ABSTRACT. We study the bordism group of stably complex $G$-manifolds in the case where $G$ is a metacyclic group of order $pq$ and $p$ and $q$ are distinct primes. This bordism group is a module over the complex bordism ring and we compute the projective dimension of this module. We develop some techniques necessary for the study of this module in case $G$ is a more general metacyclic group.

The purpose of this paper is to study the bordism theory of actions of a metacyclic group on stably complex manifolds in a special case. More general and complete results will appear in [8]. However, many of the techniques used can be isolated and simplified in this special case. We will be concerned with actions of a group $G$ which is the semidirect product of two cyclic groups of prime order.

By a family $F$ of subgroups of $G$ we will mean a subset of the subgroups of $G$ with the property that if $K$ is an element of $F$ every subgroup and every conjugate of $K$ is in $F$. $\Omega_*(G; F)$ is the bordism group of actions of $G$ on stably complex manifolds such that every isotropy group lies in the family $F$. $\Omega_*$ will denote the complex bordism ring.

Theorem A. Let $F$ be the family consisting of all subgroups of $G$. Then $\Omega_*(G; F)$ is a free $\Omega_*$ module on even dimensional generators.

Theorem B. Let $F$ be any family of subgroups of $G$. Then $\Omega_+(G; F) = \bigoplus \Omega_+(G; F)$ is a free $\Omega_*$ module. And $\Omega_-(G; F) = \bigoplus \Omega_{2i+1}(G; F)$ has projective dimension one over $\Omega_*$. 

Basic material on the groups $\Omega_*(G; F)$ can be found in [3], [10]. Our analysis largely follows the lines of [2] but was influenced by [10] and [6].

In §1 we discuss general facts about equivariant bordism groups. In §2 we discuss the cohomology of a cyclic group acting on a polynomial ring. In §3 we discuss $\Omega_*(G; F, F')$ for adjacent families $F, F'$ in $G$ and in §4 we present the proofs of the main theorems.
Basic facts. Let $p$ and $q$ be distinct primes. Let $f$ be a homomorphism of $Z_p$ into the automorphism group of $Z_q$. We denote the resulting semidirect product by $Z_p \rtimes f Z_q = G$. $G$ is generated by two elements $a$ and $b$ with relations $b^p = a^q = 1, bab^{-1} = a^r$ where $r$ is a nonzero element of $Z_q$. $f$, of course, is the homomorphism which takes $b$ to the automorphism $a \rightarrow a^r$.

Lemma (1.1). $H_{2k-1}(BG) = Z_p$ if $k$ is not a multiple of $p$ and $H_{2k-1}(BG) = Z_p \oplus Z_q$ if $k$ is a multiple of $p$.

Proof. Consider the Hochschild-Serre spectral sequence

$$E^2_{i,j} = H_i(Z_p; H_j(Z_q)).$$

$E^2_{i,j} = 0$ unless $i$ or $j$ is zero. $E^2_{i,0} = H_i(Z_p; Z) = 0$ or $Z_p$ depending upon whether $i$ is even or odd. To compute $E^2_{0,i} = H_0(Z_p; H_i(Z_q))$ we use the resolution

$$Z[Z_p] \xrightarrow{N} Z[Z_p] \xrightarrow{D} Z[Z_p]$$

where $D = 1 - b$, $N = 1 + b + \cdots + b^{p-1}$ of [9, p. 121]. Tensoring on the right with $H_j(Z_q)$, we find that $E^2_{0,i} = \text{cokernel}(D \otimes 1)$. A simple computation shows the generator $b$ of $Z_p$ acts on $H_{2j-1}(Z_q) = Z_q$ by multiplication by $i^j$ and $1 - i^j$ is a unit in $Z_q$ unless $j \equiv 0 \mod p$. It is clear that the inclusion $Z_p \subset G$ induces an injection $H_*(BZ_p) \rightarrow H_*(BG)$ onto the $p$-torsion, and the inclusion $Z_q \subset G$ induces a map $H_*(BZ_p) \rightarrow H_*(BG)$ which is onto the $q$-torsion.

Lemma (1.3). $\Omega_*(BG)$ has projective dimension one over $\Omega_*$. The left $G$-manifolds $G \times Z_p S^{2k-1}$ generate the $p$-torsion and the left $G$-manifolds $G \times Z_q S^{2kp-1}$ generate the $q$-torsion where $Z_p$ acts on $S^{2k-1}$ by multiplication of each coordinate in $C^k$ by $e^{2\pi i/p}$ and $Z_q$ acts on $S^{2kp-1}$ by multiplication by $e^{2\pi i/q}$. Let $V$ be the representation of $G$ induced from the one-dimensional representation $e^{2\pi i/q}$ of $Z_q$ (see [5, p. 333]). $V = Z[G] \otimes Z[Z_q] C^*$. Then the left $G$-manifolds $S(kV)$ also serve as generators for the $q$-torsion.

Proof. The statement about dimension follows directly from [7] since $H_*(BG)$ is concentrated in odd dimensions. The manifolds $[Z_p, S^{2k-1}]$ and $[Z_q, S^{2kp-1}]$ give rise, via the Thom map, to generators of $H_{2k-1}(BZ_p)$ and $H_{2kp-1}(BZ_q)$ and so give rise to generators of $H_*(BG)$. Thus the $G$ extensions of these manifolds yield generators of $\Omega_*(BG)$ by standard arguments [1, p. 49]. It is well known that if $l$ is an integer, $\xi_1, \ldots, \xi_l$ are primitive $q$th roots of unity, and if $Z_q$ acts on $C^l$ by $a(x_1, \ldots, x_l) = (\xi_1 x_1, \ldots, \xi_l x_l)$, then the fundamental class of $S^{2l-1}/Z_q$ is a generator of $H_{2l-1}(BZ_q)$. If $C^k = C^1 \times \cdots \times C^k$ and $Z_q$ acts on the $j$th factor by $\xi^{r-j}$ then $S^{2kp-1}$ gives rise to a generator of $H_{2kp-1}(BZ_q)$. $V$ has $1 \otimes 1, b \otimes 1, \ldots, b^{p-1} \otimes 1$ as basis and
$a(b^i \otimes 1) = \xi^{r^{-1}}(b^j \otimes 1)$. Now we claim that $G \times Z_q \times S^{2kp-1}$ and $Z_p \times S(kV)$ with the diagonal action of $G$ are $G$ diffeomorphic. It is enough to take the case $k = 1$. Let $e_j$ be a basis for the $j$th factor in $C \times \cdots \times C$. Send $[b^i, e_j]$ in $G \times Z_q \times C^p$ to $(b^i, b^i \otimes 1)$ in $Z_p \times V$. Thus the manifolds $Z_p \times S(kV)$ generate the $q$-torsion in $\Omega_*(BG)$. Let $\mu: \Omega_*(BG) \to H_*(BG)$ be the Thom map. Then

$$\mu(Z_p \times G S(kV))$$

are generators for the $q$-torsion in $H_*(BG)$ and $\mu(Z_p \times G S(kV)) = p\mu(S(kV)/G)$. Since $p$ and $q$ are relatively prime, $\mu(S(kV)/G)$ are generators for the $q$-torsion in $H_*(BG)$ and so $S(kV)/G$ give generators for the $q$-torsion in $\Omega_*(BG)$.

