ABSTRACT. We study the bordism group of stably complex G-manifolds in the case where G is a metacyclic group of order pq and p and q are distinct primes. This bordism group is a module over the complex bordism ring and we compute the projective dimension of this module. We develop some techniques necessary for the study of this module in case G is a more general metacyclic group.

The purpose of this paper is to study the bordism theory of actions of a metacyclic group on stably complex manifolds in a special case. More general and complete results will appear in [8]. However, many of the techniques used can be isolated and simplified in this special case. We will be concerned with actions of a group G which is the semidirect product of two cyclic groups of prime order.

By a family F of subgroups of G we will mean a subset of the subgroups of G with the property that if K is an element of F every subgroup and every conjugate of K is in F. \( \Omega_+^*(G; F) \) is the bordism group of actions of G on stably complex manifolds such that every isotropy group lies in the family F. \( \Omega_* \) will denote the complex bordism ring.

Theorem A. Let F be the family consisting of all subgroups of G. Then \( \Omega_+^*(G; F) \) is a free \( \Omega_* \) module on even dimensional generators.

Theorem B. Let F be any family of subgroups of G. Then \( \Omega_+^*(G; F) = \bigoplus \Omega_{2i}^*(G; F) \) is a free \( \Omega_* \) module. And \( \Omega_-^*(G; F) = \bigoplus \Omega_{2i+1}^*(G; F) \) has projective dimension one over \( \Omega_* \).

Basic material on the groups \( \Omega_+^*(G; F) \) can be found in [3], [10]. Our analysis largely follows the lines of [2] but was influenced by [10] and [6].

In §1 we discuss general facts about equivariant bordism groups. In §2 we discuss the cohomology of a cyclic group acting on a polynomial ring. In §3 we discuss \( \Omega_+^*(G; F, F') \) for adjacent families F, F' in G and in §4 we present the proofs of the main theorems.

Received by the editors September 13, 1971.


Key words and phrases. Bordism theory of actions, actions of metacyclic groups, adjacent families of subgroups, projective dimension of bordism modules, equivariant bordism, cohomology of groups, actions of cyclic groups, induced representations.

(1) Research partially supported by National Science Foundation grant GP-12639.
1. Basic facts. Let \( p \) and \( q \) be distinct primes. Let \( / \) be a homomorphism of \( Z_p \) into the automorphism group of \( Z_q \). We denote the resulting semidirect product by \( Z_p \rtimes Z_q = G \). \( G \) is generated by two elements \( a \) and \( b \) with relations \( b^p = a^q = 1 \), \( bab^{-1} = a^r \) where \( r \) is a nonzero element of \( Z_q \). \( / \), of course, is the homomorphism which takes \( b \) to the automorphism \( a \rightarrow a^r \).

Lemma (1.1). \( H_{2k-1}(BG) = Z_p \) if \( k \) is not a multiple of \( p \) and \( H_{2k-1}(BG) = Z_p \oplus Z_q \) if \( k \) is a multiple of \( p \).

Proof. Consider the Hochschild-Serre spectral sequence

\[
E^2_{i,j} = H_i(Z_p; H_j(Z_q)).
\]

\( E^2_{i,j} = 0 \) unless \( i \) or \( j \) is zero. \( E^2_{i,0} = H_i(Z_p; Z) = 0 \) or \( Z_p \) depending upon whether \( i \) is even or odd. To compute \( E^2_{0,j} = H_0(Z_p; H_j(Z_q)) \) we use the resolution

\[
Z[Z_p] \xrightarrow{N} Z[Z_p] \xrightarrow{D} Z[Z_p]
\]

where \( D = 1 - b \), \( N = 1 + b + \cdots + b^{p-1} \) of [9, p. 121]. Tensoring on the right with \( H_j(Z_q) \), we find that \( E^2_{0,j} = \text{cokernel}(D \otimes 1) \). A simple computation shows the generator \( b \) of \( Z_p \) acts on \( H_{2j-1}(Z_q) = Z_q \) by multiplication by \( r^j \) and \( 1 - r^j \) is a unit in \( Z_q \) unless \( j \) is 0 mod \( p \). It is clear that the inclusion \( Z_p \subset G \) induces an injection \( H_*(BZ_p) \rightarrow H_*(BG) \) onto the \( p \)-torsion, and the inclusion \( Z_q \subset G \) induces a map \( H_*(BZ_q) \rightarrow H_*(BG) \) which is onto the \( q \)-torsion.

