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ULTRAFILTERS AND INDEPENDENT SETS[1]

BY

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ABSTRACT. Independent families of sets and of functions are used to prove some theorems about ultrafilters. All of our results are well known to be provable from some form of the generalized continuum hypothesis, but had remained open without such an assumption. Independent sets are used to show that the Rudin-Keisler ordering on ultrafilters is nonlinear. Independent functions are used to prove the existence of good ultrafilters.

1. **General notation.** If A and B are sets, B^A is the set of functions from A into B, $\mathcal{P}(A)$ is the set of subsets of A, and $S_\omega(A)$ is the set of finite subsets of A.

We identify cardinals with initial von Neumann ordinals. We use ξ and η to range over ordinals, and κ to range over infinite cardinals. $|A|$ is the cardinality of A. If $|A| = \kappa$, $2^\kappa = |\mathcal{P}(A)|$. κ^+ is the first cardinal bigger than κ. $\xi + 1$ is the first ordinal bigger than ξ. $\omega = \aleph_0$ is the first infinite ordinal and the first infinite cardinal and the set of nonnegative integers.

A *filter* over an infinite set I is a nonempty subset $\mathcal{F}$, of $\mathcal{P}(I)$, such that $\mathcal{F}$ is closed under finite intersections and supersets. $\mathcal{P}(I)$ is the *improper filter* over I; other filters are called *proper filters.* An *ultrafilter* is a maximal proper filter. An ultrafilter, $\mathfrak{U}$, over I, is *uniform* iff $|A| = |I|$ for all $A \in \mathfrak{U}$.

If $\mathfrak{A} \subseteq \mathcal{P}(I)$, $((\mathfrak{A}))_I$ is the filter generated by $\mathfrak{A}$, i.e.

$$((\mathfrak{A}))_I = \bigcap\{\mathcal{F}: \mathfrak{A} \subseteq \mathcal{F} \subseteq \mathcal{P}(I) \ \&\ \mathcal{F} \text{ is a filter}\}.$$

The generalized *Fréchet filter*, $\mathcal{FR}_I$, is $\{X \subseteq I: |I \sim X| < |I|\}$. Thus, an ultrafilter $\mathfrak{U}$ over I is uniform iff $\mathcal{FR}_I \subseteq \mathfrak{U}$. The subscripts I will be dropped from the notations $((\mathfrak{A}))_I$ and $\mathcal{FR}_I$ when I is understood.

2. **Nonlinearity of the Rudin-Keisler ordering.** If I is any infinite set, βI denotes the set of ultrafilters over I. If $f: I \to J$, f_* or βf is the function

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from βI into βJ defined by

$$f_*(\mathfrak{U}) = (\beta f)(\mathfrak{U}) = \{Y \subseteq J: f^{-1}(Y) \in \mathfrak{U}\}.$$

Thus, if we identify $\mathfrak{U}$ with a 2-valued measure on I, $f_*(\mathfrak{U})$ is the induced measure on J in the usual measure-theoretic sense. Note that if $f: I \to J$ and $g: J \to K$, then

$$\beta(g \circ f) = (\beta g) \circ (\beta f): \beta I \to \beta K.$$

We remark on some relationships between ultrafilters and topology, although these remarks are not needed for this paper. We can consider βI to be a topological space by identifying it with the Stone space of the Boolean algebra $\mathcal{P}(I)$, or, equivalently, with the Čech compactification of the space I with the discrete topology. β is then a covariant function from the category of sets and maps to the category of compact topological spaces and continuous maps.

The ordering $\preceq$ on ultrafilters was defined independently by M. E. Rudin and H. J. Keisler as follows: If $\mathfrak{U} \in \beta I$ and $\mathfrak{V} \in \beta J$, $\mathfrak{U} \preceq \mathfrak{V}$ iff there is a function $f: J \to I$ such that $\mathfrak{U} = f_*\mathfrak{V}$. It is easy to check that $\preceq$ is transitive. So, if we define $\mathfrak{U} \approx \mathfrak{V}$ iff both $\mathfrak{U} \preceq \mathfrak{V}$ and $\mathfrak{V} \preceq \mathfrak{U}$, then $\approx$ is an equivalence relation. That $\approx$ is a reasonable notion of equivalence is indicated by

2.1. **Theorem.** *Let* $\mathfrak{U} \in \beta I$, $\mathfrak{V} \in \beta J$.