In [10] and [2] the groups $\Omega_*(G; F, F')$, where $G$ is a finite group and $F$ and $F'$ are families of subgroups of $G$, are studied. If $K$ is a subgroup of $G$ we let $F(K)$ denote the family consisting of all conjugates of subgroups of $K$ and we let $F_0(K)$ denote the family consisting of all conjugates of proper subgroups of $K$. We call $F$ and $F'$ strictly adjacent families if there is a subgroup $K$ of $G$ such that $F - F'$ consists of the conjugates of $K$.

Let $F$ and $F'$ be strictly adjacent families of a finite group $G$ which differ by the conjugates of $K$. We have well-known isomorphisms

$$\Omega_*(G; F, F') \cong \Omega_*(G; F(K), F_0(K)) \cong \Omega_*(N(K); F(K), F_0(K))$$

where $N(K)$ is the normalizer of $K$ in $G$. We enunciate these isomorphisms. Let $W$ be a stably complex left $G$-manifold with boundary. Let $M$ be the points in $W$ with isotropy group conjugate to $K$ and $N$ a closed $G$-invariant tubular neighborhood of $K$. Then the class of $W$ in $\Omega_*(G; F, F')$ is sent to the class of $N$ in $\Omega_*(G; F(K), F_0(K))$. If we let $\nu$ be the normal bundle of $M$ in $W$, $M_0$ the points in $M$ with isotropy group equal to $K$, $\nu_0$ the restriction of $\nu$ to $M_0$, then the disk bundle $D(\nu_0)$ is an $N(K)$ manifold with boundary which represents an element in $\Omega_*(N(K); F(K), F_0(K))$ as we assign the class of $D(\nu_0)$ to the class of $N$.

The analysis of these groups is carried out in [11] for unoriented bordism and in more detail in a special case in [2]. We present a treatment of $\Omega_*(G; F(K), F_0(K))$ which we will later explicitly need. Let $I_x$ denote inner automorphism by $x$. Let $\rho$ and $\rho'$ be $n$-dimensional complex representations of $K$. We will say that $\rho$ and $\rho'$ are $G$-equivalent if there are elements $g$ in $G$ and $A$ in $U(n)$ such that $\rho' = I_A \rho I_g$.

**Theorem (1.5).** Let $K$ be normal in $G$. Then

$$\Omega_*(G; F(K), F_0(K)) \cong \sum \Omega_{m-2n}(B(N(\rho)/\Gamma(\rho)))$$

where $n$ runs from 0 to $[m/2]$, $\rho$ runs over a set of representatives of $G$-equivalent $n$-dimensional representations of $K$, $\Gamma(\rho)$ is the graph of $\rho$ in $G \times U(n)$, and $N(\rho)$ is the normalizer of $\Gamma(\rho)$ in $G \times U(n)$.
Proof. Let $W$ be a manifold with boundary representing an element in $\Omega_m(G; F(K), F_0(K))$, $M$ the set of points with isotropy group equal to $K$, $v$ the normal bundle, $M_n$ the union of the $(m - 2n)$-dimensional components of $M$, $\nu_n$ the restriction of $v$ to $M_n$. Then $W$ and $\Sigma D(\nu_n)$ are bordant and the bordism relation respects this decomposition. Thus $\Omega_m(G; F(K), F_0(K)) \cong \Sigma \Psi_n$, where $\Psi_n$ is the bordism group of complex $n$-dimensional left $G$ vector bundles $E$ over compact $(m - 2n)$-dimensional left $G$-manifolds $M$ such that every point on the zero section of $E$ has isotropy group equal to $K$ and every point off the zero section has isotropy group properly contained in $K$. Let $P$ be the principal $U(n)$ bundle corresponding to $E$. As in [2], $P$ can be regarded as a right $G \times U(n)$ space such that the isotropy group of a point $e$ in $P$ is $\Gamma(\rho_e)$ for some $n$-dimensional complex representation $\rho_e$ of $K$. $H = G \times U(n)$ acts differentiably on $P$ so there is a neighborhood $V$ of $e$ in $P$ such that for $e'$ in $V$ there is a $(g, A)$ in $H$ such that $H_e \subseteq (g^{-1}, A^{-1})H_e(g, A)$. From this it follows that $\rho_{e''} = I_{A^{-1}} \rho_e I_A$. Further, if $e' = eA$ for some $A$ in $U(n)$, then $\rho_{e''} = I_{A^{-1}} \rho_e I_A$. Thus for each $x$ in $M$ there is a neighborhood $U$ of $x$ such that, for any two points $e$ and $e'$ in $P \setminus U$, $\rho_e$ and $\rho_{e'}$ are $G$-equivalent and so $H_e$ and $H_{e'}$ are conjugate in $H$. Thus $\Psi_n \cong \Sigma \Psi_{n, \rho}$ where $\Psi_{n, \rho}$ is the bordism group of compact right $G \times U(n)$ spaces $P$ such that $U(n)$ acts freely on $P$, $P/U(n) = M$ is a stably complex $G$-manifold, every point in $P$ has isotropy group conjugate, in $H$, to $\Gamma(\rho)$, and $\rho$ runs over a set of representatives of $n$-dimensional representations of $K$ under $G$-equivalence. So $M/G/K$ is a stably complex manifold. Now let $P \to M$ represent an element in $\Psi_{n, \rho}$ and let $Q \subseteq P$ be the points in $P$ with isotropy group equal to $\Gamma(\rho)$. $Q$ is a principal right $N(\rho)/\Gamma(\rho)$ bundle, $Q/N(\rho)/\Gamma(\rho) = M/G/K$, and $Q$ and $P$ determine each other since $Q \times_{N(\rho)} P \cong P$. Thus

\[ \Psi_{n, \rho} \cong \Omega_{m - 2n}(H(N(\rho)/\Gamma(\rho))). \]

(1.7) We should make a few remarks about extensions of actions. If $H$ is a subgroup of $G$ and $F$ a family of subgroups of $H$, then we let $F$ also denote that family of subgroups of $G$ consisting of all conjugates of elements of $F$. We have a map $E: \Omega_*(H; F) \to \Omega_*(G; F)$ which takes the class of a stably complex $H$-manifold $M$ to the class of $G \times_H M$. Note $G \times_H M$ has a well-defined $G$-invariant stable complex structure. In a similar manner we have $F: \Omega_*(H; F, F') \to \Omega_*(G; F, F')$. Note that if $H_1$ and $H_2$ are conjugate in $G$, then the image of $\Omega_*(H_1; F_0(H_1))$ is equal, in $\Omega_*(G; F)$, to the image of $\Omega_*(H_2; F_0(H_2))$ where $F$ is any family of $G$ containing $F_0(H_1) = F_0(H_2)$.