Lemma (1.3). \( \Omega_*(BG) \) has projective dimension one over \( \Omega_* \). The left \( G \)-manifolds \( G \times Z_p \times Z_q \) generate the \( p \)-torsion and the left \( G \)-manifolds \( G \times Z_p \times Z_q \) generate the \( q \)-torsion where \( Z_p \) acts on \( S^{2k-1} \) by multiplication of each coordinate in \( C^k \) by \( e^{2\pi i/p} \) and \( Z_q \) acts on \( S^{2kp-1} \) by multiplication by \( e^{2\pi i/q} \). Let \( V \) be the representation of \( G \) induced from the one-dimensional representation \( e^{2\pi i/q} \) of \( Z_q \) (see [5, p. 333]). \( V = Z[G] \otimes Z[q] \). Then the left \( G \)-manifolds \( S(kV) \) also serve as generators for the \( q \)-torsion.

Proof. The statement about dimension follows directly from [7] since \( H_*(BG) \) is concentrated in odd dimensions. The manifolds \( [Z_p, S^{2k-1}] \) and \( [Z_q, S^{2kp-1}] \) give rise, via the Thom map, to generators of \( H_{2k-1}(BZ_p) \) and \( H_{2kp-1}(BZ_q) \) and so give rise to generators of \( H_*(BG) \). Thus the \( G \) extensions of these manifolds yield generators of \( \Omega_*(BG) \) by standard arguments [1, p. 49]. It is well known that if \( l \) is an integer, \( \xi_1, \ldots, \xi_l \) are primitive \( q \)th roots of unity, and if \( Z_q \) acts on \( C^l \) by \( d(x_1, \ldots, x_l) = (\xi_1 x_1, \ldots, \xi_l x_l) \), then the fundamental class of \( S^{2l-1}/Z_q \) is a generator of \( H_{2l-1}(BZ_q) \). If \( C^{kp} = C^k \times \cdots \times C^k \) and \( Z_q \) acts on the \( j \)th factor by \( \xi_j \) then \( S^{2kp-1} \) gives rise to a generator of \( H_{2kp-1}(BZ_q) \). \( V \) has \( 1 \otimes 1, b \otimes 1, \ldots, b^{p-1} \otimes 1 \) as basis and
a(b_i \otimes 1) = \xi^{r_i}(b_i \otimes 1). \) Now we claim that \( G \times Z_q S^{2kp-1} \) and \( Z_p \times S(kV) \) with the diagonal action of \( G \) are \( G \) diffeomorphic. It is enough to take the case \( k = 1 \). Let \( e_j \) be a basis for the \( j \)th factor in \( C \times \cdots \times C \). Send \( [b_i, e_j] \) in \( G \times Z_q C^p \) to \( (b_i, b_i^{1+j} \otimes 1) \) in \( Z_p \times V \). Thus the manifolds \( Z_p \times S(kV) \) generate the \( q \)-torsion in \( \Omega_*(BG) \). Let \( \mu: \Omega_*(BG) \to H_*(BG) \) be the Thom map. Then \( \{\mu(Z_p \times G S(kV))\} \) are generators for the \( q \)-torsion in \( H_*(BG) \) and \( \mu(Z_p \times G S(kV)) = p\mu(S(kV))/G \). Since \( p \) and \( q \) are relatively prime, \( \{\mu(S(kV))/G\} \) are generators for the \( q \)-torsion in \( H_*(BG) \) and so \( S(kV)/G \) give generators for the \( q \)-torsion in \( \Omega_*(BG) \).

