ON THE NONSTANDARD REPRESENTATION OF MEASURES

BY

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ABSTRACT. In this paper it is shown that every finitely additive probability measure \( \mu \) on \( S \) which assigns 0 to finite sets can be given a nonstandard representation using the counting measure for some \( *\)-finite subset \( F \) of \( *S \). Moreover, if \( \mu \) is countably additive, then \( F \) can be chosen so that

\[
\int f \, d\mu = \text{st} \left( \frac{1}{\|F\|} \sum_{p \in F} *f(p) \right)
\]

for every \( \mu \)-integrable function \( f \). An application is given of such representations. Also, a simple nonstandard method for constructing invariant measures is presented.

Let \( S \) be a set in some set theoretical structure \( \mathbb{M} \) and let \( *S \) be the corresponding set in an enlargement \( *\mathbb{M} \) of \( \mathbb{M} \). Bernstein and Wattenberg have noted [2] that if \( F \) is a \( *\)-finite subset of \( *S \), then a finitely additive probability measure \( \mu_F \) can be defined for all subsets \( A \) of \( S \) by

\[
\mu_F(A) = \text{st}(\|A \cap F\|/\|F\|).
\]

They used this observation as the basis for a nonstandard proof of the theorem, due to Banach [1], which states that Lebesgue measure on \( [0, 1] \) can be extended to a totally defined (finitely additive) measure which is invariant under translations (mod 1).

This paper concerns the representation of probability measures as non-standard counting measures \( \mu_F \). Let \( \mu \) be any finitely additive probability measure which is defined on an algebra \( \mathcal{B} \) of subsets of \( S \) and which satisfies \( \mu(A) = 0 \) for each finite set \( A \) in \( \mathcal{B} \). In \( \S 1 \) it is shown that there exists a \( *\)-finite subset \( F \) of \( *S \) which satisfies \( \mu = \mu_F \) on \( \mathcal{B} \). This has the consequence that for any bounded, \( \mu \)-integrable function \( f \),

\[
\int f \, d\mu = \text{st} \left( \frac{1}{\|F\|} \sum_{p \in F} *f(p) \right).
\]
Moreover, if $\mathcal{B}$ is a $\sigma$-algebra and $\mu$ is countably additive, then $F$ can be chosen so that (2) holds for every $\mu$-integrable function.

Closely related to these results is a nonstandard representation for bounded linear functionals on the space $l^\infty$ of bounded sequences in $R$, which was given by Robinson [7]. In §2 a straightforward extension of Robinson’s result is used to give a nonstandard proof of a convergence result (Theorem 3) for bounded linear functionals on $C(X)$, where $X$ is a compact, Hausdorff space.

Also, in §3 a nonstandard construction of invariant measures is given which yields a particularly simple proof of Banach’s extension result for Lebesgue measure.

Preliminaries. The given structure $\mathbb{M}$ is assumed to have the set $R$ of real numbers as an element (thus also the set $N$ of nonnegative integers). Moreover, the embedding $x \mapsto {}^*x$ of $M$ into $^*\mathbb{M}$ is taken to be the identity on $R$. The standard part of a finite element $p$ of $^*R$ is denoted by $\text{st}(p)$. If $p, q \in {}^*R$, then $p = q$ means that $p - q$ is infinitesimal.

For each set $S$ in $\mathbb{M}$ and each $^{*}$-finite subset $F$ of $^{*}S$, $\|F\|$ is the "cardinality" of $F$, in the sense of $^{*}\mathbb{M}$. That is, if $c$ is the function assigning to each finite subset $A$ of $S$ the cardinality of $A$, then $\|F\| = c(F)$. Alternately, $\|F\|$ is the smallest element $\omega$ of $^*N$ for which there is an internal bijection between $F$ and $\{\omega' | \omega' \in \omega \}$ (For an introduction to the methods of nonstandard analysis see [5], [6] or [8]).