2. Actions of cyclic groups on polynomial rings. We will develop some algebra which will be relevant to the analysis of $\Omega_*(G; F, F')$. Let $P_1, \ldots, P_t$
be polynomial rings over the integers, \( P_i^{(p)} \) the \( p \)-fold tensor product. \( Z_p \) acts on \( P_i^{(p)} \) by letting the generator \( b \) of \( Z_p \) act by \( b(x_0 \otimes \cdots \otimes x_{p-1}) = x_{p-1} \otimes x_0 \otimes \cdots \otimes x_{p-2} \). Let \( P = P_i^{(p)} \otimes \cdots \otimes P_t^{(p)} \). \( P \) is a polynomial ring and \( Z_p \) acts on \( P \) via the tensor product of the actions on \( P_i^{(p)} \). \( \Omega_* \otimes P \) is a polynomial ring over \( \Omega_* \) on which \( Z_p \) acts. It follows directly from the definition of homology via resolutions that \( H_*(Z_p; \Omega_* \otimes P) \cong \Omega_* \otimes H_*(Z_p; P) \).

**Lemma (2.1).** \( H_i(Z_p; P) = 0 \) for \( i \) even.

**Proof.** We use the resolution of (1.2) for \( Z \) over \( Z[Z_p] \). We must calculate \( \text{kernel}(N)/\text{image}(D) \), where \( N \) and \( D \) act on \( P \). Let \( X \) be a monomial in \( P \). Let \( Z[X] \) be the subgroup of \( P \) which has basis consisting of the distinct images of \( X \) under \( Z_p \). \( P = \bigoplus Z[X] \). It is enough to show \( H_i(Z_p; Z[X]) = 0 \). Either \( bX = X \) or \( X : b \) are distinct. In the first case, the kernel of \( N \) is clearly zero. In the second case

\[
\sum_{j=0}^{p-1} m_j (b^jX) = \sum_{j=0}^{p-1} m_j N(X) = 0 \text{ implies } \sum_{j=0}^{p-1} m_j = 0.
\]

Now for any integer \( k \), \( X - b^kX = \sum_{j=0}^{k-1} (b^jX - b^{j+1}X) \). So \( X \equiv b^kX \mod \text{image}(D) \). Thus \( \sum_{j=0}^{p-1} m_j (b^jX) \equiv \sum_{j=0}^{p-1} m_j X \mod \text{image}(D) \).

**Lemma (2.2).** \( H_i(Z_p; P) \) is a free \( Z \) module.

**Proof.** \( H_0(Z_p; P) = P/\text{image}(D) \). As in (2.1) it is enough to compute \( Z[X]/\text{image}(D) \). If \( bX = X \), \( \text{image}(D) = 0 \) and \( Z[X] \) is a free module with basis \( X \).

In the other case, \( Z[X] \) has, as basis \( X \), \( (1 - b)X \), \( \cdots \), \( (1 - b)b^{p-2}X \). From the formula \( (1 - b)b^{p-1}X = (1 - b)b^{p-2}X + \cdots (1 - b)b^{p-2}X \) it follows that \( (1 - b)X \), \( \cdots \), \( (1 - b)b^{p-2}X \) is a basis for \( \text{image}(D) \) and so \( Z[X]/\text{image}(D) \) is a free \( Z \) module with basis \( X + \text{image}(D) \).

Now let us consider the map \( \phi_j: P_j \rightarrow P^{(p)}_j \)

\[
(2.3) \quad \phi_j(X) = \sum X_0 \otimes \cdots \otimes X_{p-1}
\]

where the sum is taken over all \( X_0 \otimes \cdots \otimes X_{p-1} \) such that \( X_0X_1 \cdots X_{p-1} = X \) in \( P_j \). We then let \( \bar{P} = P_1 \otimes \cdots \otimes P_t \) and let

\[
(2.4) \quad \Phi: \bar{P} \rightarrow P
\]

be the tensor product of the \( \phi_j: P_j \rightarrow P^{(p)}_j \). \( \bar{P} \) is a trivial \( Z_p \) module. \( \Phi \) is a map of \( Z_p \) modules.

**Theorem (2.5).** \( \Phi: H_{2i-1}(Z_p; \bar{P}) = \bar{P} \rightarrow H_{2i-1}(Z_p; P) \) is surjective. Let \( X = X_1 \otimes \cdots \otimes X_t \) be a monomial in \( P \) and let \( P_1 \) be the subgroup with basis consisting of those \( X \) such that each monomial \( X_j \) is a \( p \)-th power. Let \( P_2 \) be
the submodule with basis consisting of the remaining monomials. Then \( P_2 = \ker(\Phi) \) and \( \Phi: \overline{P_1} \cong H_{2i-1}(\mathbb{Z}_p; P) \).

**Proof.** Let \( H_{-i}(\mathbb{Z}_p; P) \) denote \( H_{2i-1}(\mathbb{Z}_p; P) \). Consider \( \Phi(X) = \phi(X_1) \otimes \cdots \otimes \phi(X_t) \) where \( \phi_j(X_j) = \sum X_{j0} \otimes \cdots \otimes X_{j_{p-1}} \), \( X_j \) is a \( p \)-th power in \( P \), if and only if for one term in this sum \( X_{j0} = \cdots = X_{j_{p-1}} \). Call this common value \( Y_j \). Thus \( \Phi(X) \) has a term

\[
Y_1 \otimes \cdots \otimes Y_1 \otimes \cdots \otimes Y_t \otimes \cdots \otimes Y_t = Y
\]

if and only if each \( X_j \) is a \( p \)-th power. Furthermore, a monomial in \( P \) is invariant under \( \mathbb{Z}_p \) if and only if it is such a \( Y \). Thus \( \Phi(X) = Y + \sum T \) where \( bY = Y \) and \( bT \neq T \) if each \( X_j \) is a \( p \)-th power, and \( \Phi(X) = \sum T \) where \( bT = T \) if some \( X_j \) is not a \( p \)-th power.

Now if \( bY = Y \), then \( D(Y) = 0 \) and \( N(Y) = pY \) and so \( H_{-i}(\mathbb{Z}_p; \mathbb{Z}[Y]) = Y \mod pY \). If \( bT = T \), then the kernel of \( D \) restricted to \( \mathbb{Z}[T] \) is the subgroup generated by \( T + bT + \cdots + b^{p-1}T = N(T) \) and so \( H_{-i}(\mathbb{Z}_p; \mathbb{Z}[T]) = 0 \). Thus \( H_{-i}(\mathbb{Z}_p; P) = \bigoplus H_{-i}(\mathbb{Z}_p; \mathbb{Z}[Y]) = \bigoplus Y \mod pY \) for those \( Y \) such that \( bY = Y \).

3. \( \Omega^*(G; F, F') \). Throughout this section \( G = \mathbb{Z}_p \times \mathbb{Z}_q \), \( \eta = e^{2\pi i/p} \), \( \xi = e^{2\pi i/q} \). Let \( F_0 \) be the family of all subgroups of \( G \), \( F_p \) be \( F_0(\mathbb{Z}_p) \), \( F_q \) be \( F_0(\mathbb{Z}_q) \), \( F_0 \) be the family consisting of all proper subgroups, and \( F_1 \) the family consisting of the identity subgroup

\[
\Omega_*(G; F_0, F_q) \cong \Omega_*(G; F_p, F_1) \cong \Omega_*(\mathbb{Z}_p; F(\mathbb{Z}_p), F_0(\mathbb{Z}_p)),
\]

\[
\Omega_*(G; F_0, F_p) \cong \Omega_*(G; F(\mathbb{Z}_q), F_0(\mathbb{Z}_q)) \quad \text{by (1.4).}
\]

First we study \( \Omega_*(\mathbb{Z}_p; F(\mathbb{Z}_p), F_0(\mathbb{Z}_p)) \). The first lemma is well known.