In [10] and [2] the groups \( \Omega_*(G; F, F') \), where \( G \) is a finite group and \( F \) and \( F' \) are families of subgroups of \( G \), are studied. If \( K \) is a subgroup of \( G \) we let \( F(K) \) denote the family consisting of all conjugates of subgroups of \( K \) and we let \( F_0(K) \) denote the family consisting of all conjugates of proper subgroups of \( K \). We call \( F \) and \( F' \) strictly adjacent families if there is a subgroup \( K \) of \( G \) such that \( F - F' \) consists of the conjugates of \( K \).

Let \( F \) and \( F' \) be strictly adjacent families of a finite group \( G \) which differ by the conjugates of \( K \). We have well-known isomorphisms

\[
\Omega_*(G; F, F') \cong \Omega_*(G; F(K), F_0(K)) \cong \Omega_*(N(K); F(K), F_0(K))
\]

where \( N(K) \) is the normalizer of \( K \) in \( G \). We enunciate these isomorphisms. Let \( W \) be a stably complex left \( G \)-manifold with boundary. Let \( M \) be the points in \( W \) with isotropy group conjugate to \( K \) and \( N \) a closed \( G \)-invariant tubular neighborhood of \( K \). Then the class of \( W \) in \( \Omega_*(G; F, F') \) is sent to the class of \( N \) in \( \Omega_*(G; F(K), F_0(K)) \). If we let \( \nu \) be the normal bundle of \( M \) in \( W \), \( M_0 \) the points in \( M \) with isotropy group equal to \( K \), \( \nu_0 \) the restriction of \( \nu \) to \( M_0 \), then the disk bundle \( D(\nu_0) \) is an \( N(K) \) manifold with boundary which represents an element in \( \Omega_*(N(K); F(K), F_0(K)) \) as we assign the class of \( D(\nu_0) \) to the class of \( N \).

The analysis of these groups is carried out in [11] for unoriented bordism and in more detail in a special case in [2]. We present a treatment of \( \Omega_*(G; F(K), F_0(K)) \) which we will later explicitly need. Let \( I_x \) denote inner automorphism by \( x \). Let \( \rho \) and \( \rho' \) be \( n \)-dimensional complex representations of \( K \). We will say that \( \rho \) and \( \rho' \) are \( G \)-equivalent if there are elements \( g \) in \( G \) and \( A \) in \( U(n) \) such that \( \rho' = I_A \rho I_g \).

**Theorem (1.5).** Let \( K \) be normal in \( G \). Then

\[
\Omega_*(G; F(K), F_0(K)) \cong \sum \Omega_{m-2n}(B(N(\rho)/\Gamma(\rho)))
\]

