $G_1(\sigma)$.

Moving to Abelian theorems of the final value type for the Laplace transform of a singular distribution, a slightly more specific representation for distributions of compact support is more suitable. The various existing representation theorems of this type do not always get the support of the measures contained in the support of the distribution, so in going to other representations it may be necessary to sacrifice this desirable support condition to get stronger conditions on the measures. In fact, some results from Treves [6] do exactly this. Each distribution of compact support is of finite order. Theorem 24.4 on p. 259 of [6] gives a representation for distributions of finite order as the finite sum of derivatives of Radon measures whose supports are contained in an arbitrary open set containing the support of the distribution. Then Theorem 24.5 on p. 262 in turn says that for every Radon measure μ there is a locally L^∞ function f whose support is contained in an arbitrary open set containing the support of μ such that $T_\mu = DT_f$. Since locally each L^∞ function is L^1, this gives us the desired representation of a distribution of compact support as a finite sum of derivatives of locally L^1 functions, yet the supports of these functions are contained in some arbitrary open set containing the support of the distribution.

We now will also need a representation for tempered distributions and Schwartz [4, p. 239] supplies a suitable one. A distribution $T \in \mathcal{S}'$ can be represented as the jth order derivative of a continuous tempered function. That is, $T = D^j T_{(1+x^2)^{k/2} f(x)}$, where f is bounded and continuous. As in the representation theorems for distributions of compact support, there is some flexibility here also. If the support of T does not contain the origin, the representation can be written $T = D^j T_{x^k f(x)}$. It is also possible to get a representation with tempered measures. With these representations at our disposal we are now ready to prove the final value type result for the Laplace transform.

Theorem 4.2. *Let $T \in \mathcal{S}'_+$ such that for some real t, $e^{-tx}T \in \mathcal{S}'_+$, and let $T = S + V$ where S has support in $[-b, a]$ and V has support in $[a - \delta, \infty)$ for positive numbers δ, a, and b such that $\delta < a$. Then given any open neighborhood Ω of $[-b, a]$ there exists a nonnegative integer m such that $S = \sum_{i=0}^m D^i T_{f_i}$ where each f_i is in $L^1(\Omega)$, having support in Ω. Also, there exist nonnegative integers l and k such that $V = D^l T_{x^k g(x)}$ where g is bounded and continuous, having support containing $[a - \delta, \infty)$. Then if $\bar{\Omega} = [-d, c]$ and if $g(x) = x^{-\alpha} h(x)$ for $x > y$ where $y > a$, $\alpha < 1 + k$, and $h(\infty)$ exists, we have*

$$G(\sigma) \sim \Gamma(1 - \alpha + k) h(\infty) \sigma^{l+\alpha-k-1} + \sum_{i=0}^{m} K_i \sigma^i \qquad (\sigma \to 0^+)$$

where $K_i = \int_{-d}^{c} f_i(x)\,dx$.

Proof. Again $\mathfrak{L}[T] = \mathfrak{L}[S] + \mathfrak{L}[V]$. The representation for S is justified by the remarks preceding the theorem, so

$$\mathfrak{L}[S] = \sum_{i=0}^{m} \sigma^i \mathfrak{L}[T_{f_i}] = \sum_{i=0}^{m} \sigma^i \mathfrak{L}[f_i] = \sum_{i=0}^{m} \sigma^i \int_{-d}^{c} e^{-\sigma x} f_i(x)\, dx.$$

Since we are considering $\sigma \to 0^+$, assume that $0 < \sigma < \xi$ for some positive number ξ. Then

$$|e^{-\sigma x} f_i(x)| \leq e^{d\xi} |f_i(x)| \in L^1(-d, c).$$

Then by Lebesgue's Dominated Convergence Theorem we have

$$\int_{-d}^{c} e^{-\sigma x} f_i(x)\, dx \to \int_{-d}^{c} f_i(x)\, dx \qquad (\sigma \to 0^+).$$

Then

$$\mathfrak{L}[V] = \mathfrak{L}[D^l T_{x^k g(x)}] = \sigma^l \mathfrak{L}[T_{x^k g(x)}] = \sigma^l \mathfrak{L}[x^k g(x)].$$

Let $m = -k$, so $m \leq 0$. The last expression then equals

$$\sigma^l \mathfrak{L}[x^{-(m+\alpha)} h(x)] \sim \Gamma(1 - m - \alpha) h(\infty) \sigma^{l+\alpha+m-1} \qquad (\sigma \to 0^+)$$

if $m + \alpha < 1$ or $\alpha < 1 - m = 1 + k$ by the classical result.