(a) $\mathfrak{U} \approx \mathfrak{V}$ *iff there are* $X \in \mathfrak{U}$, $Y \in \mathfrak{V}$, *and* $f: I \to J$ *such that* $\mathfrak{V} = f_*\mathfrak{U}$ *and* f *restricted to* X *is* 1-1 *onto* Y.

(b) *If* $\mathfrak{U} \approx \mathfrak{V}$ *and* $|I| = |J|$, *there is an* $f: I \to J$ *such that* f *is* 1-1 *and onto*, $\mathfrak{V} = f_*\mathfrak{U}$, *and* $\mathfrak{U} = (f^{-1})_*\mathfrak{V}$.

This theorem is proved by an easy modification of methods in M. E. Rudin [7], to which we refer the reader for more details on $\preceq$ and other orderings of ultrafilters.

It is reasonable to confine one's study to uniform ultrafilters. Indeed, if $\mathfrak{U} \in \beta I$ is not uniform let $J \subset I$ be an element of $\mathfrak{U}$ of least cardinality. Then $\mathfrak{U}$ is equivalent under $\approx$ to the uniform ultrafilter $\mathfrak{U} \cap \mathcal{P}(J) \in \beta J$.

Let $\beta_u I$ be the set of uniform ultrafilters over I. $\beta_u I$ is a closed subspace of βI and may be identified with the Stone space of the Boolean algebra $\mathcal{P}(I)/\mathfrak{F}\mathfrak{R}$.

The main result of this section is that $\preceq$ restricted to $\beta_u I$ is not linear, i.e.

2.2. **Theorem.** *If* I *is infinite, there are* $\mathfrak{U}, \mathfrak{V} \in \beta_u I$ *such that* $\mathfrak{U} \not\preceq \mathfrak{V}$ *and* $\mathfrak{V} \not\preceq \mathfrak{U}$.

Before proving this theorem, we interject some technical remarks. If $|I| = \kappa$ and $2^\kappa = \kappa^+$, Theorem 2.2 is established by a trivial transfinite induction (see below). In fact, it is well known that in this case there is a family of 2^{2^κ}

noncomparable *minimal* elements in $\beta_u(I)$. However, the assumption here that $2^\kappa = \kappa^+$ cannot be omitted, since, for example, one cannot prove, without the continuum hypothesis, that any minimal elements exist for countable I (there are none in the model obtained by adjoining $\aleph_2$ random reals to a model of set theory plus the continuum hypothesis).

When I is countable, $\beta_u I$ is the same as $\beta I - I$ (the space of nonprincipal ultrafilters over I). For any I, ultrafilters minimal in $\beta I - I$ are known as selective, or Ramsey ultrafilters. Such ultrafilters are very rare. Not only need they not exist for countable I, but, if $|I| = \kappa > \aleph_0$, they exist iff κ is a measurable cardinal; in this case, the selective ultrafilters are exactly those equivalent to normal ultrafilters on κ, and the statement that there are nonequivalent selective ultrafilters is both consistent with (see [6, §2]) and independent from (see [5, §6]) the axioms of set theory.

Now, to prove Theorem 2.2, we must construct $\mathfrak{U}$, $\mathfrak{V}$ in $\beta_u I$ such that, for every function $f\colon I \to I$, $\mathfrak{V} \neq f_*\mathfrak{U}$ and $\mathfrak{U} \neq f_*\mathfrak{V}$, so we must have, for every such f,

$$(*)\qquad \exists X \in \mathfrak{U}[(I \sim f^{-1}(X)) \in \mathfrak{V}] \;\&\; \exists Y \in \mathfrak{V}[(I \sim f^{-1}(Y)) \in \mathfrak{U}].$$