**Lemma (3.1).** Let \( \rho \) be an \( n \)-dimensional representation of \( \mathbb{Z}_p \). Let \( N(\rho) \) be the normalizer of \( \Gamma(\rho) \) in \( \mathbb{Z}_p \times U(n) \). Then

\[
\Omega_*(B(N(\rho)/\Gamma(\rho))) \cong \Omega_*(BU(k_1) \times \cdots \times BU(k_{p-1})).
\]

The isomorphism \( \Omega_*(\mathbb{Z}_p; F(\mathbb{Z}_p), F_0(\mathbb{Z}_p)) \cong \Omega_*(G; F_0, F_q) \) of (1.4) is induced by extension.

**Proof.** Let \( \rho(b) = k_1 \eta \oplus \cdots \oplus k_{p-1} \eta^{p-1} \). Then \( N(\rho) = \mathbb{Z}_p \times C(\rho) \), where
C(\rho) is the centralizer of \rho in U(n). By Schur’s lemma, \( C(\rho) = U(k_1) \times \cdots \times U(k_{p-1}) \). \( N(\rho)/\Gamma(\rho) \cong U(k_1) \times \cdots \times U(k_{p-1}) \), an explicit isomorphism is obtained by sending \( (b_1, A_1, \ldots, A_{p-1}) \) to \( (\eta^{-1}A_1, \ldots, \eta^{-(p-1)}A_{p-1}) \). Now let \( W \) represent an element of \( \Omega^s(Z_p; F(Z_p), F_0(Z_p)) \). \( G \times_{Z_p} W \) is the extension. Let \( M \) be the fixed point set of \( Z_p \) in \( W \) and \( D \) an invariant tubular neighborhood. Then \( G \times_{Z_p} M = G/Z_p \times M \) is the points with isotropy group conjugate to \( Z_p \), \( G \times_{Z_p} D = D' \) is an invariant tubular neighborhood of \( G \times_{Z_p} M \). The points with isotropy group equal to \( Z_p \) is \( Z_p \times Z_p M = M \) and \( D \) restricted to \( M \) is \( D' \). So, the last statement follows.

Now we study \( \Omega^s(G; F(Z_q), F_0(Z_q)) \). Let \( \sigma \) be an \( n \)-dimensional representation of \( Z_q \). Suppose \( \sigma(a) = k_1 \xi \oplus \cdots \oplus k_1 \xi^{q-1} \). Then \( \sigma(1) = 1, 2, \ldots, p-1 \), are all \( G \)-equivalent to \( \sigma \). Let \( C(\sigma) \) be the centralizer of \( \sigma \) in \( U(n) \). Then \( N(\sigma) = \Gamma(\sigma), 1 \times C(\sigma) \). (Here \( [\cdot] \) denotes the group generated by \( \cdot \)).

**Lemma (3.2).** \( (b, B) \) is in \( N(\sigma) \) for some \( B \) in \( U(n) \) if and only if \( \sigma(1) = 1_B \sigma \), and this holds if and only if \( k_j = k_{jr} \) for all \( j \) and all \( s = 1, 2, \ldots, p-1 \).

**Proof.** \( (b, B) \Gamma(\sigma)(b^{-1}, B^{-1}) = \Gamma(\sigma) \) means that \( B \sigma(a) B^{-1} = \sigma(\sigma) = \sigma(1) \). So \( \sigma(1) = 1_B \sigma \). Let \( V = C^n \) be a representation space for \( \sigma \), and \( V_j \) the \( \xi^j \)-eigenspace for \( \sigma \). Suppose \( \sigma(\sigma) = B^{-1} \sigma(\sigma) = B^{-1} \sigma(\sigma) \). Then \( x \) is in \( V_j \) if and only if \( \sigma(\sigma) = B^{-1} \sigma(\sigma) = B^{-1} \sigma(\sigma) \). Thus \( B^{-1}: V_j \cong V_j \), and this can hold only if \( k_j = k_{jr} \). On the other hand, if \( k_j = k_{jr} \) for all \( j \), choose a basis for \( V \) by choosing a basis for each \( V_j \). Let \( B^{-1} \) be the permutation matrix relative to this basis which takes the basis in \( V_j \) to the basis in \( V_{jr} \). Then \( (b, B) \) is in \( N(\sigma) \).

**Lemma (3.3).** If \( \sigma(1) = 1_B \sigma \) for some \( B \), then
\[
N(\sigma) = [(b, B), \Gamma(\sigma), 1 \times C(\sigma)].
\]
If not, \( N(\sigma) = [\Gamma(\sigma), 1 \times C(\sigma)] \). \( C(\sigma) \cong U(k_1) \times \cdots \times U(k_{q-1}) \). If \( A = (A_1, \ldots, A_{q-1}) \) is the matrix of an element in \( C(\sigma) \) relative to the basis indicated in (3.2), we can choose \( B \) to be the permutation matrix of (3.2) and then
\[
(b, B)(1, A_1, \ldots, A_{q-1})(b^{-1}, B^{-1}) = (1, A'_1, \ldots, A'_{q-1})
\]
where \( A'_{j} = A_{jr} \).

**Proof.** It is clear that the \( B \) of (3.2) gives us an element \( (b, B) \) in \( N(\sigma) \). If \( (b, B) \) is in \( N(\sigma) \), then \( B^{-1}B' \) is in \( C(\sigma) \) and so \( (b, B') = (b, B)(1, B^{-1}B') \).

Let \( Z_q^* \) be the multiplicative group of nonzero elements of \( Z_q \) and \( [r] \) the subgroup generated by \( r \). Let \( \{s_j, j = 1, \ldots, (q-1)/p \} \), be representatives for \( Z_q^*/[r] \). Then
Inner automorphism by $B$ in (3.3) sends $\prod_{j=0}^{p-1} U(k_j)$ to itself and takes $(A_0, \ldots, A_{p-1})$ to $(A_1, \ldots, A_{p-1}, A_0)$. Thus

**Lemma (3.4).** If $ab = l \sigma$, then

$$\frac{N(\sigma)}{\Gamma(\sigma)} \cong Z_p \times \prod_{j=1}^{q-1/p} \prod_{\ell=0}^{p-1} U(l_j)$$

where $l$ is the common value of $k_j$ and $t$, and inner automorphism by the generator $b$ of $Z_p$ preserves $\prod_{j=0}^{p-1} U(l_j)$ and takes $(A_0, \ldots, A_{p-1})$ to $(A_1, \ldots, A_{p-1}, A_0)$.

