HORN CLASSES AND REDUCED DIRECT PRODUCTS

BY

RICHARD MANSFIELD

ABSTRACT. Boolean-valued model theory is used to give a direct proof that an $EC^\Delta$ model class closed under reduced direct products can be characterized by a set of Horn sentences. Previous proofs by Keisler and Galvin used either the G. C. H. or involved axiomatic set theory.

We shall give a direct proof of the theorem that an $EC^\Delta$ model class is closed under reduced direct products iff it is characterizable by a set of Horn sentences. This was first proven by Keisler as a consequence of the continuum hypothesis. Galvin then proved that in $ZF$ set theory it is provably equivalent to a certain arithmetical statement. From these two results, it follows that the theorem is true in the constructible universe for set theory and is consequently true. This indirect proof of a simple proposition of model theory seems overly ornate. We shall carry out the main features of Keisler's argument within the system developed in [3] and prove the theorem without any use of axiomatic set theory. Subsequent to this proof Shelah has given another direct proof of this same theorem [5]. His methods do not use Boolean-valued models as do mine, but rather closely follow his proof that elementarily equivalent models have isomorphic ultrapowers.

1. A major tool in our proof is the theory of first order Boolean-valued models. Since the standard notation for model theory becomes cumbersome in the Boolean case, we give an alternate system; a model is identified with its truth function. For $\mathcal{L}$ a finitary language without function symbols, an $\mathcal{L}$-structure is a set of constant symbols $\{\mathcal{I}\}$ containing all the constant symbols of $\mathcal{L}$ together with a function $\mathcal{I}$ from $\mathcal{I}(\mathcal{I})$ into a complete Boolean algebra satisfying the conditions:

1. $\mathcal{U}(a = b) = 1$,
2. $\mathcal{U}(a = b) \leq \mathcal{U}(b = a)$,
3. $\mathcal{U}(a = b) \wedge \mathcal{U}(b = c) \leq \mathcal{U}(a = c)$,
4. For $\phi$ an atomic sentence, $\mathcal{U}(\phi(a)) \wedge \mathcal{U}(a = b) \leq \mathcal{U}(\phi(b))$,
5. $\mathcal{U}(\phi \lor \psi) = \mathcal{U}(\phi) \lor \mathcal{U}(\psi)$,
6. $\mathcal{U}(\neg \phi) = \neg \mathcal{U}(\phi)$,
7. $\mathcal{U}[\exists x \phi(x)] = \mathcal{U}[\phi(a)]$.

Received by the editors January 15, 1971.

AMS (MOS) subject classifications (1969). Primary 1250; Secondary 0242.
When the language \( \mathcal{L} \) contains function symbols, these must first be interpreted by actual functions from the appropriate powers of \(|\mathcal{M}| \) into \(|\mathcal{M}| \) before proceeding as above. An \( \mathcal{L} \)-structure satisfies the maximum principle if the truth value of any existential statement is always equal to the truth value of some instance. Any \( \mathcal{L} \)-structure has a canonical elementary extension satisfying the maximum principle.

Various basic operations of model theory can be generalized to Boolean model theory. If \( \{\mathcal{M}_i\}_{i \in I} \) is a collection of \( \mathcal{L} \)-structures with corresponding algebras \( \{B_i\}_{i \in I} \), \( \Pi_i \mathcal{M}_i B_i \) can be defined as a \( \Pi_i B_i \)-valued model. The set of constant symbols for \( \Pi_i \mathcal{M}_i \) is just the usual cartesian product of the component symbols and truth is defined by the equation

\[
\Pi_i [\phi(a_i)]_{i \in I} = (\Pi_i [\phi(a_i)])_{i \in I}.
\]

This definition should actually be called the covariant direct product. It has the drawback that it does not specialize to the traditional definition in the two-valued case; the product of a pair of two-valued models is a four-valued model. The traditional definition will be a special case of our definition of a reduced direct product. The contravariant direct product, which is defined using the algebra \( \Sigma_i B_i \), does not have this drawback and has a much better claim to the name "direct product." However it is the covariant product which is useful for the purposes of this paper.