Given a set $S$, $\mathcal{P}(S)$ is the algebra of all subsets of $S$. Also, $l^\infty(S)$ is the linear space of all bounded, real valued functions on $S$, furnished with the sup norm. In this paper $\mu$ is a measure on $S$ if it is a nonnegative, finitely additive set function defined on an algebra of subsets of $S$. If $\mu$ is normalized to satisfy $\mu(S) = 1$, then it is a probability measure. The notation $A \Delta B$ will be used for the symmetric difference, $(A \sim B) \cup (B \sim A)$, of two subsets of $S$.

1. Nonstandard representations. Let $\mu$ be a probability measure on $\mathcal{P}(S)$ and let $\phi$ be the linear functional on $l^\infty(S)$ defined by integration with respect to $\mu$. Then $\phi$ is a positive linear functional of norm 1. Therefore, by the principal result of [7], there exist a $^{*}$-finite subset $F$ of $^{*}S$ and an internal function $\lambda$ from $F$ to $^{*}R$ which satisfy

$$\text{st}\left(\sum_{p \in F} |\lambda(p)|\right) = 1$$

and, for each $f$ in $l^\infty(S)$,

$$\phi(f) = \text{st}\left(\sum_{p \in F} \lambda(p)^*f(p)\right).$$
Robinson's result [7] only covers the case $S = N$ explicitly, but his argument is easily extended to cover the general case.) Therefore the measure $\mu$ has the representation

$$\mu(A) = \text{st} \left( \sum_{p \in A \cap F} \lambda(p) \right).$$

Theorem 1 below states that, if $\mu(\{s\}) = 0$ for every $s \in S, \quad (1)$ then $F$ can be chosen so that $\mu$ is represented as in (3), but with every $\lambda(p)$ equal to $1/\|F\|$. That is, $\mu(A) = \mu_F(A)$ for every $A \subset S$.

**Theorem 1.** If $\mu$ is a probability measure on $\mathcal{P}(S)$ which satisfies $\mu(\{s\}) = 0$ for each $s \in S$, then there is a *-finite set $F \subset S$ for which $\mu = \mu_F$.

**Proof.** Since $\mathcal{M}$ is an enlargement of $\mathcal{M}$, there exists a *-finite subset $U$ of $\mathcal{P}(S)$ which satisfies $\star a \in \star S$ for each $A \subset S$. For each internal subset $\mathcal{F}$ of $\mathcal{A}$, define

$$E(\mathcal{F}) = \bigcap \{E \mid E \in \mathcal{F} \cap \bigcap \{\star s \sim E \mid E \in \mathcal{A} \sim \mathcal{F}\},$$

so that the function taking $\mathcal{F}$ to $E(\mathcal{F})$ is internal. Let $\mathcal{A}' = \{E(\mathcal{F})\}$ if is an internal subset of $\mathcal{A}$, so that $\mathcal{A}'$ is a *-finite set. Moreover, $\mathcal{A}'$ is a partition of $\star S$, and each member of $\mathcal{A}'$ is the union of an internal subset of $\mathcal{A}'$.

Let $\omega = \|\mathcal{A}'\|$ and choose $\tau \in \star N$ so that $\omega^{2/\tau}$ is infinitesimal. For each $E$ in $\mathcal{A}'$ define $r(E)$ in $\star N$ by the inequalities

$$r(E)/\tau \leq \star \mu(E) \leq (r(E) + 1)/\tau. \quad (4)$$

Then the function $E \mapsto r(E)$ on $\mathcal{A}'$ is internal. Moreover, if $E$ is a *-finite element of $\mathcal{A}'$, then $\star \mu(E) = 0$, from which it follows that $r(E) = 0$. Therefore there exists an internal function $f$ which is defined on $\mathcal{A}'$ and which satisfies: For each $E$ in $\mathcal{A}'$, $f(E)$ is a *-finite subset of $E$ and $\|f(E)\| = r(E)$.