In the preceding initial value type theorem the distributions and measures were restricted to have support in $[0, \infty)$ by the growth of $e^{-\sigma x}$ for negative x and large positive σ. Later we will investigate refinements of the basic theorems and see that we can get rid of this restriction for the Laplace transform. This problem is not present in the basic results for the Fourier transform, but restrictions of a different nature occur.

Theorem 4.3. *Let T be a tempered distribution and write $T = U + S + V$ where the supports of U, S, and V are $(-\infty, -b]$, $[-b-\epsilon, a+\delta]$, and $[a, \infty)$, respectively, where a, b, ϵ and δ are positive numbers such that $\epsilon < b$ and $\delta < a$. Then there exist functions $g \in L^1(b, \infty)$ and $h \in L^1(a, \infty)$ such that $U = D^m((-x)^k g(-x))$ and $V = D^n(x^l h(x))$. Assume g and h have support in $[b, \infty)$ and $[a, \infty)$, respectively. Also $S = \Sigma_{j=0}^{r} D^j T_{f_j}$ where each f_j has support in $\bar{\Omega}$ and is in $L^1(\bar{\Omega})$, with Ω some open interval containing $[-b-\epsilon, a+\delta]$, say $(-d, c)$. If $l = 0$, $k = 0$, $g(x) = x^{-\beta} q(x)$ for $x \in [b, \infty)$, and $h(x) = x^{-\alpha} p(x)$ for $x \in [a, \infty)$, where $0 < \alpha < 1$, $0 < \beta < 1$, $g \in BV[b, \infty)$ and $h \in BV[a, \infty)$, then the Fourier transform of T is a regular distribution defined by the function Φ such that*

$$\Phi(x) \sim (2\pi)^{-1/2} q(\infty)\Gamma(1-\beta)e^{\pi i \operatorname{sgn}(x)(m+1-\beta)/2}|x|^{m+\beta-1}$$
$$+ (2\pi)^{-1/2} \sum_{j=0}^{r} e^{j \operatorname{sgn}(x)\pi i/2} \int_{-d}^{c} f_j(t)\,dt\,|x|^j$$
$$+ (2\pi)^{-1/2} p(\infty)\Gamma(1-\alpha)e^{\pi i \operatorname{sgn}(x)(n+\alpha-1)/2}|x|^{n+\alpha-1} \qquad (x \to \pm 0).$$

Proof. The representations for U, S, and V are justified by the remarks preceding the results for the Laplace transform. For S we have $\hat{S} = \Sigma_{j=0}^{r} (ix)^j \hat{T}_{f_j} = \Sigma_{j=0}^{r} (ix)^j T_{\hat{f}_j}$. But $(ix)^j = |x|^j e^{j \operatorname{sgn}(x)\pi i/2}$, and

$$\hat{f}_j(x) = (2\pi)^{-1/2} \int_{-d}^{c} e^{-ixt} f_j(t)\,dt \sim (2\pi)^{-1/2} \int_{-d}^{c} f_j(t)\,dt \qquad (x \to \pm 0),$$

again by Lebesgue's Dominated Convergence Theorem. For U,

$$\hat{U} = (ix)^m \hat{T}_{(-x)^k g(-x)} = (ix)^m D^k \hat{T}_{g(-x)}.$$

We now cannot infer the behavior of $D^k \hat{g}$ from the behavior of $\hat{g}$, hence the necessity of the assumption that $k = 0$. In this case, $\hat{U} = (ix)^m T_{\widehat{g(-x)}}$. But by Theorem 3.2(iv),

$$\widehat{g(-x)} \sim (2\pi)^{-1/2} q(\infty)\Gamma(1-\beta)e^{\pi i \operatorname{sgn}(x)(1-\beta)/2}|x|^{\beta-1} \qquad (x \to \pm 0).$$

The argument for $\hat{V}$ is similar and the result follows from the fact that $\hat{T} = \hat{U} + \hat{S} + \hat{V}$.

Using slightly different initial conditions and the alternate representation for S, it is possible to get an initial value result for the same type of distribution.