Say $|I| = \kappa$. The construction will be carried out by transfinite induction over the ordinals $\eta < 2^\kappa$. Thus, we shall construct an increasing sequence of filters $\mathfrak{F}_\eta$, $\mathfrak{G}_\eta$ $(\eta < 2^\kappa)$ and take $\mathfrak{U}$, $\mathfrak{V}$ to be ultrafilters extending $\bigcup\{\mathfrak{F}_\eta\colon \eta < 2^\kappa\}$, $\bigcup\{\mathfrak{G}_\eta\colon \eta < 2^\kappa\}$, respectively. Fix an enumeration f_η $(\eta < 2^\kappa)$ of all the functions from I into I. At the ηth stage in the construction, we shall insure that $(*)$ holds for the function f_η. More precisely, we do our construction so that the following hold:

(i) For each $\eta < 2^\kappa$, $\mathfrak{F}_\eta$ and $\mathfrak{G}_\eta$ are filters over I.

(ii) For $\xi < \eta < 2^\kappa$, $\mathfrak{F}_\xi \subseteq \mathfrak{F}_\eta$ and $\mathfrak{G}_\xi \subseteq \mathfrak{G}_\eta$.

(iii) $\mathfrak{F}_0 = \mathfrak{G}_0 = \mathfrak{F}\mathfrak{R}$.

(iv) If η is a limit ordinal, $\mathfrak{F}_\eta = \bigcup\{\mathfrak{F}_\xi\colon \xi < \eta\}$ and $\mathfrak{G}_\eta = \bigcup\{\mathfrak{G}_\xi\colon \xi < \eta\}$.

(v) $\exists X \in \mathfrak{F}_{\eta+1}[(I \sim f_\eta^{-1}(X)) \in \mathfrak{G}_{\eta+1}] \;\&\; \exists Y \in \mathfrak{G}_{\eta+1}[(I \sim f_\eta^{-1}(Y)) \in \mathfrak{F}_{\eta+1}]$.

Conditions (i)–(iv) present no problem, but (v) may become impossible at some stage η. For example, if η is a limit ordinal, the construction before stage η determines what $\mathfrak{F}_\eta$ and $\mathfrak{G}_\eta$ must be, and it might happen that they are already ultrafilters and that $\mathfrak{F}_\eta = f_{\eta *}(\mathfrak{G}_\eta)$. In the special case that $2^\kappa = \kappa^+$, we could always arrange for $\mathfrak{F}_\eta$ and $\mathfrak{G}_\eta$ to be generated by no more than κ sets, and a simple diagonal argument would show that the construction could be carried out at each stage. In the general case, we enlist the aid of the concept of independent sets.

2.3. **Definition.** A family $\mathcal{S} \subseteq \mathcal{P}(I)$ is *independent* iff, for each n and m, whenever $X_1, \cdots, X_n, Y_1, \cdots, Y_m$ are distinct elements of $\mathcal{S}$,

$$X_1 \cap \cdots \cap X_n \cap (I \sim Y_1) \cap \cdots \cap (I \sim Y_m) \neq 0.$$

The following theorem was proved first by Fichtenholz and Kantorovitch [3, esp. p. 80] for $|I|$ equal $\aleph_0$ or $2^{\aleph_0}$, and then, for all I and by a much easier proof, by Hausdorff [4]. In the next section we shall prove a more general result, due to Engelking and Karłowicz [2].

2.4. **Theorem (Fichtenholz-Kantorovitch, Hausdorff).** *For any infinite I, if $|I| = \kappa$, then there is an independent family $\mathcal{S} \subseteq \mathcal{P}(I)$ such that $|\mathcal{S}| = 2^\kappa$.*

As a generalization of the notion of independence,

2.5. **Definition.** If $\mathcal{S} \subseteq \mathcal{P}(I)$ and $\mathcal{F}$ is a filter over I, $\mathcal{S}$ is *independent* (mod $\mathcal{F}$) iff, whenever $X_1, \cdots, X_n, Y_1, \cdots, Y_m$ are distinct elements of $\mathcal{S}$, $(I \sim X_1) \cup \cdots \cup (I \sim X_n) \cup Y_1 \cup \cdots \cup Y_m \notin \mathcal{F}$.