It will be shown that the set $F$ defined by

$$F = \bigcup \{f(E) \mid E \in \mathcal{A}'\}$$

satisfies the condition $\mu = \mu_F$. Since the elements of $\mathcal{A}'$ are pairwise disjoint, the elements of $\{f(E) \mid E \in \mathcal{A}'\}$ have the same property, and therefore,

$$\|F\| = \sum_{E \in \mathcal{A}'} r(E).$$

Moreover, since the function $\star \mu$ is *-finitely additive,

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\(\dagger\) The added condition on $\mu$ is only slightly more restrictive than necessary. Indeed, if $F$ is infinite and $s \in S$, then $\mu_F(\{s\}) \leq \text{st}(1/\|F\|) = 0$. If $F$ is finite, say with $k$ elements, then $\mu_F$ is of the form $\mu = k^{-1}(\mu_1 + \cdots + \mu_k)$, where each of the measures $\mu_j$ takes on as values only 0 and 1.
Therefore, from the inequalities (4) follows
$$\|F\|/\tau \leq \|F\|/\tau + \omega/\tau,$$
by summing over $E$. That is, by the choice of $\tau$, $\omega(\|F\|/\tau - 1)$ is infinitesimal.

Now let $A$ be any element of $\mathcal{G}$ and let $\mathcal{F}$ be the collection of $E$ in $\mathcal{G}'$ which are subsets of $A$. Therefore $A$ is the union of $\mathcal{F}$, by the construction of $\mathcal{G}'$. It follows that
$$\|A \cap F\| = \sum_{E \in \mathcal{F}} r(E), \text{ and } \mu(A) = \sum_{E \in \mathcal{F}} \mu(E).$$
Therefore
$$\mu(A) - \frac{\|A \cap F\|}{\|F\|} = \sum_{E \in \mathcal{F}} \left( \mu(E) - \frac{r(E)}{\|F\|} \right).$$

But for each $E$ in $\mathcal{G}'$,
$$|\mu(E) - r(E)/\|F\|| \leq |\mu(E) - r(E)/\tau| + |r(E)/\tau - r(E)/\|F\|| \leq 1/\tau + |\|F\||/\tau - 1| \leq 1/\tau + |\|F\||/\tau - 1|.$$
Thus (5) implies
$$|\mu(A) - \|A \cap F\|/\|F\|| \leq \omega/\tau + \omega|\|F\||/\tau - 1|$$
which is infinitesimal. In particular, for each $A \subseteq S$,
$$\mu(A) = \mu(A) = \text{st}(\|A \cap F\|/\|F\|) = \mu_F(A).$$
This completes the proof.

While Theorem 1, as stated, applies only to totally defined measures, it is valid for any probability measure $\mu$ which is defined on an algebra of subsets of $S$ and which assigns measure 0 to any finite set in its domain. This is because any such measure can be extended to a measure which satisfies the conditions of Theorem 1.

A different nonstandard representation for measures, based on partitions of $\mathcal{S}$ rather than *finite subsets, has been developed and applied by Peter Loeb [5], [6].

Lemma 1. Let $E$ be any *finite subset of $\mathcal{S}$ and let $F$ be an internal subset of $E$ which satisfies $\|F\|/\|E\| = 1$. Then $\mu_F = \mu_E$ on $\mathcal{S}$ and
$$\int f \, d\mu_F = \text{st}\left( \frac{1}{\|F\|} \sum_{p \in F} \mu_F(p) \right)$$
for each $f$ in $L_\infty(S)$. 
Proof. Let $A$ be any subset of $S$. Then
\[ |\|A \cap E\|/\|E\| - \|A \cap F\|/\|F\| | \leq \|E \sim F\|/\|E\| = 1/0. \]
Therefore
\[ \mu_E(A) = \text{st}(\|F\|/\|E\| \cdot \|A \cap F\|/\|F\|) = \mu_F(A). \]

Now let $f$ be any element of $l_\infty(S)$, and define
\[ \psi(f) = \text{st} \left( \frac{1}{\|F\|} \sum_{p \in F} *f(p) \right). \]
Then $\psi$ is a bounded linear functional on $l_\infty(S)$. Also, if $V$ is the subspace of $l_\infty(S)$ generated by the characteristic functions, then $\psi$ agrees with the $\mu_E$-integral on $V$. The fact that $V$ is norm-dense in $l_\infty(S)$ implies that $\psi$ and the $\mu_E$-integral are equal on all of $l_\infty(S)$.