Now we will analyze $\Omega_*(B(N(a)/\Gamma(a)))$ for such $a$. Let

$$U = \prod_{j=1}^{q-1/p} \prod_{\ell=0}^{p-1} U(l_j).$$

$\Omega^*(BU)$ is a power series ring in Chern classes.

$$\Omega^*(BU) = \text{Hom}_{\Omega^*}(\Omega^*(BU); \Omega^n).$$

For each monomial $C_\alpha$ in the Chern classes, let $X_\alpha$ be the element in $\Omega_*(BU)$ such that $\langle X_\alpha, C_\beta \rangle = \delta_\alpha^\beta$. Then $\Omega_*(BU)$ is a free $\Omega_*$ module with basis $\{X_\alpha\}$. We can make $\Omega_*(BU)$ into a polynomial ring by defining $X_\alpha X_\beta = X_\gamma$ where $X_\gamma$ is dual to $C_\alpha C_\beta$. And $\Omega_*(BU) \cong \Omega_* \otimes P$ where $P$ is the polynomial ring over the integers with monomials $X_\alpha$. Consider the diagonal map $U(l_j) \to \prod_{j=0}^{p-1} U(l_j)$.

Let

$$U = \prod_{j=1}^{q-1/p} \prod_{\ell=0}^{p-1} U(l_j)$$

and let

$$\Psi : U \to U$$

be the product of the diagonal maps. Let $\Psi$ also denote the induced map $BU \to BU$. Let $t = (q - 1)/p$.

**Lemma (3.8).** The map $\Omega_*(BU(l)) \to \bigotimes_{0}^{p-1} \Omega_*(BU(l))$ induced by the diagonal takes a monomial $X$ in $\Omega_*(BU(l))$ to $\sum X_0 \otimes X_t \otimes X_{p-1}$ where the sum is taken over all $X_0 \otimes \cdots \otimes X_{p-1}$ such that $X_0 \cdots X_{p-1} = X$ in $\Omega_*(BU(l))$. The map $\Omega_*(BU) \to \Omega_*(BU)$ is the tensor product of these maps.

**Proof.** Consider the diagonal map $\Delta : BU(l) \to \prod_{0}^{p-1} BU(l)$. Let $X$ be a monomial in $\Omega_*(BU(l))$. Then $\Delta_* (X) = \sum M(X_0 \otimes \cdots \otimes X_{p-1})$ where $X_i$ is in...
\[ \Omega_*(BU(l)) \] and \( M \) is in \( \Omega_* \). Consider \( C_{a_0} \otimes \cdots \otimes C_{a_{p-1}} \) in \( \Omega^*(\Pi_0^{p-1} BU(l)) \).

\[ \langle \Delta_*(X), C_{a_0} \otimes \cdots \otimes C_{a_{p-1}} \rangle = \langle X, \Delta^*(C_{a_0}) \cdots \Delta^*(C_{a_{p-1}}) \rangle \]

\[ \langle X, C_{a_0} \cdots C_{a_{p-1}} \rangle = 1 \]

if \( X \) is dual to \( C_{a_0} \cdots C_{a_{p-1}} \) and zero otherwise. A term \( M \langle X_0 \otimes \cdots \otimes X_{p-1}\rangle \) is one only if \( M = 1 \) and \( X_j \) is dual to \( C_{a_j} \). Then by definition, \( X_0 \cdots X_{p-1} \) is dual to \( C_{a_0} \cdots C_{a_{p-1}} \) and so equals \( X \). Now the lemma follows.

**Remark.** Let \( P_j = \Omega_*(BU(l)_j) \), \( \Omega_* \otimes P(p)_j = \Omega_*(\Pi_0^{p-1} BU(l)_j) \), and \( \Omega_* \otimes P_1(p) \otimes \cdots \otimes P_t(p) = \Omega_* \otimes P = \Omega_*(BU) \).

\[ \Omega_* \otimes P_1 \otimes \cdots \otimes P_t = \Omega_* \otimes \overline{P} = \Omega_*(BU) \]

And the map \( \Omega_*(BU) \to \Omega_*(BU) \) is precisely the map \( 1 \otimes \Phi \) where \( \Phi \) is the map of \( (2.4) \).

The map \( (3.7) \) induces a group homomorphism \( Z_p \times \overline{U} \to Z_p \times_f U \) and so a map

\[ (3.9) \quad BZ_p \times BU \to B(Z_p \times_f U) \]

**Lemma (3.10).** The maps \( \Omega_-(BZ_p \times BU) \to \Omega_-(B(Z_p \times_f U)) \) and \( H_-(BZ_p \times BU) \to H_-(B(Z_p \times_f U)) \) are surjective.

**Proof.** Let us consider the two Atiyah spectral sequences

\[ E^2_{i,*} = H_i(Z_p; \Omega_*(BU)) \Rightarrow \Omega_*(B(Z_p \times \overline{U})) \]

\[ E^2_{i,*} = H_i(Z_p; \Omega_*(BU)) \Rightarrow \Omega_*(B(Z_p \times_f U)) \]

The map \( (3.9) \) induces a map of these spectral sequences \( \overline{E}_{i,*} \to E_{i,*} \). In the remark following \( (3.8) \) we noted that \( \Omega_*(BU) = \Omega_* \otimes \overline{P} \), \( \Omega_*(BU) = \Omega_* \otimes P \) and the map \( H_i(Z_p; \Omega_* \otimes \overline{P}) \to E_{i,*} \) is the map \( 1 \otimes \Phi \). \( E_{i,j} = 0 \) if \( i \) is even, \( i \neq 0 \) by \( (2.1) \). \( E_{i,j} = 0 \) if \( j \) is odd, \( j \neq 0 \), since \( \Omega_*(BU) \) is concentrated in even dimensions. \( E_{i,j} = 0 \) if \( i \) is odd, \( j \) even. \( \Omega_*(BU) \) is free abelian for \( j \) even, and \( E_{0,j} = H_0(Z_p; \Omega_* \otimes \overline{P}) \) is free abelian by \( (2.2) \). For \( j \) odd \( E_{0,j} = 0 \). Thus no differential could possibly be nonzero and so both spectral sequences collapse. Consider the filtrations \( F_{*,*} \) on \( \Omega_*(BZ_p \times BU) \) and \( F_{*,*} \) on \( \Omega_*(B(Z_p \times_f U)) \).

\[ \overline{F}_{i,j} / \overline{F}_{i-1,j+1} = F_{i,j} / F_{i-1,j+1} = 0 \]

for \( i + j \) even and \( i \neq 0 \). For \( m \) odd,
\[ \Omega_m(BZ_p \times B \mathcal{U}) = F_{m,0} \supset \cdots \supset F_{1,m-1} \supset 0, \]

\[ \Omega_m(B(Z_p \times_f U)) = F_{m,0} \supset F_{m-2,2} \supset \cdots \supset F_{1,m-1} \supset 0. \]

And

\[ F_{i,j} / F_{i-2,j+2} = H_i(Z_p ; \Omega_j(B \mathcal{U})), \quad F_{i,j} / F_{i-2,j+2} = H_i(Z_p ; \Omega_j(B \mathcal{U})) \]

for \( i + j \) odd. We have maps

\[
0 \rightarrow F_{i-2,j+2} 
\downarrow \alpha' \downarrow \alpha \\
F_{i,j} 
\downarrow \alpha'' \\
0 
\rightarrow F_{i-2,j+2} 
\rightarrow F_{i,j} 
\rightarrow H_i(Z_p ; \Omega_j(B \mathcal{U})) 
\rightarrow 0
\]

For \( i = 1, j = m - 1, \) \( F_{1,m-1} = H_1(Z_p ; \Omega_{m-1}(B \mathcal{U})) \rightarrow H_1(Z_p ; \Omega_{m-1}(B \mathcal{U})) = F_{1,m-1} \) is onto. By induction \( \alpha'' \) is surjective, and \( \alpha'' \) is surjective by computation. Hence \( \alpha \) is surjective. Thus by induction \( F_{m,0} \rightarrow F_{m,0} \) is surjective. The same argument applied to \( H_*(BZ_p \times B \mathcal{U}) \) and \( H_*(B(Z_p \times_f U)) \) gives the corresponding conclusion for homology.