The Boolean power of a two-valued model is the structure that was used in the construction of Boolean ultrapowers in [3]. For \( \mathcal{M} \) a two-valued model and \( B \) a complete Boolean algebra the \( B \)-valued power \( \mathcal{M}^B \) is defined as follows. The constant symbols for \( \mathcal{M}^B \) is the set of all functions from the constants of \( \mathcal{M} \) into \( B \) whose ranges partition \( B \), i.e.

\[
\{f \in B^\mathcal{M}; a_1 \neq a_2 \rightarrow f(a_1) \land f(a_2) = 0 \text{ and } \bigvee_a f(a) = 1\}.
\]

Truth is defined by the equation

\[
\mathcal{M}^B[\phi(f_1 \cdots f_n)] = \bigvee \left\{ \prod_{i=1}^n f_i(a_i); \mathcal{M} \models \phi(a_1, \ldots, a_n) \right\}.
\]

For a more extensive treatment of this structure the reader is referred to [3, §1] where it is discussed in necessary detail. In [3] it is shown that \( \mathcal{M}^B \) is an elementary extension of \( \mathcal{M} \). Again our definition does not specialize to the traditional one when \( B \) is a power set algebra; we will need to first reduce by a filter.

If \( A \) and \( B \) are both complete Boolean algebras, we define an \( A \)-valued filter on \( B \) to be a function \( D \) from \( B \) into \( A \) such that \( D(b_1 \land b_2) = \)
If \( D(b_1) \land D(b_2) \) and \( D(1) = 1 \). If in addition \( D(\neg b) = \neg D(b) \), \( D \) is an ultrafilter. \( D \) is proper if \( D(0) = 0 \). A function \( E \) from \( B \) into \( A \) has the finite intersection property if \( E(0) = 0 \) and also \( \bigwedge_{i=1}^n b_i = 0 \) implies \( \bigwedge_{i=1}^n E(b_i) = 0 \). Just as with two-valued filters any function with the finite intersection property uniquely generates a filter. This is accomplished by the definition

\[
D(b) = \bigvee \left\{ \bigwedge_{i=1}^n E(b_i) : \bigwedge_{i=1}^n b_i \leq b \right\}.
\]

If \( \{\mathcal{U}_i\}_{i \in I} \) is a collection of \( \mathcal{L} \)-structures satisfying the maximum principle and \( D \) is a \( B \)-valued filter on the product algebra, we define the Boolean reduced product \( \prod_{i \in I} \mathcal{U}_i / D \) as a \( B \)-valued model. The set of constant symbols is the same as in \( \prod_{i \in I} \mathcal{U}_i \) and truth for atomic \( \phi \) is defined by

\[
\prod_{i \in I} \mathcal{U}_i / D(\phi) = D \left[ \prod_{i \in I} \mathcal{U}_i(\phi) \right].
\]

Truth for arbitrary sentences is then defined by induction according to conditions 5, 6, 7 of the definition of an \( \mathcal{L} \)-structure.

In the special case that \( D \) is an ultrafilter, it can be shown that, for arbitrary \( \phi \), \( \prod_{i \in I} \mathcal{U}_i / D(\phi) = D[\prod_{i \in I} \mathcal{U}_i(\phi)] \) but in the general case this is not so [3]. However, when \( \phi \) is a Horn sentence it is an easy exercise to prove that

\[
\prod_{i \in I} \mathcal{U}_i / D(\phi) \geq D \left[ \prod_{i \in I} \mathcal{U}_i(\phi) \right].
\]

This shows that Horn sentences are preserved by reduced direct products.

Since we are allowing the use of Boolean-valued models, nontrivial use of the above definition can be made even when only one model is involved. \( \mathcal{U}^{(B)} / D \), the application of the above definition to just the one model \( \mathcal{U}^{(B)} \), is a reduced direct power of the two-valued model \( \mathcal{U} \). When \( D \) is a two-valued ultrafilter, this structure is just the Boolean ultrapower studied in [3].

If \( D \) is a two-valued filter on 2\(^I\) and each \( \mathcal{U}_i \) is a two-valued model, \( \prod_{i \in I} \mathcal{U}_i / D \) is the traditional reduced direct product of the \( \mathcal{U}_i \)'s. If \( D \) is the trivial filter \( \{1\} \), \( \prod_{i \in I} \mathcal{U}_i / D \) is just the traditional cartesian product and \( \mathcal{U}^{(2^I)} / D \) is canonically isomorphic to the traditional cartesian power of \( \mathcal{U} \).