where \( n \) runs from 0 to \( [m/2] \), \( \rho \) runs over a set of representitives of \( G \)-equivalent \( n \)-dimensional representations of \( K \), \( \Gamma(\rho) \) is the graph of \( \rho \) in \( G \times U(n) \), and \( N(\rho) \) is the normalizer of \( \Gamma(\rho) \) in \( G \times U(n) \).
Proof. Let $W$ be a manifold with boundary representing an element in $\Omega_m(G; F(K), F_0(K))$, $M$ the set of points with isotropy group equal to $K$, $\nu$ the normal bundle, $M_n$ the union of the $(m-2n)$-dimensional components of $M$, $\nu_n$ the restriction of $\nu$ to $M_n$. Then $W$ and $\Sigma D(\nu_n)$ are bordant and the bordism relation respects this decomposition. Thus $\Omega_m(G; F(K), F_0(K)) \cong \Sigma \Psi$, where $\Psi$ is the bordism group of complex $n$-dimensional left $G$ vector bundles $E$ over compact $(m-2n)$-dimensional left $G$-manifolds $M$ such that every point on the zero section of $E$ has isotropy group equal to $K$ and every point off the zero section has isotropy group properly contained in $K$. Let $P$ be the principal $U(n)$ bundle corresponding to $E$. As in [2], $P$ can be regarded as a right $G \times U(n)$ space such that the isotropy group of a point $e$ in $P$ is $\Gamma(\rho_e)$ for some $n$-dimensional complex representation $\rho_e$ of $K$. $H = G \times U(n)$ acts differentiably on $P$ so there is a neighborhood $V$ of $e$ in $P$ such that for $e'$ in $V$ there is a $(g, \Lambda)$ in $H$ such that $H_{e'} \subset (g^{-1}, A^{-1})H_e(g, \Lambda)$. From this it follows that $\rho_{e'} = l_A^{-1} \rho_e l_x$. Further, if $e' = e\Lambda$ for some $\Lambda$ in $U(n)$, then $\rho_{e'} = l_A^{-1} \rho_e l_{x'}$. Thus for each $x$ in $M$ there is a neighborhood $U$ of $x$ such that, for any two points $e$ and $e'$ in $P \setminus U$, $\rho_e$ and $\rho_{e'}$ are G-equivalent and so $H_e$ and $H_{e'}$ are conjugate in $H$. Thus $\Psi \cong \Sigma \Psi_{n, \rho}$ where $\Psi_{n, \rho}$ is the bordism group of compact right $G \times U(n)$ spaces $P$ such that $U(n)$ acts freely on $P$, $P/U(n) = M$ is a stably complex $G$-manifold, every point in $P$ has isotropy group conjugate, in $H$, to $\Gamma(\rho)$, and $\rho$ runs over a set of representatives of $n$-dimensional representations of $K$ under $G$-equivalence. So $M/G/K$ is a stably complex manifold. Now let $P \to M$ represent an element in $\Psi_{n, \rho}$ and let $Q \subset P$ be the points in $P$ with isotropy group equal to $\Gamma(\rho)$. $Q$ is a principal right $N(\rho)/\Gamma(\rho)$ bundle, $Q/N(\rho)/\Gamma(\rho) = M/G/K$, and $Q$ and $P$ determine each other since $Q \times_{N(\rho)} \Gamma(\rho) = P$. Thus

\[(1.6) \] \[
\Psi_{n, \rho} \cong \Omega_{m-2n}(H(N(\rho)/\Gamma(\rho))).
\]

(1.7) We should make a few remarks about extensions of actions. If $H$ is a subgroup of $G$ and $F$ a family of subgroups of $H$, then we let $F$ denote that family of subgroups of $G$ consisting of all conjugates of elements of $F$. We have a map $E: \Omega_*(H; F) \to \Omega_*(G; F)$ which takes the class of a stably complex $H$-manifold $M$ to the class of $G \times_H M$. Note $G \times_H M$ has a well-defined $G$-invariant stable complex structure. In a similar manner we have $E: \Omega_*(H; F, F') \to \Omega_*(G; F, F')$. Note that if $H_1$ and $H_2$ are conjugate in $G$, then the image of $\Omega_*(H_1; F_0(H_1))$ is equal, in $\Omega_*(G; F)$, to the image of $\Omega_*(H_2; F_0(H_2))$ where $F$ is any family of $G$ containing $F_0(H_1) = F_0(H_2)$.

2. Actions of cyclic groups on polynomial rings. We will develop some algebra which will be relevant to the analysis of $\Omega_*(G; F, F')$. Let $P_1, \ldots, P_t$
be polynomial rings over the integers, \( P^p \) the \( p \)-fold tensor product. \( Z_p \) acts on \( P^p \) by letting the generator \( b \) of \( Z_p \) act by \( b(x_0 \otimes \cdots \otimes x_{p-1}) = x_{p-1} \otimes x_0 \otimes \cdots \otimes x_{p-2} \). Let \( P = P^p \otimes \cdots \otimes P^p \). \( P \) is a polynomial ring and \( Z_p \) acts on \( P \) via the tensor product of the actions on \( P^p \). \( \Omega_* \otimes P \) is a polynomial ring over \( \Omega_* \) on which \( Z_p \) acts. It follows directly from the definition of homology via resolutions that \( H_*(Z_p; \Omega_* \otimes P) \cong \Omega_* \otimes H_*(Z_p; P) \).