Thus, $\mathcal{S}$ is independent iff $\mathcal{S}$ is independent (mod $\{I\}$). Note that if $\mathcal{S}$ is independent (mod $\mathcal{F}$) and $\mathcal{S} \neq 0$, then $\mathcal{F}$ is a proper filter and not an ultrafilter. Also, if $\mathcal{S}$ is independent (mod $\mathcal{F}$) and $\mathcal{A} \subseteq \mathcal{S}$, then $\mathcal{S} \sim \mathcal{A}$ is independent (mod $((\mathcal{F} \cup \mathcal{A}))$). Furthermore, the $\mathcal{S}$ of Theorem 2.4 can be taken to be independent (mod $\mathcal{F}\mathfrak{R}$). To see this, let $g: I \to I$ be such that $|g^{-1}(\{i\})| = \kappa$ for each $i \in I$. Then if $\mathcal{S}$ satisfies Theorem 2.4, let $\mathcal{S}' = \{g^{-1}(X) : X \in \mathcal{S}\}$. $\mathcal{S}'$ has cardinality 2^κ and is independent (mod $\mathcal{F}\mathfrak{R}$).

In order to prove Theorem 2.2, we keep a large family of sets, $\mathcal{S}_\eta$, independent (mod $\mathcal{F}_\eta$) and (mod $\mathcal{G}_\eta$). Thus, in addition to (i)–(v), we arrange for the following:

(vi) For each $\eta < 2^\kappa$, $\mathcal{S}_\eta$ is independent (mod $\mathcal{F}_\eta$) and independent (mod $\mathcal{G}_\eta$).

(vii) For $\xi < \eta < 2^\kappa$, $\mathcal{S}_\xi \supseteq \mathcal{S}_\eta$.

(viii) $|\mathcal{S}_\eta| = 2^\kappa$ for each $\eta < 2^\kappa$.

(ix) If η is a limit ordinal, $\mathcal{S}_\eta = \bigcap \{\mathcal{S}_\xi : \xi < \eta\}$.

(x) Each $\mathcal{S}_\eta \sim \mathcal{S}_{\eta+1}$ is finite.

Note that (viii) will be assured by (ix) and (x), provided that we have $|\mathcal{S}_0| = 2^\kappa$; but this is possible by Theorem 2.4. The inductive definition is carried out at successor stages by applying Lemma 2.6 twice.

2.6. **Lemma.** *Let $\mathcal{H}$, $\mathcal{K}$ be filters over I. Let $\mathcal{T}$ be infinite and independent* (mod $\mathcal{H}$) *and* (mod $\mathcal{K}$). *Let $f: I \to I$. Then there are filters $\mathcal{H}' \supseteq \mathcal{H}$, $\mathcal{K}' \supseteq \mathcal{K}$, and a family $\mathcal{T}' \subseteq \mathcal{T}$ such that $\mathcal{T}'$ is independent* (mod $\mathcal{H}'$) *and* (mod $\mathcal{K}'$), *$\mathcal{T} \sim \mathcal{T}'$ is finite, and, for some $B \in \mathcal{H}'$, $(I \sim f^{-1}(B)) \in \mathcal{K}'$.*

Proof. Fix $A \in \mathcal{T}$.

Case I. $\mathcal{T} \sim \{A\}$ is independent (mod $((\mathcal{K} \cup \{I \sim f^{-1}(A)\}))$). Take $\mathcal{T}' = \mathcal{T} \sim \{A\}$, $\mathcal{K}' = ((\mathcal{K} \cup \{I \sim f^{-1}(A)\}))$, $\mathcal{H}' = ((\mathcal{H} \cup \{A\}))$, and $B = A$.

Case II. Not Case I. Then there are distinct $X_1, \cdots, X_n, Y_1, \cdots, Y_m$ in $\mathcal{T} \sim \{A\}$ and $K \in \mathcal{K}$ such that

$$(I \sim X_1) \cup \cdots \cup (I \sim X_n) \cup Y_1 \cup \cdots \cup Y_m \supseteq K \cap (I \sim f^{-1}(A)).$$

Hence,

$$X_1 \cap \cdots \cap X_n \cap (I \sim Y_1) \cap \cdots \cap (I \sim Y_m) \cap K \subseteq f^{-1}(A),$$

so if we take $\mathcal{K}' = ((\mathcal{K} \cup \{X_1, \cdots, X_n, I \sim Y_1, \cdots, I \sim Y_m\}))$, then $I \sim f^{-1}(I \sim A) = f^{-1}(A) \in \mathcal{K}'$. Thus, we can take $\mathcal{H}' = ((\mathcal{H} \cup \{I \sim A\}))$, $\mathcal{T}' = \mathcal{T} \sim \{A, X_1, \cdots, X_n, I \sim Y_1, \cdots, I \sim Y_m\}$, and $B = I \sim A$.