We shall conclude this section by stating a lemma which shall be used in the main argument. This lemma follows easily from

**Theorem 1.1.** If \( \mathcal{U} \) is a \( B \)-valued \( \mathcal{L} \)-structure and \( B \) satisfies the \( \prec K_1, \infty \) distribution law and \( \mathcal{L} \) is countable, \( \mathcal{U} \) has a countable substructure \( \mathcal{U}^\mathbb{B} \) for which there is a nonzero \( b \) in \( B \) with

\[
\mathcal{U}(\phi) \land b = \mathcal{U}(\phi) \land b
\]

for every \( \phi \) in \( \mathcal{L}(\mathbb{B}) \).
Since this paper is not meant to be a treatise on Boolean-valued model theory, we are leaving the proof of this theorem to the reader. Very briefly, in order to prove it one must first pass to a certain elementary extension $U'$ of $U$ in which the truth value of any existential statement is equal to the truth value of one of its instances [4]. In the extension the Löwenheim-Skolem argument can be applied exactly as in two-valued logic. Since the extension I have in mind satisfies the condition that, for any $a' \in |U'|$, $\forall a \in U | U'(a = a') = 1$, the distributivity law produces the desired element $b$ and countable structure $B$.

**Lemma 1.2.** If $\{U_i\}_{i \in I}$ is a collection of two-valued models and $D$ is a $B$-valued filter on $2^I$ and $B$ satisfies the $< \aleph_1, \infty$ distribution law, there is a two-valued ultrafilter $\mu$ on $B$ such that for any sentence $\phi$ without parameters

$$\prod U_i/D \circ \mu(\phi) = \mu \left( \prod U_i/D(\phi) \right).$$

**Proof.** There is a countable $\mathfrak{B} \subseteq \prod U_i/D$ and an element $b \in B$ satisfying Theorem 1.1. Let $\mu$ be an ultrafilter on $B$ containing $b$ and preserving all of the countably many sups used to evaluate sentences in $\mathcal{L}(\mathfrak{B})$. The existence of such a $\mu$ is guaranteed by the Rasiowa-Sikorski homomorphism theorem. $\mu$ is easily seen to satisfy the lemma.

2.

**Theorem 2.1.** If $K$ is a model class closed under elementary equivalence and reduced direct products, $K$ can be characterized by a set of Horn sentences.

We stress that despite all the Boolean constructions of the previous section this is a two-valued theorem; the models in $K$ are two-valued and the reduced products are the traditional ones. The proof, however, will be quite Boolean. What we shall actually prove is that any model for the Horn theory of $K$ is elementarily equivalent to a reduced product of $K$, and hence $K$ can be characterized by its Horn theory.

For the sake of completeness we give a definition of the class of Horn formulae. A basic Horn formula is a formula in the form $\bigwedge_{i=1}^n \phi_i \rightarrow \phi_0$ where each of the $\phi_i$ for $0 \leq i \leq n$ is atomic (true and false are counted as atomic sentences). A Horn formula is a formula in prenex normal form whose matrix is a conjunction of basic Horn formulae. The Horn theory of $K$ is the set of Horn sentences true in every member of $K$.

Let $\mathfrak{B}$ be a model for the Horn theory of $K$. By taking an elementary extension if necessary we may assume that $\mathfrak{B}$ is $\aleph_1$-saturated. (For a definition of $\aleph_1$-saturation, see [2, p. 310].) Let $\{U_i\}_{i \in I}$ be an indexed collection from $K$ such
that any Horn sentence true in all but finitely many \( \mathbb{B}_i \) is also true in \( \mathbb{B} \). Such a collection can easily be constructed since a Horn sentence false in \( \mathbb{B} \) must also be false in at least one element of \( \mathcal{K} \) and this element can be included infinitely many times in the collection.