Theorem 4.4. *Let $T = U + S + V$ as in the preceding theorem, again with $U = D^m((-x)^k g(-x))$ and $V = D^n(x^l h(x))$. Let $S = \Sigma_{j=0}^{s} D^j T_{\mu_j}$ where each μ_j is an absolutely continuous measure whose support is contained in the support of S, so that $d\mu_j = f_j dx$ where $f \in L^1[-b, a]$. If $f_j(x) = x^{-\eta_j} u_j(x)$, $0 < x < a$, and $f_j(-x) = x^{-\nu_j} v_j(x)$, $0 < x < b$, for each j, and if $h(x) = (x-a)^{-\zeta} w(x)$, $a < x < \infty$, and $g(-x) = (x-b)^{-\xi} z(x)$, $b < x < \infty$, where $w \in BV(a, c)$, $z \in BV(b, d)$, $0 < \zeta < 1$, $0 < \xi < 1$, and $0 < \eta_j < 1$, $0 < \nu_j < 1$, $u_j \in BV(0, a)$ and $v_j \in BV(0, b)$ for each j between 0 and s, then the Fourier transform of T is a regular distribution defined by a function Φ such that*

$$\Phi(x) \sim (2\pi)^{-1/2} z(b^+)\Gamma(1-\xi)e^{-ibx+\pi i \operatorname{sgn}(x)(m+1-\xi)/2}|x|^{m+\xi-1}$$
$$+ \sum_{j=0}^{s} [(2\pi)^{-1/2} v_j(0^+)\Gamma(1-\nu_j)e^{\pi i \operatorname{sgn}(x)(j+1-\nu_j)/2}|x|^{j+\nu_j-1}$$
$$+ (2\pi)^{-1/2} u_j(0^+)\Gamma(1-\eta_j)e^{\pi i \operatorname{sgn}(x)(j+\eta_j-1)/2}|x|^{j+\eta_j-1}]$$
$$+ (2\pi)^{-1/2} w(a^+)\Gamma(1-\zeta)e^{-iax+\pi i \operatorname{sgn}(x)(n+\zeta-1)/2}|x|^{n+\zeta-1} \qquad (x \to \pm\infty).$$

Proof. For V we have $\hat{V} = (ix)^n \hat{T}_h = (ix)^n T_{\hat{h}}$. But

$$\hat{h}(x) = (2\pi)^{-1/2} \int_a^\infty (t-a)^{-\zeta} w(t) e^{-ixt}\, dt$$

$$= (2\pi)^{-1/2} e^{-iax} \int_0^\infty u^{-\zeta} w(u+a) e^{-iux}\, du$$

$$\sim (2\pi)^{-1/2} w(a^+) \Gamma(1-\zeta) e^{-iax + \pi i \operatorname{sgn}(x)(\zeta-1)/2} |x|^{\zeta-1} \qquad (x \to \pm\infty)$$

by Theorem 3.2. Combined with $(ix)^n$ we obtain the last term of the result. The first term follows similarly from $\hat{U}$. Finally, S gives

$$\hat{S} = \sum_{j=0}^{s} (ix)^j \hat{T}_{\mu_j} = \sum_{j=0}^{s} (ix)^j T_{\hat{\mu}_j} = \sum_{j=0}^{s} (ix)^j T_{\hat{f}_j}.$$

The behavior of each $\hat{f}_j$ is given by $\hat{f}_j = \hat{f}_j^- + \hat{f}_j^+$ and Theorem 3.2.

Note on Theorems 4.3 *and* 4.4. In the representations for U and V the absolutely integrable functions g and h might not have support in $[b, \infty)$ and $[a, \infty)$, respectively. If the support of h, for example, is contained in a neighborhood of $[a, \infty)$, the nature of h in the complement of $[a, \infty)$ must be taken into account in both theorems. This additional contribution to the representation is a function integrable over a relatively compact set, hence in analyzing the asymptotic behavior of T it may be lumped together with the representation for S and treated accordingly. In Theorem 4.3 the local integrability of h is sufficient and from the argument for S we see that the contribution to the final result is the term

$$(2\pi)^{-1/2} e^{n \operatorname{sgn}(x)\pi i/2} \int_{a-\epsilon}^{a} h(t)\, dt\, |x|^n$$

where $[a-\epsilon, \infty)$ contains the support of h. The situation is not as simple, however, in Theorem 4.4. The integrability of h over $[a-\epsilon, a]$ is not a sufficient condition to determine asymptotic behavior of the transform of the distribution, so we must assume a singularity at the point a. If $h(x) = (a-x)^{-\lambda} f(x)$ over the interval $(a-\epsilon, a)$ where $0 < \lambda < 1$ and $f \in BV(a-\epsilon, a)$, then Φ will contain the additional term

$$(2\pi)^{-1/2} f(a^-) \Gamma(1-\lambda) e^{-ixa + \pi i \operatorname{sgn}(x)(\lambda-1)/2} |x|^{\lambda-1}.$$

To see this note that $h(x) = (2\pi)^{-1/2} \int_{a-\epsilon}^{a} (a-t)^{-\lambda} f(t) e^{-ixt}\, dt$. The result then follows by using the methods of the proof of Theorem 3.2(i). U and g should be treated similarly.