Now let $\mathcal{B}$ be a $\sigma$-algebra of subsets of $S$ and let $\mu$ be a countably additive probability measure on $\mathcal{B}$ which satisfies $\mu(A) = 0$ for each finite set $A$ in $\mathcal{B}$. There exists an extension $\tilde{\mu}$ of $\mu$ to $\mathcal{P}(S)$ which satisfies $\tilde{\mu}(|s\}) = 0$ for $s \in S$. By Theorem 1, there exists a $*\mu$-finite subset $F$ of $*S$ which satisfies $\tilde{\mu} = \mu_F$, and thus $\mu(A) = \mu_F(A)$ for every $A$ in $\mathcal{B}$.

For any bounded, $\mu$-integrable function $f$, $\int f \, d\mu = \int f \, d\tilde{\mu}$. Therefore, by Lemma 1,
\[ \int f \, d\mu = \text{st} \left( \frac{1}{\|F\|} \sum_{p \in F} *f(p) \right). \]
However, for unbounded, $\mu$-integrable functions (6) may not be true. (Indeed, if $f$ is any unbounded function on $S$, then $F$ may be chosen satisfying $\mu = \mu_F$ on $\mathcal{B}$, but such that the sum $\|F\|^{-1} \sum_{p \in F} *f(p)$ is infinite.) It is possible, nonetheless, to choose $F$ in such a way that (6) is true for every $\mu$-integrable function.

It is convenient to assume that $^*\mathcal{N}$ is $\kappa$-saturated (in the sense of [7]), where $\kappa$ is any cardinal number greater than the number of functions from $S$ to $R$. The remainder of this section is devoted to showing that, under this assumption, it is possible to represent $\mu$ on $\mathcal{B}$ in such a way that (6) holds for every $\mu$-integrable function.

Given $n \in \mathbb{N}$ and a function $f$ from $S$ to $R$, define $f_n$ on $S$ by
\[ f_n(x) = \begin{cases} f(x) & \text{if } |f(x)| \leq n, \\ 0 & \text{otherwise.} \end{cases} \]
Each $f_n$ is a bounded function, and it is measurable whenever $f$ is. Also, if $\omega \in *N$ and $p \in *S$, then

$$\star f_{\omega}(p) = \begin{cases} \star f(p) & \text{if } |\star f(p)| \leq \omega, \\ 0 & \text{otherwise}. \end{cases}$$

**Lemma 2.** Let $E$ be any *-finite subset of *$S$ which satisfies $\mu = \mu_E$ on $\mathcal{B}$ and let $f$ be a nonnegative, $\mu$-integrable function. There exists an internal subset $F_f$ of $E$ which satisfies $\|F_f\|/\|E\| = 1$ and, for any internal subset $F$ of $F_f$,

$$\frac{\|F\|}{\|E\|} = 1 \rightarrow \int f d\mu = \operatorname{st} \left( \frac{1}{\|F\|} \sum_{p \in F} \star f(p) \right).$$

**Proof.** For each $n \in \mathbb{N}$, let $A_n = \{x \mid f(x) > n\}$. Then $\{A_n \mid n \in \mathbb{N}\}$ is a decreasing chain of sets in $\mathcal{B}$ and $\bigcap\{A_n \mid n \in \mathbb{N}\} = \emptyset$. Thus the sequence $\{\mu(A_n)\}$ decreases monotonically to 0. Since $\mu = \mu_E$ on $\mathcal{B}$, it follows that for each $\delta > 0$ in $\mathbb{R}$, there exists $n_0 \in \mathbb{N}$ which satisfies

$$\|A^{n_0} \cap E\|/\|E\| < \delta.$$

If $\omega$ is an infinite member of $*\mathbb{N}$, then $*A_{\omega} \subset *A_n$, so $\|*A_{\omega} \cap E\|/\|E\| < \delta$. This shows that for every such $\omega$,

$$\|*A_{\omega} \cap E\|/\|E\| = 0.$$  \hspace{1cm} (7)