We let the notation \( f: \Pi \mathbb{U}_i \to \mathbb{B} \) mean that \( f \) is a partial function from \( \Pi \mathbb{U}_i \) into \( \mathbb{B} \) such that whenever the Horn sentence \( \phi(a_1, \ldots, a_n) \) is true in all but finitely many \( \mathbb{U}_i \) and \( \{a_1, \ldots, a_n\} \subseteq \text{dom} f \), \( \phi[f(a_1), \ldots, f(a_n)] \) is true in \( \mathbb{B} \). (Sometimes when the parameters of \( \phi \) are not explicitly listed, we shall use the notation \( f(\phi) \) for the image formula.) Our first step is to construct a certain Boolean algebra. This will be done by using the regular open subsets of a topological space. Let \( T \) be

\[
\left\{ f: \Pi \mathbb{U}_i \to \mathbb{B} \text{ and } |f| = \mathcal{K} \right\}.
\]

Each countable \( Q: \Pi \mathbb{U}_i \to \mathbb{B} \) defines a subset of \( T \), namely \( [Q] = \{ f \in T : Q \subseteq f \} \). We give \( T \) the topology generated by the \( [Q] \)'s, and let \( B \) be the regular open algebra of that topology. In order to show that \( B \) is nontrivial we must prove that \( T \) is nonempty.

**Lemma 2.2.** If \( Q: \Pi \mathbb{U}_i \to \mathbb{B} \) is countable, there is an \( f \in [Q] \) with \( a \in \text{dom} f \) and \( b \in \text{rng} f \) for any \( a \) in \( \Pi \mathbb{U}_i \) and \( b \in \mathbb{B} \).

**Proof.** The discerning reader will realize that this lemma exactly corresponds to Keisler's lemma [2, Theorem 3.1]; not surprisingly it has the same proof. We first find a countable \( Q_0: \Pi \mathbb{U}_i \to \mathbb{B} \) extending \( Q \) with \( a \in \text{dom} Q_0 \). Let \( \Gamma \) be the set of Horn formulae with one free variable and parameters from \( \text{dom} Q \) such that for all but finitely many \( i, \mathbb{U}_i \models \phi(a(i)) \). Then for \( \Delta \) a finite subset of \( \Gamma \) the sentence \( 3x \bigwedge \Delta \) is true in all but finitely many \( \mathbb{U}_i \) and is a Horn sentence. Thus \( Q(3x \bigwedge \Delta) \) is true in \( \mathbb{B} \), i.e., \( Q(\Gamma) \) is finitely satisfiable in \( \mathbb{B} \). Therefore, by the \( \mathcal{K}_1 \)-saturation of \( \mathbb{B} \), there is a \( b' \in \mathbb{B} \) which satisfies every formula in \( Q(\Gamma) \). Clearly \( Q_0 \cup \{(a, b')\} \) is the desired extension.

We will now use a parallel argument to find a \( Q_1: \Pi \mathbb{U}_i \to \mathbb{B} \) extending \( Q_0 \) with \( b \in \text{rng} Q_1 \). This time let \( \Gamma \) be the set of Horn formulae \( \phi(x) \) with one free variable and parameters from \( \text{dom} Q_0 \) such that \( Q_0[\phi(b)] \) is false in \( \mathbb{B} \). For each \( \phi \) in \( \Gamma \), let \( I_\phi = \{ i: \mathbb{U}_i \models 3x \to \phi(x) \} \). Since \( Q_0[\phi(b)] \) is false, \( I_\phi \) is infinite. Therefore, by a lemma of Keisler [2, Lemma 1.3], there is a pairwise disjoint collection \( \bigcup f \phi \in \Gamma \) of infinite sets with \( f_\phi \subseteq I_\phi \) for each \( \phi \) in \( \Gamma \). Now pick \( a' \) in \( \Pi \mathbb{U}_i \) such that \( i \in f_\phi \) implies \( \mathbb{U}_i \models \phi(a'(i)) \). Then \( Q_0 \cup \{(a', b')\} \) is the desired extension.

We finally show that \( Q_1 \) can be extended to an element of \( T \). Since \( \Pi \mathbb{U}_i \) is
uncountable, we have just shown that any countable \( Q : \Pi \mathbb{U}_i \rightarrow \mathbb{B} \) can be properly extended; thus a canonical use of Zorn's Lemma gives the desired result.