We can now examine in more precise terms the remark in the introduction that, roughly speaking, transforms of distributions are better behaved at zero and worse behaved at infinity than are the transforms of functions.

Given an L^1 function f, the Riemann-Lebesgue lemma tells us that $\hat{f}(x)$

tends to zero as x tends to infinity. In the special case discussed in §3, Titchmarsh demonstrates the rate of decrease. The Laplace transform for a similar case behaves in the same manner. We also considered theorems for both transforms which yielded growth at zero like $Kx^{\alpha-1}$ where $0 < \alpha < 1$.

For distributions, $e^{-cx}T \in \mathcal{S}'$ with support in $[0, \infty)$ implies $e^{-cx}T = D^m T_{(1+x^2)^{k/2}g(x)}$ for nonnegative integers k, m and bounded and continuous g. Thus in the special case $c = 0$, $\mathcal{L}[T]$ behaves like the function Kx^{m-k-1}, which may tend to zero or infinity at either zero or infinity depending upon the relative sizes of m and k. As mentioned previously, the Fourier transform of a distribution of compact support is an entire function, hence bounded at zero and possibly unbounded at infinity. A theorem of Paley and Wiener [6, Chapter 29] indicates that if $T \in \mathcal{E}'$ then $\hat{T}$ is regular, defined by a function which is bounded by a polynomial on the real axis. The special cases for the Laplace transform of semiregular distributions had behavior identical to that of the defining functions, but the Fourier transform of a semiregular distribution of that form contained the contribution from an entire function so it had undetermined growth at infinity. Then the generalization to transforms of singular distributions yielded both Fourier and Laplace transforms which behave like sums of terms of the form Kx^{m-k-1}.

In all but a few of the cases mentioned for distributions we have no assurance that the transforms will not blow up rapidly at infinity. Since we are assured of such "nice" behavior of Fourier transforms and reasonably "nice" behavior of Laplace transforms of L^1 functions, the transforms of distributions clearly behave much "worse" at infinity than do the transforms of functions.

The behavior of transforms at zero is another matter, since the polynomials and entire functions contribute to, rather than detract from, the "good" behavior there. The special cases of both Laplace and Fourier transforms of functions behaved like $Kx^{\alpha-1}$ at zero. By comparison, many of the transforms of distributions mentioned above have behavior like polynomials or entire functions, hence the comment about the "better" behavior of distribution transforms at zero is a reasonable one.

5. **Improved Abelian theorems.** There are now two final questions which remain to be investigated. As mentioned earlier, the restriction of the support of the distributions to $[0, \infty)$ in the initial value type theorem for the Laplace transform is not necessary, so we need to investigate the effect of allowing the support to contain the origin as an interior point. Also, throughout the development of the Abelian theorems for distributions, no special properties of the particular classical Abelian theorems have been utilized. This suggests that our method of generalization would apply readily to other classical results, and this turns out to be true.

We have already considered the problem of singularities of a function at points other than zero for the Fourier transform of distributions. In Theorem 4.4 we had $T = U + S + V$ and an appropriate representation theorem for each of the three elements of the decomposition. For example, the support of V is $[a, \infty)$ and $V = D^n(x^l h(x))$ where $h \in L^1(0, \infty)$. Then $\hat{V} = (ix)^n T_{\hat{h}(x)}$ and

$$\hat{h}(x) = (2\pi)^{-1/2} \int_0^\infty h(t) e^{-ixt}\, dt \to 0 \text{ as } x \to \infty.$$

This implies that $\hat{V} = o(x^n)$ as $x \to \infty$, but this estimate is rather crude. The assumption of a singularity of $h(x)$ at a gives a more precise result:

$$\hat{V} \sim (2\pi)^{-1/2} w(a^+) \Gamma(1 - \zeta) e^{-iax + \pi i \operatorname{sgn}(x)(n+\zeta-1)/2} |x|^{n+\zeta-1} \qquad (x \to \pm\infty).$$

Thus we see that the expression we obtain by taking $h(x) = (x - a)^{-\zeta} w(x)$ on (a, c) with $0 < \zeta < 1$ and $w \in BV(a, c)$ differs from the expression resulting from a singularity at zero by the factor e^{-iax} and contains $w(a^+)$ rather than the limit at zero from the right of some function. We can consider distributions whose representing functions have a finite number of singularities of this kind, and each will make a similar contribution to the behavior of the transform of the distribution at ∞.