**Lemma (3.11).** \( \Omega_+(B \mathcal{U}) \rightarrow \Omega_+(B(Z_p \times_f U)) \) and \( H_+(B \mathcal{U}) \rightarrow H_+(B(Z_p \times_f U)) \) are surjective.

**Proof.** In the spectral sequence \( E_{*,*} \) of (3.10), \( E^2_{0,*} = H_0(Z_p ; \Omega_*(B \mathcal{U})) \) and \( E^2_{i,j} = 0 \) for \( i + j \) even, \( i \neq 0 \). Thus \( \Omega_+(B(Z_p \times_f U)) = H_0(Z_p ; \Omega_*(B \mathcal{U})) \) and \( \Omega_*(B \mathcal{U}) \rightarrow H_0(Z_p ; \Omega_*(B \mathcal{U})) \) is surjective by definition of \( H_0 \). Similarly for homology.

**Theorem (3.12).** \( \Omega_-(B(Z_p \times_f U)) \) has projective dimension one over \( \Omega_* \).

**Proof.** First consider

\[
\Omega_+(B \mathcal{U}) \rightarrow \Omega_+(B(Z_p \times_f U)) \qquad \Omega_+(B \mathcal{U}) \rightarrow \Omega_+(B(Z_p \times_f U))
\]

\[
\begin{array}{ccc}
\mu & \downarrow \\
H_+(B \mathcal{U}) & \rightarrow & H_+(B(Z_p \times_f U))
\end{array}
\]

where \( \bar{\mu} \) and \( \mu \) are the Thom maps. \( \bar{\mu} \) is surjective since \( \Omega_*(B \mathcal{U}) \) is a free \( \Omega_* \) module, and the two horizontal maps are surjective. Hence \( \mu \) is surjective. Now consider

\[
\Omega_-(BZ_p \times \mathcal{U}) \rightarrow \Omega_-(B(Z_p \times_f U)) \qquad \Omega_-(BZ_p \times \mathcal{U}) \rightarrow \Omega_-(B(Z_p \times_f U))
\]

\[
\begin{array}{ccc}
\bar{\mu} & \downarrow \\
H_-(BZ_p \times \mathcal{U}) & \rightarrow & H_-(B(Z_p \times_f U))
\end{array}
\]
Now \( \Omega_-(BZ_p \times \widetilde{BU}) \cong \bigoplus \Omega_-(BZ_p) \) and so has projective dimension one. Then from [4], [7], \( \mu \) is onto, and since the horizontal maps are onto, \( \mu \) is onto. Thus \( \mu: \Omega_*(B(Z_p \times \widetilde{U})) \to H_*(B(Z_p \times \widetilde{U})) \) is onto and so again by [4], [7], \( \Omega_*(B(Z_p \times \widetilde{U})) \) and \( \Omega_-(B(Z_p \times \widetilde{U})) \) have projective dimension one over \( \Omega_* \).

Corollary (3.12). \( \Omega_-(G; F(Z_q), F_0(Z_q)) \) has projective dimension one over \( \Omega_* \).

Proof. From (1.5), \( \Omega_-(G; F(Z_q), F_0(Z_q)) \cong \bigoplus \Omega_-(B(N(o)/\Gamma(o))) \). From (3.3), \( \Omega_-(B(N(o)/\Gamma(o))) \cong \Omega_-(BU) = 0 \) or \( \Omega_-(B(Z_p \times \widetilde{U})) \) which has dimension one over \( \Omega_* \).

As an indication of a technique necessary in the case of the more general metacyclic group we give more information on the module \( \Omega_-(B(Z_p \times \widetilde{U})) \) of (3.12). As in the remark following (3.8), \( \Omega_*(\widetilde{BU}) = \Omega_* \otimes \widetilde{P} \), and as in (2.5) let \( \widetilde{P}_1 \) be the free \( Z \)-module with basis consisting of \( X_1 \otimes \cdots \otimes X_t \) where each \( X_t \) is a \( p \)-th power in \( \Omega_*(\widetilde{BU}) \).

Theorem (3.13). \( \Omega_-(BZ_p) \otimes \Omega_\star \otimes \widetilde{P}_1 \cong \Omega_-(B(Z_p \times \widetilde{U})) \).

Proof. Let \( \Lambda = \Omega_\star \otimes \widetilde{P}_1 \). Let \( b_\ast(X) \) be the homology theory \( \Omega_\star(X) \otimes \Lambda \), \( b_\ast''(X) \) the homology theory \( \Omega_\star(X) \otimes \Omega_\star(\widetilde{BU}) \), and \( b_\ast''(X) = \Omega_\star(X \times \widetilde{BU}) \). We have a natural transformation \( b_\ast(X) \to b_\ast''(X) \) induced by \( \Lambda \subset \Omega_\star(\widetilde{BU}) \), and \( b_\ast''(X) \to b_\ast''(X) \) induced by \( [M, f] \otimes [W, g] \to [M \times W, f \times g] \). So we have induced maps on the skeleton filtrations

\[
(Fb'')_{\ast,*} (X) \to (Fb''')_{\ast,*} (X) \to (Fb''')_{\ast,*} (X).
\]

The second natural transformation is a natural isomorphism. Now consider the fibration \( \widetilde{BU} \to X \times \widetilde{BU} \to X \). The spectral sequence of this fibration is \( \widetilde{E}^2(X) = H_\ast(X; \Omega_\star(\widetilde{BU})) \Rightarrow \Omega_\star(X \times \widetilde{BU}) \) and the associated filtration on \( \Omega_\star(X \times \widetilde{BU}) \) is precisely \( (Fb'')_{\ast,*} (X) \). Now let \( X = BZ_p \) and consider the map of fibrations of (3.9).

\[
\begin{array}{ccc}
B\widetilde{U} & \longrightarrow & BU \\
\downarrow & & \downarrow \\
BZ_p \times \widetilde{BU} & \longrightarrow & B(Z_p \times \widetilde{U}) \\
\downarrow & & \downarrow \\
BZ_p & \longrightarrow & BZ_p
\end{array}
\]

gives rise to a map \( \widetilde{E}^2(BZ_p) = \widetilde{E}^2 \to E^2 = H_\ast(Z_p; \Omega_\star(\widetilde{BU})) \) and we know from (3.10) that both spectral sequences collapse. Let
All of these spectral sequences collapse. We have a map of spectral sequences $E'^2 \to E''^2 \to E''^2 = E^2 \to E^2$ where the first two maps are induced by the natural transformations. Thus we have a map $E'^{i,j}_{i,j} \to E''^{i,j}_{i,j}$. Now $E'^{i,j}_{i,j} = E^2_{i,j}$ if $i$ is even and $i \neq 0$, or if $j$ is odd. If $i$ is odd and $j$ is even, this map is an isomorphism (2.5). Thus $E'^{i,j}_{i,j} = E^2_{i,j}$ unless $i = 0$ and $j$ is even. Let $F_{*,*}$ be the filtration on $b_*(BZ_p)$ and $F_{*,*}$ on $\Omega_*(B(Z_p \times_f U))$. Then for $m$ odd,

$$0 \to F_{m-(2k+2),*} \to F_{m-2k,*} \to H_{m-2k}(BZ_p; b'^{i}_*) = E^{i}_{m-2k,*} \to 0$$

$$0 \to F_{m-(2k+2),*} \to F_{m-2k,*} \to H_{m-2k}(Z_p; \Omega_*(BU)) = E^2_{m-2k,*} \to 0$$