Since $f$ is nonnegative, the sequence of integrals $\int f_n d\mu$ is increasing. By the monotone convergence theorem, the supremum of this sequence is $\int f d\mu$. If $\int f d\mu = \int f_n d\mu$ for some $n \in \mathbb{N}$, then $\mu(A_n) = 0$ and hence

$$\|E \sim *A_n\|/\|E\| = 1.$$

In this case let $F_f = E \sim *A_n$. If $F \subset F_f$ and $\|F\|/\|E\| = 1$, then

$$\int f d\mu = \int f_n d\mu = \operatorname{st} \left( \frac{1}{\|F\|} \sum_{p \in F} \star f(p) \right)$$

since $\star f = f_n$ on $F$ and $\mu_F = \mu_E$.

Therefore it may be assumed that $\int f_n d\mu < \int f d\mu$ for all $n \in \mathbb{N}$. Thus

$$\frac{1}{\|E\|} \sum_{p \in E} \star f_n(p) < \int f d\mu$$

for all $n \in \mathbb{N}$. It follows that there is an infinite $\omega$ in $*\mathbb{N}$ which satisfies

$$\frac{1}{\|E\|} \sum_{p \in E} \star f_{\omega}(p) < \int f d\mu.$$
\[ \int f \, d\mu \leq \text{st} \left( \frac{1}{\|F\|} \sum_{p \in F} *f(p) \right) \]
\[ \leq \text{st} \left( \frac{1}{\|E\|} \sum_{p \in E} *f_\omega(p) \right) = \int f \, d\mu, \]

using Lemma 1 and the fact that \(*f = *f_\omega\) on \(F\). By the monotone convergence theorem

\[ \text{st} \left( \frac{1}{\|F\|} \sum_{p \in F} *f(p) \right) = \int f \, d\mu, \]

completing the proof.

Theorem 2. Let \(\mathcal{B}\) be an \(\sigma\)-algebra of subsets of \(S\) and let \(\mu\) be a countably additive probability measure on \(\mathcal{B}\) which satisfies \(\mu(A) = 0\) for each finite set \(A\) in \(\mathcal{B}\). There exists a \(*\)-finite subset \(F\) of \(*S\) which satisfies \(\mu = \mu_F\) on \(\mathcal{B}\) and

\[ \int f \, d\mu = \text{st} \left( \frac{1}{\|F\|} \sum_{p \in F} *f(p) \right) \]

for every \(\mu\)-integrable function \(f\).

Proof. Let \(I\) be the set of nonnegative, \(\mu\)-integrable functions. Since each \(\mu\)-integrable function is the difference of two elements of \(I\), it suffices to find an \(F\) which satisfies the conditions of the theorem for every \(f\) in \(I\). By Theorem 1 (and the remarks following) there exists a \(*\)-finite subset \(E\) of \(*S\) which satisfies \(\mu = \mu_E\) on \(\mathcal{B}\). For each \(f \in I\), let \(F_f\) be a subset of \(E\) which satisfies the conditions of Lemma 2. Given \(n \in \mathbb{N}\) and \(f \in I\), define

\[ A(n, f) = \{ F \mid F \text{ is an internal subset of } F_f \text{ and } \|F\|/\|E\| > n/(n+1) \}. \]

This family of internal sets has cardinality \(\text{card}(N \times I)\), which is less than \(\kappa\). Moreover, the family has the finite intersection property. \((F_{f_1} \cap \cdots \cap F_{f_n})\) is an element of \(A(m_1, f_1) \cap \cdots \cap A(m_n, f_n)\) whenever \(m_1, \ldots, m_n \in \mathbb{N}\) and \(f_1, \ldots, f_n \in I\). Since \(*\mathcal{B}\) is \(\kappa\)-saturated, there exists a \(*\)-finite set \(F\) which satisfies \(F \in A(n, f)\) for every \(n \in \mathbb{N}\) and \(f \in I\) (Theorem 2.7.12 of [5]). That is, \(F \subset F_f\) for every \(f \in I\), and \(\|F\|/\|E\| = 1\). It follows by Lemma 2 that \(F\) satisfies the conditions of the theorem.