Lemma 2.6 concludes the proof of Theorem 2.2. By a similar argument, one can show

2.7. **Theorem.** *If* $|I| = \kappa \geq \aleph_0$, *there is a family of* 2^κ *elements of* $\beta_u I$ *which are pairwise incomparable under* $\preceq$.

As another application of independent sets, we now prove

2.8. **Theorem.**[3] *If* $|I| = \kappa \geq \aleph_0$, *there is a* $\mathcal{U} \in \beta_u I$ *such that* $\mathcal{U}$ *is not generated by any subset of itself of cardinality less than* 2^κ.

Proof. Let $\mathcal{S}$ have cardinality 2^κ and be independent (mod $\mathcal{FR}$). Let $\mathfrak{S}$ be the set of sets of the form $I \sim \bigcap \{A_n : n \in \omega\}$, where the A_n are distinct elements of $\mathcal{S}$. Let $\mathcal{U}$ be any ultrafilter such that $\mathcal{FR} \cup \mathfrak{S} \cup \mathcal{S} \subseteq \mathcal{U}$.

Remarks. No $\mathcal{U} \in \beta_u I$ can be generated by less than κ^+ elements of $\mathcal{U}$. In the case $|I| = \aleph_0$, it is consistent with the axioms of set theory that $2^{\aleph_0} > \aleph_1$ and that there is a $\mathcal{U} \in \beta_u I$ generated by $\aleph_1$ elements of $\mathcal{U}$. Thus $\mathcal{U}$ can in fact be selective (such a $\mathcal{U}$ exists, for example, in the model obtained by adding $\aleph_2$ mutually Sacks-generic reals to a model of set theory plus the continuum hypothesis). It is also easy to check that in the standard Cohen model violating GCH at a regular κ, no $\mathcal{U} \in \beta_u \kappa$ can be generated by less than 2^κ elements of $\mathcal{U}$.

3. **The existence of good ultrafilters.** If $p: S_\omega(I) \to \mathcal{P}(I)$, p is *multiplicative* iff whenever $s, t \in S_\omega(I)$, $p(s \cup t) = p(s) \cap p(t)$; p is *monotone* iff whenever $s \subseteq t \in S_\omega(I)$, $p(s) \supseteq p(t)$. If $p, q: S_\omega(I) \to \mathcal{P}(I)$, $p \leq q$ iff, for all $s \in S_\omega(I)$, $p(s) \subseteq q(s)$.

3.1. **Definition (Keisler).** $\mathcal{U}$ in βI is *good* iff whenever $p: S_\omega(I) \to \mathcal{U}$ is

(3) **Added in proof.** It has come to our attention that Theorem 2.8 was first proved by B. Pospíšil (Publ. Fac. Sci. Univ. Masaryk, 1939, no. 270) by a topological argument. For more on this type of topological question, see the paper of I. Juhász in Comment. Math. Univ. Carolinae 8 (1967), 231–247 (MR 35 #7300). Also, the same combinatorial proof presented here was discovered earlier by Juhász and Hajnal (unpublished).

monotone, there is a multiplicative $q: S_\omega(I) \to \mathfrak{U}$ such that $q \leq p$.

Also, Keisler proved the following theorem under the assumption that $2^\kappa = \kappa^+$ (see [1] for details).

3.2. **Theorem.** *If $|I| = \kappa \geq \aleph_0$, there is a good, countably incomplete ultrafilter over I.*

We now present a proof which does not assume $2^\kappa = \kappa^+$. Our proof uses the notion of an independent family of functions.