**Lemma (2.1).** \( H_i(Z_p; P) = 0 \) for \( i \) even.

**Proof.** We use the resolution of (1.2) for \( Z \) over \( Z[Z_p] \). We must calculate \( \text{ker}(N)/\text{image}(D) \), where \( N \) and \( D \) act on \( P \). Let \( X \) be a monomial in \( P \). Let \( Z[X] \) be the subgroup of \( P \) which has basis consisting of the distinct images of \( X \) under \( Z_p \). \( P = \bigoplus Z[X] \). It is enough to show \( H_i(Z_p; Z[X]) = 0 \). Either \( bX = X \) or \( bX, \ldots, b^{p-1}X \) are distinct. In the first case, the kernel of \( N \) is clearly zero. In the second case

\[
N\left( \sum_{j=0}^{p-1} m_j(b^jX) \right) = \sum_{j=0}^{p-1} m_j N(X) = 0 \quad \text{implies} \quad \sum_{j=0}^{p-1} m_j = 0.
\]

Now for any integer \( k \), \( X - b^kX = \sum_{j=0}^{k-1} (b^jX - b^{j+1}X) \). So \( X \equiv b^kX \mod \text{image}(D) \). Thus \( \sum_{j=0}^{p-1} m_j(b^jX) \equiv \sum_{j=0}^{p-1} m_j X \mod \text{image}(D) \).

**Lemma (2.2).** \( H_0(Z_p; P) \) is a free \( Z \) module.

**Proof.** \( H_0(Z_p; P) = P/\text{image}(D) \). As in (2.1) it is enough to compute \( Z[X]/\text{image}(D) \). If \( bX = X \), \( \text{image}(D) = 0 \) and \( Z[X] \) is a free module with basis \( X \).

In the other case, \( Z[X] \) has, as basis \( X, (1 - b)X, \ldots, (1 - b)b^{p-2}X \). From the formula \( (1 - b)b^{p-1}X = -(1 - b)X - (1 - b)bX - \cdots - (1 - b)b^{p-2}X \) it follows that \( (1 - b)X, \ldots, (1 - b)b^{p-2}X \) is a basis for \( \text{image}(D) \) and so \( Z[X]/\text{image}(D) \) is a free \( Z \) module with basis \( X + \text{image}(D) \).

Now let us consider the map \( \phi_j: P_j \rightarrow P_j^p \)

\[
(2.3) \quad \phi_j(X) = \sum X_0 \otimes \cdots \otimes X_{p-1}
\]

where the sum is taken over all \( X_0 \otimes \cdots \otimes X_{p-1} \) such that \( X_0X_1 \cdots X_{p-1} = X \) in \( P_j \). We then let \( \bar{P} = P_1 \otimes \cdots \otimes P_t \) and let

\[
(2.4) \quad \Phi: \bar{P} \rightarrow P
\]

be the tensor product of the \( \phi_j: P_j \rightarrow P_j^p \). \( \bar{P} \) is a trivial \( Z_p \) module. \( \Phi \) is a map of \( Z_p \) modules.

**Theorem (2.5).** \( \Phi^*: H_{2i-1}(Z_p; \bar{P}) \rightarrow H_{2i-1}(Z_p; P) \) is surjective. Let \( X = X_1 \otimes \cdots \otimes X_t \) be a monomial in \( \bar{P} \) and let \( \bar{P}_1 \) be the subgroup with basis consisting of those \( X \) such that each monomial \( X_i \) is a \( p \)-th power. Let \( \bar{P}_2 \) be
the submodule with basis consisting of the remaining monomials. Then \( P_2 = \ker(\Phi_*) \) and \( \Phi_*: P_1 \cong H_{2i-1}(Z_p; P) \).