For \( a \in \Pi \mathbb{U}_i \) and \( b \in \mathbb{B} \), define
\[
(a, b) = \text{interior} (\text{closure} (\{ f \in T : f(a) = b \}))
\]
Then \((a, b)\) is a regular open subset of \( T \) and hence is a member of \( B \). Note that \([Q] \subseteq (a, b)\) implies \( Q \cup \{(a, b)\} : \Pi \mathbb{U}_i \rightarrow \mathbb{B} \). We can now define a function \( j \) from \( \Pi \mathbb{U}_i \) into \( \mathbb{B}(B) \) by \( j(a)(b) = (a, b) \). We must first show that, for each \( a \), \( j(a) \) is actually a member of \( \mathbb{B}(B) \). Clearly, for \( b_1 \neq b_2 \), \( j(a)(b_1) \land j(a)(b_2) = 0 \). Suppose that \( \bigvee_b j(a)(b) < 1 \). Then there would be a countable \( Q : \Pi \mathbb{U}_i \rightarrow \mathbb{B} \) with \([Q] \land \bigvee_b (a, b) = 0 \). We have just shown that there is a \( Q_0 \) extending \( Q \) with \( a \in \text{dom} Q_0 \). Then,
\[
0 < [Q_0] \leq [(a, Q_0(a))] \leq (a, Q_0(a)) \leq \bigvee_b (a, b)
\]
In similar fashion it can be shown that \( \bigvee_a (a, b) = 1 \).

Lemma 2.3. If \( \phi \) is a Horn sentence with parameters from \( \Pi \mathbb{U}_i \) and \( \{ i : \mathbb{U}_i \models \phi \} \) is cofinite, \( \mathbb{B}(B)[j(\phi)] = 1 \).

Proof. If \( a_1, \ldots, a_n \) are all the parameters of \( \phi \) and \( Q : \Pi \mathbb{U}_i \rightarrow \mathbb{B} \) and \( \{a_1, \ldots, a_n\} \subseteq \text{dom} Q \), then \( \mathbb{B} \models \phi[Q(a_1), \ldots, Q(a_n)] \). Therefore
\[
\bigwedge_{i=1}^n j(a_i)(b_i) > 0 \quad \text{implies} \quad \mathbb{B} \models \phi(b_1, \ldots, b_n).
\]
Consequently,
\[
\bigvee_{\phi(b_1, \ldots, b_n)} \bigwedge_{i=1}^n j(a_i)(b_i) = \bigvee_{b_1 \cdots b_n} \bigwedge_{i=1}^n j(a_i)(b_i) = 1
\]
but L. H. S. is \( \mathbb{B}(B)[j(\phi)] \).

Now let \( \mathbb{B}' = \text{rng} j \).

Lemma 2.4. For every \( h \) in \( \mathbb{B}(B) \),
\[
\bigvee \mathbb{B}(B)(h = f) : f \in \mathbb{B}' = 1
\]
Proof. Suppose otherwise; then there is a \( Q \) with \([Q] \land \bigvee \mathbb{B}(B)(h = f) : f \in \mathbb{B}' = 0 \). Since \( \bigvee_b b(b) = 1 \) there is a \( b \) in \( \mathbb{B} \) with \([Q] \land b(b) > 0 \). Then since \( \bigvee_a (a, b) = 1 \) there is an \( a \) with \( Q \land h(b) \land (a, b) > 0 \). But \( h(b) \land (a, b) \leq \mathbb{B}(B)[h = j(a)] \).

In [3, §1] it was shown that \( \mathbb{B}(B) \) is an elementary extension of \( \mathbb{B} \), i.e., a sentence is true in \( \mathbb{B} \) iff it has value one in \( \mathbb{B}(B) \). We now use Lemma 2.4 to show that \( \mathbb{B}(B) \) is elementarily equivalent to \( \mathbb{B}' \).
Lemma 2.5. If $\phi$ is any sentence in $\mathcal{L}(\mathcal{B}')$, $\mathcal{B}'(\phi) = \mathcal{B}^{(B)}(\phi)$.