The left most map is an isomorphism by induction, and the right most by computation. Thus, the middle map is an isomorphism and so

$$b'_*(BZ_p) \cong \Omega_*(B(Z_p \times_f U))$$

We need a description of the image of $\Omega_*(B(Z_p \times_f U))$ in $\Omega_*(G; F(Z_q), F_0(Z_q))$. In view of (3.10) we need only describe the image of $\Omega_*(BZ_p \times BU(l_1) \times \cdots \times BU(l_t))$.

**Lemma (3.14).** Let $P_0$ be a left principal $Z_p$ bundle, and $P_i$ a right principal $U(l_i)$ bundle. Let $E_i = P_i \times U(l_i)$ $C^{l_i}$. Then $P_0 \times P_1 \times \cdots \times P_t$ in $\Omega_*(BZ_p \times BU)$ goes under the map

$$\Omega_*(BZ_p \times BU) \to \Omega_*(B(Z_p \times_f U)) \to \Omega_*(G; F(Z_q), F_0(Z_q))$$

to the $G$ vector bundle $P_0 \times \bigoplus_{i=0}^{t-1} E_i \to P_0$ where $G$ acts on $P_0$ by $G \to G/Z_q = Z_p$ acting on $P_0$, and $G$ acts on $\bigoplus_{i=0}^{t-1} E_i$ by $b(x_0, \ldots, x_{p-1}) = (x_1, x_2, \ldots, x_{p-1}, x_0)$, $a(x_0, \ldots, x_{p-1}) = (\xi^{a_1}x_0, \ldots, \xi^{a_t}x_{p-1})$.

**Proof.** Let $P = P_0 \times P_1 \times \cdots \times P_t$. $P$ is a right $Z_p \times U$ bundle. Let $Q = P \times Z_p \times_f U$. It is easy to see that $Q \cong P_0 \times \prod_{i=0}^{t-1} (P_i \times U(l_i)) (U(l_i))^p$, where $Z_p$ acts diagonally on the right and on $P_i \times U(l_i) (U(l_i))^p$, $Z_p$ acts by $[e; A_0, \ldots, A_{p-1}] = [e; A_{p-1}, A_0, \ldots, A_{p-2}]$. Now $N(\sigma)/\Gamma(\sigma) \cong Z_p \times_f U$, the isomorphism takes $(b, B)\Gamma(\sigma)$ to $b$ and $(1 \times U)\Gamma(\sigma)$ to $1 \times U$. Thus $Q$ becomes a principal $N(\sigma)/\Gamma(\sigma)$ bundle. Then $Q \times N(\sigma) (G \times U(n)) (U(n))^C$ is a left $G$ vector bundle, and from (1.5) is the element in $\Omega_*(G; F(Z_q), F_0(Z_q))$ coming from $P$. By a series of obvious isomorphisms,

$$Q \times N(\sigma) (G \times C^n) \cong Q \times_U C^n \cong P_0 \times \prod_{i=1}^{t} P_i \times U(l_i) C^{pl_i}$$
where, following the left action of $G$ along, we find $G$ acting on $P_0 \times \Pi P_j$ by acting diagonally. $G$ acts on $P_0$ by $G \to G/Z_q = Z_p$ and on $P_j \times U(l_j) \subset C^{pl_j}$ by

\[ b[e; x_0, \ldots, x_{p-1}] = [e; x_1, x_2, \ldots, x_{p-1}, x_0], \]
\[ a[e; x_0, \ldots, x_{p-1}] = [e; \xi_s x_0, \ldots, \xi_s x_{p-1}]. \]

Finally $P_j \times U(l_j) \subset C^{pl_j} = \bigoplus_{0}^{p-1} E_j$, with the indicated action.

**Corollary (3.15).** $\Omega_*(G; F(Z_q), F_0(Z_q))$ is generated by the $G$ vector bundles $S^{2k-1} \times \bigoplus_{0}^{p-1} E_1 \times \cdots \times \bigoplus_{0}^{p-1} E_t \to S^{2k-1}$ where $G$ acts on $S^{2k-1}$ by $G \to Z_p$ and $Z_p$ acts by $b(x_1, \ldots, x_k) = (\eta x_1, \ldots, \eta x_k)$ and $G$ acts on $\bigoplus_{0}^{p-1} E_j$ as in (3.14).

**Proof.** This follows from the fact that $S^{2k-1}$ with the indicated action of $Z_p$ for $k = 1, 2, \cdots$ are generators for $\Omega_*(BZ_p)$.

**Remark. (3.16).** If $G$ is any finite group, $F$ the family of all subgroups, $F_0$ the family of all proper subgroups, then it is well known that $\Omega_*(G; F, F_0)$ is the bordism group of $G$ vector bundles $E$ over trivial $G$ manifolds. $E \cong \bigoplus_{\pi} V_{\pi} \otimes \text{Hom}_G(\pi, E)$ where $V_{\pi}$ runs over the irreducible representation of $G$. So $\Omega_*(G; F, F_0) \cong \bigoplus \Omega_*(BU(k_1) \times \cdots \times BU(k_l))$ is a free $\Omega_*$-module.

4. Proofs of Theorems A and B. We start with the families $F_p$ and $F_1$ which are adjacent:

\[ \Omega_+(G; F_1) = \Omega_+, \quad \Omega_-(G; F_p, F_1) = 0. \]

The long exact sequence

\[ (4.1) \rightarrow \Omega_+(G; F') \rightarrow \Omega^*(G; F) \rightarrow \Omega^*(G; F, F') \rightarrow \Omega^*(G; F') \rightarrow \]

for any pair of families breaks up as

\[ 0 \rightarrow \Omega_+ \rightarrow \Omega_+(G; F_p) \rightarrow \Omega_+(G; F_p, F_1) \rightarrow \Omega_+(G; F_1) \rightarrow \Omega_+(G; F) \rightarrow 0. \]

Now we have, from (1.3) and (3.1),

\[ \Omega_+(G; F_p, F_1) \rightarrow \Omega_+(G; F_1) = \Omega_-(Z_p; F_0) \oplus \text{Image}(\Omega_-(Z_q; F_0)) \]

\[ \Omega_+(Z_p; F(Z_p), F_0(Z_p)) \rightarrow \Omega_-(Z_p; F_0). \]