**Proof.** Let \( H^{-}(Z_p; P) \) denote \( H_{2i-1}(Z_p; P) \). Consider \( \Phi(X) = \phi(X)_{i-1} \theta \cdots \theta \phi(X_1) \) where \( \phi_j(X) = \sum X_j \theta \cdots \theta X_{j-p+1} \). \( X_j \) is a \( p \)-th power in \( P \); if and only if for one term in this sum \( X_{j0} = \cdots = X_{j-p+1} \). Call this common value \( Y_j \). Thus \( \Phi(X) \) has a term

\[
Y_1 \otimes \cdots \otimes Y_1 \otimes \cdots \otimes Y_t \otimes \cdots \otimes Y_t = Y
\]

if and only if each \( X_j \) is a \( p \)-th power. Furthermore, a monomial in \( P \) is invariant under \( Z_p \) if and only if it is such a \( Y \). Thus \( \Phi(X) = Y + \sum T \) where \( bY = Y \) and \( bT = T \) if each \( X_j \) is a \( p \)-th power, and \( \Phi(X) = \sum T \) where \( bT \neq T \) if some \( X_j \) is not a \( p \)-th power.

Now if \( bY = Y \), then \( D(Y) = 0 \) and \( \ker(D^k) = \ker(Y) \) for \( k \) such that \( bY = Y \). If \( Y \) is a \( p \)-th power, then \( \Phi(X) = Y + \sum T \) where \( bY = Y \) and \( bT = T \) if each \( X_j \) is a \( p \)-th power, and \( \Phi(X) = \sum T \) where \( bT \neq T \) if some \( X_j \) is not a \( p \)-th power.

3. \( \Omega_*(G; F, F') \). Throughout this section \( G = Z_p \times Z_q \), \( \eta = e^{2\pi i/p} \), \( \xi = e^{2\pi i/q} \). Let \( F_0 \) be the family of all subgroups of \( G \), \( F_p \) be \( F_0(Z_p) \), \( F_q \) be \( F_0(Z_q) \), \( F_0 \) be the family consisting of all proper subgroups, and \( F_1 \) the family consisting of the identity subgroup

\[
\begin{align*}
\Omega_*(G; F_0, F_q) &\cong \Omega_*(G; F_p, F_1) \\
\Omega_*(G; F_0, F_p) &\cong \Omega_*(G; F(Z_q), F_0(Z_q)) \quad \text{by (1.4).}
\end{align*}
\]

First we study \( \Omega_*(Z_p; F(Z_p), F_0(Z_p)) \). The first lemma is well known.

**Lemma (3.1).** Let \( \rho \) be an \( n \)-dimensional representation of \( Z_p \). Let \( N(\rho) \) be the normalizer of \( \Gamma(\rho) \) in \( Z_p \times U(n) \). Then

\[
\Phi_*(B(N(\rho)/\Gamma(\rho))) \cong \Phi_*(BU(k_1) \times \cdots \times BU(k_{p-1})).
\]

The isomorphism \( \Omega_*(Z_p; F(Z_p), F_0(Z_p)) \cong \Omega_*(G; F_0, F_q) \) of (1.4) is induced by extension.