Proof. We proceed by induction on the logical depth of $\phi$. The lemma is true by definition for atomic formula. For negations and disjunctions it follows instantly from the inductive hypothesis without any use of Lemma 2.4. So we assume $\phi = 3x \psi(x)$. Then

$$\mathcal{B}'(\phi) = \bigvee_{f \in \mathcal{B}'} \mathcal{B}'(\psi(f)) \leq \bigvee_{f \in \mathcal{B}(B)} \mathcal{B}^{(B)}(\psi(f)) = \mathcal{B}^{(B)}(\phi).$$

We must show that the reverse inequality also holds. For each $f$ in $\mathcal{B}'$ and $h$ in $\mathcal{B}(B)$, $\mathcal{B}(B)(\psi(h) \land f = h) \leq \mathcal{B}^{(B)}(\psi(f))$. Therefore,

$$\bigvee_{f \in \mathcal{B}'} \mathcal{B}^{(B)}(\psi(h)) \land \mathcal{B}^{(B)}(f = h) \leq \mathcal{B}'(\phi).$$

Thus for each $h$ in $\mathcal{B}(B)$, $\mathcal{B}(B)(\psi(h)) \leq \mathcal{B}'(\phi)$ and taking the sup over $h$ gives the desired result.

We now define a $B$-valued filter on $2^I$. For each atomic $\phi$ in $\mathcal{L}(\Pi \mathcal{U}_i / D)$, let $I_\phi = \{i: \mathcal{U}_i \models \phi\}$. Then let $E(I_\phi) = \mathcal{B}'(\phi) (\phi)$; $E(J) = 0$ for any $J \subseteq I$ which is not an $I_\phi$. It is straightforward to show using the technique of the next lemma that $E$ has the finite intersection property and thus generates a proper $B$-valued filter $D$.

Lemma 2.6. $j$ is an isomorphism from $\Pi \mathcal{U}_i / D$ onto $\mathcal{B}'$.

Proof. We show that, for any atomic sentence,

$$\prod \mathcal{U}_i / D(\phi(a_1, \ldots, a_n)) = \mathcal{B}'(\phi(j(a_1), \ldots, j(a_n))).$$

From the definition of $E$ and $D$ it follows that

$$\prod \mathcal{U}_i / D(\phi(a_1, \ldots, a_n)) = D\{i: \mathcal{U}_i \models \phi(a_1(i), \ldots, a_n(i))\}$$

$$\geq E\{i: \mathcal{U}_i \models \phi(a_1(i), \ldots, a_n(i))\} = \mathcal{B}'(\phi(j(a_1), \ldots, j(a_n))).$$

In order to prove that equality holds suppose $\{\phi_k\}_{k=1}^n$ is a finite set of atomic sentences in $\mathcal{L}(\Pi \mathcal{U}_i)$ with $\{i: \mathcal{U}_i \models \bigwedge \mathcal{U}_i \phi_k\} \subseteq \{i: \mathcal{U}_i \models \phi\}$. Then for every $i$, $\mathcal{U}_i \models \bigwedge \mathcal{U}_i \phi_k \models \phi$ and this is a Horn sentence; thus by Lemma 2.3 it is valid in $\mathcal{B}'$, i.e., $\bigcap \mathcal{U}_i \phi_k \models \phi$ implies $\bigwedge \mathcal{U}_i E(I_{\phi_k}) < E(I_{\phi})$ and thus $E(I_{\phi}) = D(I_{\phi})$.

Lemma 2.7. $B$ satisfies the $< \aleph_1$, $\infty$ distribution law.

Proof. From Lemma 2.2 any countable decreasing infimum of base sets is nonzero. The distribution law follows in a standard manner from this fact.

We have now nearly completed the proof of Theorem 2.1. By Lemmas 1.2 and 2.7 there is an ultrafilter $\mu$ on $B$ with $\Pi \mathcal{U}_i / D \mu(\phi) = \mu(\Pi \mathcal{U}_i / D(\phi))$ for every
sentence $\phi$. Then every sentence true in $\mathcal{B}$ has value one in $\mathcal{B}(B)$ [3], value one in $\mathcal{B}'$ (Lemma 2.5), value one in $\Pi \mathcal{U}_i/D$ (Lemma 2.6) and hence is true in $\Pi \mathcal{U}_i/D \circ \mu$ by the above equation. That is to say, $\mathcal{B}$ is elementarily equivalent to $\Pi \mathcal{U}_i/D \circ \mu$.

BIBLIOGRAPHY(1)


DEPARTMENT OF MATHEMATICS, PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PENNSYLVANIA 16802

(1) For a complete bibliography on the subject of Horn classes, the reader is referred to [1].