Remark. Theorem 2 is true even if \(*\mathcal{B}\) is not \(\kappa\)-saturated, but the proof of that fact is somewhat more complicated. The proof given here proves the stronger result that \(F\) can be chosen as a subset of any given set \(E\) which satisfies \(\mu = \mu_E\) on \(\mathcal{B}\).
2. An application. The following standard result can be proved easily using the Riesz Representation Theorem. The nonstandard proof given here uses the extension to $l_\infty(S)$ of Robinson's representation result [9] instead.

Theorem 3. Let $X$ be a compact, Hausdorff space, $\{f_n\}$ a sequence in $C(X)$ and $\phi$ a bounded linear functional on $C(X)$. If $\{f_n\}$ is uniformly bounded on $X$ and converges to 0 pointwise, then $\phi(f_n) \to 0$.

Proof. Let $\phi$ be any bounded linear functional on $C(X)$. By the Hahn-Banach theorem, $\phi$ may be extended to a bounded linear functional $\tilde{\phi}$ on $l_\infty(X)$. By the extension to $l_\infty(X)$ of the principal result of [9], there exist a *-finite subset of $^*X$ and an internal function $\lambda$ from $F$ into $^*R$ which satisfy

$$\tilde{\phi}(f) = st \left( \sum_{p \in F} \lambda(p) * f(p) \right)$$

for every $f$ in $l_\infty(X)$, and $\sum_{p \in F} |\lambda(p)|$ is finite.

Let $\{f'_n\}$ be a sequence in $C(X)$ which is uniformly bounded on $X$ by 1, and which converges to 0, pointwise. If $\phi(f'_n)$ does not converge to 0, then it may be assumed (by taking a subsequence) that for some $\delta > 0$ in $R$, $|\phi(f'_n)| > \delta$ for every $n \in N$. Let $M = st (\sum_{p \in F} |\lambda(p)|) + 1$. For $n \in N$, define

$$A_n = \{x | x \in X \text{ and } |f'_n(x)| \geq \delta/2M\}.$$

Therefore,

$$\delta < \left| \sum_{p \in F} \lambda(p) * f'_n(p) \right| \leq \sum_{p \in *A_n \cap F} |\lambda(p)*f'_n(p)| + \sum_{p \in F \cap *A_n} |\lambda(p)*f'_n(p)| \leq \sum_{p \in *A_n \cap F} |\lambda(p)| + \frac{\delta}{2}.$$

Thus, for each $n \in N$, $\sum_{p \in *A_n \cap F} |\lambda(p)| > \delta/2$.

Now define $\mu'$ on $\mathcal{P}(X)$ by

$$\mu'(A) = st \left( \sum_{p \in *A \cap F} |\lambda(p)| \right)$$

for each $A \subseteq X$. Then $\mu'$ is a measure on $\mathcal{P}(X)$, and $\mu'(A_n) > \delta/2$ for every $n \in N$. It follows that there is an infinite subset $K$ of $N$ such that $\{A_n | n \in K\}$ has the finite intersection property (see Lemma 17.9 of [4]). Since $^*\mathbb{N}$ is an enlargement, there is an element $p$ of $^*X$ which satisfies $|\phi(p)| \geq \delta/2M$ for all $n \in K$. $X$ is compact, so $p$ is near-standard to some $x \in X$. In particular, $^*f'_n(p) = 1 f'_n(x)$ for every $n \in N$. This implies $|f'_n(x)| \geq \delta/2M$ for every $n \in K$, \[ \text{for every } x \in X. \]
which contradicts the assumption that $f_n(x)$ converges to 0. Therefore $\phi(f_n)$ must converge to 0.