3.3. **Definition.** If $\mathcal{S} \subseteq I^I$ and $\mathcal{F}$ is a filter over I, $\mathcal{S}$ is *independent* (mod $\mathcal{F}$) iff, whenever $f_1, \cdots, f_n$ are distinct members of $\mathcal{S}$ and $i_1, \cdots, i_n \in I$,

$$I \sim \{j: f_1(j) = i_1 \ \& \cdots \& \ f_n(j) = i_n\} \notin \mathcal{F}.$$

$\mathcal{S}$ is *independent* iff $\mathcal{S}$ is independent (mod $\{I\}$).

Note that if $\mathcal{S}$ is independent and $0 \subset J \subset I$, then $\{f^{-1}(J): f \in \mathcal{S}\}$ is an independent family of sets. Also, if $\mathcal{S}$ is independent and infinite, $\mathcal{S}$ is independent (mod $\mathcal{F}\mathcal{R}$).

3.4. **Theorem (Engelking-Karłowicz [2]).** *If $|I| = \kappa \geq \aleph_0$, there is an independent $\mathcal{S} \subseteq I^I$ such that $|\mathcal{S}| = 2^\kappa$.*

Proof. Let $\langle\langle s_i, r_i\rangle: i \in I\rangle$ enumerate $\{\langle s, r\rangle: s \in S_\omega(I) \ \& \ r \in I^{\mathcal{P}(s)}\}$. Let $\mathcal{S} = \{f_A: A \subseteq I\}$, where $f_A(i) = r_i(A \cap s_i)$.

Let A_η $(\eta < 2^\kappa)$ enumerate $\mathcal{P}(I)$. Let p_η $(\eta < 2^\kappa)$ enumerate all monotone functions from $S_\omega(I)$ into $\mathcal{P}(I)$ so that each monotone $p: S_\omega(I) \to \mathcal{P}(I)$ is listed 2^κ times. To prove Theorem 3.2, we construct $\mathcal{F}_\eta$ $(\eta < 2^\kappa)$ and $\mathcal{S}_\eta$ $(\eta < 2^\kappa)$ to satisfy the following:

(i) For each $\eta < 2^\kappa$, $\mathcal{F}_\eta$ is a filter over I, $\mathcal{S}_\eta \subseteq I^I$ and $\mathcal{S}_\eta$ is independent (mod $\mathcal{F}_\eta$).

(ii) For $\xi < \eta < 2^\kappa$, $\mathcal{F}_\xi \subseteq \mathcal{F}_\eta$ and $\mathcal{S}_\xi \supseteq \mathcal{S}_\eta$.

(iii) Each $|\mathcal{S}_\eta| = 2^\kappa$.

(iv) If η is a limit ordinal, $\mathcal{F}_\eta = \bigcup\{\mathcal{F}_\xi: \xi < \eta\}$ and $\mathcal{S}_\eta = \bigcap\{\mathcal{S}_\xi: \xi < \eta\}$.

(v) Each $\mathcal{S}_\eta \sim \mathcal{S}_{\eta+1}$ is finite.

(vi) $\mathcal{F}_0$ is generated by sets B_n $(n < \omega)$ such that $\bigcap\{B_n: n < \omega\} = 0$.

(vii) For $\eta < 2^\kappa$, either A_η or $I \sim A_\eta$ is in $\mathcal{F}_{\eta+1}$.

(viii) For $\eta < 2^\kappa$, if $p_\eta: S_\omega(I) \to \mathcal{F}_\eta$, then there is a multiplicative $q: S_\omega(I) \to \mathcal{F}_{\eta+1}$ such that $q \leq p_\eta$.

By (vii), $\mathfrak{U} = \bigcup\{\mathcal{F}_\eta: \eta < 2^\kappa\}$ will be an ultrafilter. If $p: S_\omega(I) \to \mathfrak{U}$ is monotone then, since $cf(2^\kappa) > \kappa$, $p: S_\omega(I) \to \mathcal{F}_\xi$ for some ξ. Applying (viii) to some $\eta > \xi$ such that $p_\eta = p$ shows that there is multiplicative $q: S_\omega(I) \to \mathcal{F}_{\eta+1} \subseteq \mathfrak{U}$ such that $q \leq p$. Thus, $\mathfrak{U}$ will be good. Condition (vi) insures that $\mathfrak{U}$ will be countably incomplete. To make (vi) hold, take $\mathcal{S}_0 \cup \{f\}$ to be independent and of power 2^κ. Take $B_n = \{i \in I: n < f(i) < \omega\}$ and $\mathcal{F}_0 = ((\{B_n: n < \omega\}))$.