**Proof.** Let \( \rho(b) = k_1 \eta \oplus \cdots \oplus k_{p-1} \eta^{p-1} \). Then \( N(\rho) = Z_p \times C(\rho) \), where
C(\rho) is the centralizer of \rho in U(n). By Schur's lemma, \( C(\rho) = U(k_1) \times \cdots \times U(k_{p-1}) \). \( N(\rho)/\Gamma(\rho) \cong U(k_1) \times \cdots \times U(k_{p-1}) \), an explicit isomorphism is obtained by sending \((b^i, A_1, \ldots, A_{p-1})\) to \((\eta^{-1}A_1, \ldots, \eta^{-1(p-1)}A_{p-1})\). Now let \( W \) represent an element of \( \Omega_s(Z_p^r; F(Z_p^r), F_0(Z_p^r)) \). \( G \times Z_p^r \) is the extension. Let \( M \) be the fixed point set of \( Z_p^r \) in \( W \) and \( D \) an invariant tubular neighborhood. Then \( G \times Z_p^r \) is the points with isotropy group conjugate to \( Z_p^r \), \( G \times Z_p^r \) is an invariant tubular neighborhood of \( G \times Z_p^r M \). The points with isotropy group equal to \( Z_p^r \) is \( Z_p^r \times Z_p^r M = M \) and \( D \) restricted to \( M \) is \( D \). So, the last statement follows.

Now we study \( \Omega_s(G; F(Z_q^r), F_0(Z_q^r)) \). Let \( \sigma \) be an \( n \)-dimensional representation of \( Z_q^r \). Suppose \( \sigma(x) = k_1 \xi \oplus \cdots \oplus k_{q-1} \xi^{q-1} \). Then \( \sigma^{(j,s)} \), \( j = 0, 1, \ldots, p-1 \), are all \( G \)-equivalent to \( \sigma \). Let \( C(\sigma) \) be the centralizer of \( \sigma \) in \( U(n) \). Then \( \Gamma(\sigma), 1 \times C(\sigma) \) is the group generated by \( \sigma \).

**Lemma (3.2).** \((b, B) \) is in \( N(\sigma) \) for some \( B \) in \( U(n) \) if and only if \( \sigma l_b = l_B \sigma \), and this holds if and only if \( k_j = k_j^{(s,r)} \) for all \( j \) and all \( s = 0, 1, \ldots, p-1 \).

**Proof.** \((b, B) \) is in \( \Gamma(\sigma) \) if and only if \( \sigma l_b^{-1} = l_B^{-1} \sigma \sigma \). So \( \sigma l_b = l_B \sigma \). Let \( V = C^n(\sigma) \) be a representation space for \( \sigma \), and \( V_j \) the \( \xi_j \) eigen-space for \( \sigma(x) \). Suppose \( \sigma(x) = B^{-1} \sigma(x) \). Then \( x \) is in \( V_j \) if and only if \( \sigma(x) x = \xi_j B^{-1} \sigma(x) \). Thus \( B^{-1} : V_j \cong V_{j^{(s,r)}} \), and this can hold only if \( k_j = k_j^{(s,r)} \). On the other hand, if \( k_j = k_j^{(s,r)} \) for all \( j \), choose a basis for \( V \) by choosing a basis for each \( V_j \) and this basis which takes the basis in \( V_j \) to the basis in \( V_{j^{(s,r)}} \). Then \( b, B \) is in \( N(\sigma) \).

**Lemma (3.3).** If \( \sigma l_b = l_B \sigma \) for some \( B \), then
\[
N(\sigma) = [(b, B), \Gamma(\sigma), 1 \times C(\sigma)].
\]

If not, \( N(\sigma) = [\Gamma(\sigma), 1 \times C(\sigma)] \). \( C(\sigma) \cong U(k_1) \times \cdots \times U(k_{q-1}) \). If
\[
A = (A_1, \ldots, A_{q-1})
\]

is the matrix of an element in \( C(\sigma) \) relative to the basis indicated in (3.2), we can choose \( B \) to be the permutation matrix of (3.2) and then
\[
(b, B)(1, A_1, \ldots, A_{q-1})(b^{-1}, B^{-1}) = (1, A'_1, \ldots, A'_{q-1})
\]
where \( A'_j = A_{j^{(s,r)}} \).