3. Constructing invariant measures. Let $G$ be a group of permutations on $S$, and assume that $G$ satisfies Følner's condition:

For each $a_1, \ldots, a_n \in G$ and $k \in N$, there exists a finite set $A \subseteq G$ which satisfies $\|A \triangle Aa_j\|/\|A\| < 1/(k + 1)$ for each $j = 1, \ldots, n$.

To apply the corresponding statement in $\mathcal{N}$, let $E$ be a $\ast$-finite subset of $\ast G$ which contains $\{g \mid g \in G\}$ and let $\omega$ be an infinite member of $\ast N$. Then there is a $\ast$-finite set $F \subseteq \ast G$ which satisfies $\|F \Delta Fp\|/\|F\| < 1/\omega$ for every $p \in E$. In particular,

$$g \in G \rightarrow \|F \triangle F^*g\|/\|F\| = 0.$$  

If $F$ satisfies (8), then $\mu_F$ is a probability measure on $\mathcal{P}(G)$ and $\mu_F$ is invariant under the action of $G$ on itself by right multiplication. The principal result of [3] is, essentially, that the converse holds: If there is such a measure on $\mathcal{P}(G)$, then $G$ satisfies Følner's condition.

**Theorem 4.** Let $G$ be a group of permutations of $S$ and let $F$ be a $\ast$-finite subset of $\ast G$ which satisfies (8). Let $\mu$ be any measure on $\mathcal{P}(S)$ and define $\tilde{\mu}$ by

$$\tilde{\mu}(A) = st \left( \frac{1}{\|F\|} \sum_{p \in F} \mu(p^*A) \right)$$

for $A \subseteq S$. Then $\tilde{\mu}$ is a $G$-invariant measure on $\mathcal{P}(S)$. Moreover, if $A \subseteq S$ satisfies $\mu(gA) = \mu(A)$ for every $g \in G$, then $\tilde{\mu}(A) = \mu(A)$.

**Proof.** Each element of $\ast G$ is a permutation of $\ast S$. Thus if $A, B$ are disjoint subsets of $S$, then $p^*A, p^*B$ are disjoint subsets of $\ast S$ for each $p \in \ast G$. Thus $\mu(p(\ast A \cup \ast B)) = \mu(p^*A) + \mu(p^*B)$. From this the finite additivity of $\tilde{\mu}$ is immediate.

Given $A$ in $\mathcal{P}(S)$ and $g$ in $G$,

$$|\tilde{\mu}(gA) - \tilde{\mu}(A)| = \left| \frac{1}{\|F\|} \sum_{p \in F} (\mu(p^*gA) - \mu(p^*A)) \right|$$

$$\leq \frac{1}{\|F\|} \sum_{p \in F \triangle F^*g} \mu(p^*A)$$

$$\leq \mu(S) \cdot \|F \triangle F^*g\|/\|F\| = 0.$$  

Therefore $\tilde{\mu}(gA) = \tilde{\mu}(A)$, so that $\tilde{\mu}$ is $G$-invariant.

Finally, suppose $A$ is a subset of $S$ which satisfies $\mu(gA) = \mu(A)$ for every $g \in G$. Then $\mu(p^*A) = \mu(A)$ for every $p \in \ast G$. Therefore
\[ \hat{\mu}(A) = \mu \left( \frac{1}{\|F\|} \sum_{\rho \in F} \rho(A) \right) = \mu(A). \]

To prove Banach's extension result, let \( G \) be the group of all translations (mod 1) of \([0, 1]\), and let \( \mu \) be any extension of Lebesgue measure to \( \mathcal{P}([0, 1]) \). It is well known, and easy to prove using the decomposition theorem for finitely generated abelian groups, that every abelian group satisfies Følner's condition. Since \( G \) is abelian, Theorem 4 can be applied to obtain a \( G \)-invariant measure \( \hat{\mu} \) on \( \mathcal{P}([0, 1]) \). If \( A \) is a Lebesgue measurable subset of \([0, 1]\), then \( \mu(gA) = \mu(A) \) for every \( g \in G \). Theorem 4 thus asserts that \( \hat{\mu}(A) = \mu(A) \); that is, \( \hat{\mu} \) is an extension of Lebesgue measure.

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