The effect of considering singularities at nonzero points for the Laplace transform is somewhat similar. If f is a Laplace transformable function having support in $[a, \infty)$ such that $f(x) = (x - a)^\alpha g(x)$ in (a, b) for some $b > a$, where $\alpha < 1$ and $g(a^+)$ exists, then

$$G(\sigma) = \int_{-\infty}^\infty (x - a)^{-\alpha} g(x) e^{-\sigma x}\, dx.$$

By a change of variable,

$$G(\sigma) = e^{-\sigma a} \int_{-\infty}^\infty u^{-\alpha} g(u + a)\, du \sim \Gamma(1 - \alpha) g(a^+) e^{-\sigma a} \sigma^{\alpha - 1} \qquad (\sigma \to \infty).$$

So the additional contribution to the asymptotic behavior in the case of the initial value type theorem for the Laplace transform is a factor $e^{-\sigma a}$, and just as for the Fourier transform we may consider any function that has a finite number of singularities of this kind.

For the final value type Abelian theorem we can essentially reverse this procedure. That is, if $f(x) = e^{ax} x^{-\beta} h(x)$ on $[b, \infty)$ for some real b where $\beta < 1$ and $h(\infty)$ exists, then $G(\sigma) \sim \Gamma(1 - \beta) h(\infty)(\sigma - a)^{\alpha - 1}$ $(\sigma \to a^+)$ if $G(\sigma)$ exists [2, p. 459]. Again we can consider functions that are a finite sum of functions of this nature.

These minor generalizations have been based on the fact that the Abelian theorems for transforms depend directly on the behavior of the representing functions. If we begin with different Abelian theorems they will imply

distribution results if the functions concerned satisfy the conditions imposed by the particular distribution representation. Even if they are not immediately satisfied, we may still be able to get a distribution theorem since there is a certain amount of flexibility in the representations. For example, if we have the representation $T = D^m T_\mu$ where μ is a measure, we can integrate μ one or more times to get a function with the desired properties at the sacrifice of increasing m by the corresponding number.

As an example of an improved Abelian theorem we will consider one from Doetsch [2, p. 460] and use it to obtain a distribution result. Theorem 4.2 generalizes to the following:

Theorem 5.1. *Let $e^{-cx}T \in \mathcal{S}'_+$ for some real number c, and let $T = S + V$ where S has support in $[-b, a]$ and V has support in $[a - \delta, \infty)$ for positive numbers δ, a, and b such that $\delta < a$. Then S is of compact support and hence given any open neighborhood Ω of $[-b, a]$ there exists a positive integer m such that $S = \sum_{i=0}^m D^i T_{f_i}$ where each f_i is in $L^1(\Omega)$, having support in Ω. Also, there exist positive integers l and k such that $V = D^l T_{x^k g(x)}$ where g is bounded and continuous, having support in $[d, \infty)$ for some $d < a$. Then if $\overline{\Omega} = [-y, t]$, g is absolutely integrable over $[d, \infty)$ and if $g(x) \sim Ax^{-\alpha}h(x)$ $(x \to \infty)$ for complex numbers A and α such that $\mathfrak{Re}(\alpha) < 1$, and if $h(x)$ satisfies $h(ux)/h(x) \to 1$ for any $u > 0$ we have*

$$G(\sigma) \sim A\Gamma(1-\alpha)h(1/\sigma)\sigma^{l+u+\alpha-1} + \sum_{i=0}^{m} K_i\sigma^i \qquad (\sigma \to 0^+)$$

where $K_i = \int_{-y}^{t} f_i(x)\,dx$.

In the classical Abelian theorem that was generalized to yield Theorem 4.1 we had $f(x) \sim K_1 x^{-\alpha}$ $(\alpha < 1)$ as $x \to 0^+$ implies $\mathfrak{L}[f] \sim K_2\sigma^{\alpha-1}$ as $\sigma \to 0^+$. If $K_2 = 0$, the theorem says only that $f(x) = O(x^{-\alpha})$ as $x \to 0^+$ implies $\mathfrak{L}[f](\sigma) = o(\sigma^{\alpha-1})$ as $\sigma \to \infty$. To get more precise information than this we need to compare f to something other than powers. The function $h(x)$ provides greater flexibility in comparison.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, DAVIS, CALIFORNIA 95616