Conditions (i)–(iv) will take care of themselves. To get (v), (vii) and (viii), we apply, at each stage η, Lemmas 3.5 and 3.6 successively.

3.5. **Lemma.** *If* $\mathcal{S}$ *is independent* (mod $\mathcal{F}$) *and* $A \subseteq I$, *there are* $\mathcal{S}' \subseteq \mathcal{S}$ *and* $\mathcal{F}' \supseteq \mathcal{F}$ *such that* $\mathcal{S}'$ *is independent* (mod $\mathcal{F}'$), $\mathcal{S} \sim \mathcal{S}'$ *is finite and either* A *or* $I \sim A$ *is in* $\mathcal{F}'$.

Proof.

Case I. $\mathcal{S}$ is independent (mod $((\mathcal{F} \cup \{A\}))$). Take $\mathcal{S}' = \mathcal{S}$, $\mathcal{F}' = ((\mathcal{F} \cup \{A\}))$.

Case II. Not Case I. Let $f_1, \cdots, f_n$ be distinct members of $\mathcal{S}$ and $i_1, \cdots, i_n \in I$ such that

$$I \sim \{j: f_1(j) = i_1 \ \& \cdots \& \ f_n(j) = i_n\} \in ((\mathcal{F}, A)).$$

Let $\mathcal{S}' = \mathcal{S} \sim \{f_1, \cdots, f_n\}$,

$$\mathcal{F}' = ((\mathcal{F} \cup \{\{j: f_1(j) = i_1 \ \& \cdots \& \ f_n(j) = i_n\}\})).$$

Note that $I \sim A \in \mathcal{F}'$.

3.6. **Lemma.** *If* $\mathcal{S}$ *is independent* (mod $\mathcal{F}$) *and* $p: S_\omega(I) \rightarrow \mathcal{F}$ *is monotone, then there are* $\mathcal{S}' \subseteq \mathcal{S}$, $\mathcal{F}' \supseteq \mathcal{F}$, *and multiplicative* $q: S_\omega(I) \rightarrow \mathcal{F}'$ *such that* $\mathcal{S}'$ *is independent* (mod $\mathcal{F}'$), $\mathcal{S} \sim \mathcal{S}'$ *is finite, and* $q \leq p$.

Proof. Fix $g \in \mathcal{S}$. Let $\mathcal{S}' = \mathcal{S} \sim \{g\}$. For each $t \in S_\omega(I)$, let

$$q_t(s) = \begin{cases} 0 & \text{if } s \not\subseteq t, \\ p(t) & \text{if } s \subseteq t. \end{cases}$$

Let t_i $(i \in I)$ enumerate $S_\omega(I)$. Let $q(s) = \bigcup \{q_{t_i}(s) \cap g^{-1}(\{i\}): i \in I\}$ and $\mathcal{F}' = ((\mathcal{F} \cup \text{range } q))$.

This concludes the proof of Theorem 3.2. The interest of good ultrafilters in model theory is that they make ultrapowers saturated. If A and B are two infinite structures of power $\leq \kappa$ and $\mathfrak{U}$ is a good, countably incomplete ultrafilter over κ, then, as Keisler showed (see [1]), the ultrapowers $A^\kappa/\mathfrak{U}$ and $B^\kappa/\mathfrak{U}$ have power 2^κ and are κ^+-saturated. Thus, if $2^\kappa = \kappa^+$ and A and B are elementarily equivalent, then $A^\kappa/\mathfrak{U}$ and $B^\kappa/\mathfrak{U}$ are isomorphic.

Shelah [8] has shown, without any assumption about 2^κ, that there is an ultrafilter over 2^κ which makes ultrapowers of elementarily equivalent models of power κ isomorphic. It is unknown, however, whether any ultrafilter over κ has this property.

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