The map $\Omega_+(Z_p; F(Z_p), F_0(Z_p)) \rightarrow \Omega_-(Z_p; F_0)$ is well known to be onto. Thus we get short exact sequences
\[ 0 \to \Omega_+ \to \Omega_+(G; F_p) \to \Omega_+(G; F_q; F_1) \to \Omega_-(Z_p; F_0) = \Omega_-(BZ_p) \to 0, \]

\[ \text{Image}(\Omega_-(Z_q; F_0)) \cong \Omega_-(G; F_p). \]

Now \( \text{Image}(\Omega_-(Z_q; F_0)) \) being a summand of \( \Omega_-(BG) \) has projective dimension one. Hence so does \( \Omega_-(G; F_p) \). Now \( \Omega_-(BZ_p) \) has dimension one and \( \Omega_+(G; F_p; F_1) \) is free by (3.1), hence \( \text{Image}(\gamma) \) is projective over \( \Omega_+ \) and hence free [4, 3.2]. Thus \( 0 \to \Omega_+ \to \Omega_+(G; F_p) \to \text{Image}(\gamma) \to 0 \) and so \( \Omega_+(G; F_p) \) is free.

Next consider the families \( F_q, F_1 \) and apply (4.1). \( \Omega_-(G; F_q, F_1) \) is all torsion by (3.10). Thus

\[ 0 \to \Omega_+ \to \Omega_+(G; F_q) \to \Omega_+(G; F_q; F_1) \to \Omega_-(G; F_q) \to 0. \]

Consider

\[ \Omega_+(G; F_q; F_1) \to \Omega_+(G; F_q) = \text{Image}(\Omega_-(Z_q; F_0)) + \Omega_+(Z_p; F_0) \]

\[ \begin{array}{c}
\Omega_+(Z_q; F_q; F_0) \to \Omega_-(Z_q; F_0) \to 0.
\end{array} \]

From (3.1) and (3.11) it follows that the map \( E \) is onto, and we also know, from (3.11), that \( \Omega_+(G; F_q, F_1) \) is free. Thus we get

\[ 0 \to \Omega_+ \to \Omega_+(G; F_q) \to \Omega_+(G; F_q; F_1) \to \text{Image}(\Omega_-(Z_q; F_0)) \to 0 \]

\[ 0 \to \Omega_-(Z_p; F_0) \to \Omega_-(G; F_q) \to \Omega_-(G; F_q; F_1) \to 0. \]

Now \( \Omega_-(Z_p; F_0) \) has dimension one and, by (3.12), \( \Omega_-(G; F_q, F_1) \) has dimension one and thus \( \Omega_-(G; F_q) \) has dimension one. By the same argument as for \( F_p, F_1 \), \( \text{Image}(\gamma) \) is free and hence \( \Omega_+(G; F_q) \) is free.

Now consider the families \( F_q, F_0 \). By (1.4), \( \Omega_+(G; F_q, F_0) \cong \Omega_+(G; F_q, F_1) \). Thus we get the sequence of (4.1)

\[ 0 \to \Omega_+(G; F_q) \to \Omega_+(G; F_q; F_1) \to \Omega_-(G; F_p) \to \Omega_-(G; F_0) \to 0, \]

\[ \begin{array}{c}
\Omega_+(Z_q; F_q, F_1) \to \Omega_-(Z_q; F_0) \to 0.
\end{array} \]

Since we have previously shown \( \Omega_-(G; F_q) \cong \text{Image}(\Omega_-(Z_q; F_0)) \) we get \( \Omega_-(G; F_q) \cong \Omega_-(G; F_q, F_1) \) and

\[ 0 \to \Omega_+(G; F_q) \to \Omega_+(G; F_q; F_1) \to \text{Image}(\Omega_+(BZ_q)) \to 0. \]
From the first isomorphism and (3.12), $\Omega_-(G; F_0)$ has projective dimension one. The isomorphism takes $W$ in $\Omega_-(G; F_0)$ to the $G$ vector bundle $E \to M$ where $M$ is the set of points in $W$ with isotropy group $Z_q$, and $E$ is the normal bundle of $M$ in $W$. From the second sequence we conclude, since $\Omega_+(G; F_q, F_1)$ is free and $\text{Image}(\Omega_-(BZ_q))$ has dimension one, that $\text{Image}(y)$ is free. Now

$$0 \to \Omega_+(G; F_p, F_0) \to \Omega_+(G; F_0) \to \text{Image}(y) \to 0.$$

Since $\Omega_+(G; F_p)$ and $\text{Image}(y)$ are free, it follows that $\Omega_+(G; F_0)$ is free.

Finally consider the families $F_\alpha$ and $F_0$ of $G$:

$$0 \to \Omega_+(G; F_0) \to \Omega_+(G; F_\alpha) \to \Omega_+(G; F_\alpha, F_0) \to \Omega_-(G; F_0) \to \Omega_-(G; F_\alpha) \to 0.$$

We want to show $\Omega_+(G; F_\alpha, F_0) \to \Omega_-(G; F_0)$ is onto. (3.15) tells us generators $S^{2k-1} \times \bigoplus E_1 \times \cdots \times \bigoplus E_t$ represents an element in $\Omega_+(G; F_\alpha, F_0)$. $S(E)$ represents an element in $\Omega_-(G; F_0)$. The set of points with isotropy group $Z_q$ is precisely $S(C^k)$ and the normal bundle to $S^{2k-1}$ is $S^{2k-1} \times \bigoplus E_1 \times \cdots \times \bigoplus E_t$. To check that this is the generator $S^{2k-1} \times \bigoplus E_1 \times \cdots \times \bigoplus E_t$ consider $V_j \otimes E_j$.

We can take as a basis of $V_j, 1 \otimes 1, b^{-1} \otimes 1, \ldots, b^{-(p-1)} \otimes 1$ where $a(b^{-i} \otimes 1) = \xi^j \pi^j(b^{-i} \otimes 1)$. The general element of $V_j \otimes E_j = (x_0, \ldots, x_{p-1}) = 1 \otimes 1 \otimes x_0 + \cdots + 1 \otimes b^{-1} \otimes x_{p-1}, b^j(x_0, \ldots, x_{p-1}) = 1 \otimes 1 \otimes x_1 + b^{-1} \otimes 1 \otimes x_2 + \cdots + 1 \otimes b^{-(p-1)} \otimes x_{p-1}$, and $a(x_0, \ldots, x_{p-1}) = \sum \xi^{j \pi^j} x_0, \ldots, x_{p-1}$. Thus $\Omega_+(G; F_\alpha, F_0)$ is onto and so $\Omega_-(G; F_0) = 0$. Our exact sequence becomes

$$0 \to \Omega_+(G; F_0) \to \Omega_+(G; F_\alpha) \to \Omega_-(G; F_0) \to 0.$$

We know $\Omega_-(G; F_0)$ has projective dimension one and so $\text{Image}(y)$ is free. We get the exact sequence

$$0 \to \Omega_+(G; F_0) \to \Omega_+(G; F_\alpha) \to \text{Image}(y) \to 0.$$

$\Omega_+(G; F_0)$ and $\text{Image}(y)$ are free, hence $\Omega_+(G; F_\alpha)$ is free. This concludes the proof of Theorems A and B.
BIBLIOGRAPHY


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