**Proof.** It is clear that the \( B \) of (3.2) gives us an element \( (b, B) \) in \( N(\sigma) \). If \( (b, B') \) is in \( N(\sigma) \), then \( B^{-1} B' \) is in \( C(\sigma) \) and so \( (b, B') = (b, B)(1, B^{-1} B') \).

Let \( Z_q^* \) be the multiplicative group of nonzero elements of \( Z_q^r \) and \( [r] \) the subgroup generated by \( r \). Let \([s,j], j = 1, \ldots, (q-1)/p \), be representatives for \( Z_q^*[r] \). Then
Inner automorphism by \( B \) in (3.3) sends \( \prod_{t=0}^{p-1} U(k_{s,t}) \) to itself and takes \((A_0, \ldots, A_{p-1})\) to \((A_1, \ldots, A_{p-1}, A_0)\). Thus

**Lemma (3.4).** \( \alpha l_b = l_0 \sigma \), then

\[
\frac{N(\sigma)}{\Gamma(\sigma)} \cong \mathbb{Z}_p \times \prod_{j=1}^{q-1/p} \prod_{0}^{p-1} U(l_j)
\]

where \( l \) is the common value of \( k_{s,t} \), and inner automorphism by the generator \( b \) of \( \mathbb{Z}_p \) preserves \( \prod_{0}^{p-1} U(l_j) \) and takes \((A_0, \ldots, A_{p-1})\) to \((A_1, \ldots, A_{p-1}, A_0)\).

Now we will analyze \( \Omega^*(\mathbb{B}(N(\sigma)/\Gamma(\sigma))) \) for such \( \sigma \). Let

(3.5)

\[
U = \prod_{j=1}^{q-1/p} \prod_{0}^{p-1} U(l_j).
\]

\( \Omega^*(\mathbb{B}) \) is a power series ring in Chern classes.

\[
\Omega^*(\mathbb{B}) = \text{Hom}_{\mathbb{Z}}(\Omega^*(\mathbb{B})); \Omega^*).
\]

For each monomial \( C_\alpha \) in the Chern classes, let \( X_\alpha \) be the element in \( \Omega^*(\mathbb{B}) \) such that \( \langle X_\alpha, C_\beta \rangle = \delta^\beta_\alpha \). Then \( \Omega^*(\mathbb{B}) \) is a free \( \Omega^* \) module with basis \( \{X_\alpha\} \).

We can make \( \Omega^*(\mathbb{B}) \) into a polynomial ring by defining \( X_\alpha X_\beta = X_\gamma \) where \( X_\gamma \) is dual to \( C_\alpha C_\beta \). And \( \Omega^*(\mathbb{B}) \cong \Omega^* \otimes P \) where \( P \) is the polynomial ring over the integers with monomials \( X_\alpha \).

Consider the diagonal map \( U(l_j) \rightarrow \prod_{0}^{p-1} U(l_j) \). Let

(3.6)

\[
\overline{U} = \prod_{j=1}^{q-1/p} U(l_j)
\]

and let

(3.7)

\[
\Psi: \overline{U} \rightarrow U
\]

be the product of the diagonal maps. Let \( \Psi \) also denote the induced map \( \overline{\mathbb{B(U)}} \rightarrow \mathbb{B(U)} \). Let \( t = (q - 1)/p \).

**Lemma (3.8).** The map \( \Omega^*(\mathbb{B}(U(l))) \rightarrow \bigotimes_{1}^{p-1} \Omega^*(\mathbb{B}(U(l))) \) induced by the diagonal takes a monomial \( X \) in \( \Omega^*(\mathbb{B}(U(l))) \) to \( \sum X_0 \otimes \cdots \otimes X_{p-1} \) where the sum is taken over all \( X_1 \otimes \cdots \otimes X_{t} \) such that \( X_0 \cdots X_{p-1} = X \) in \( \Omega^*(\mathbb{B}(U(l))) \).

The map \( \Omega^*(\mathbb{B}) \rightarrow \Omega^*(\mathbb{B(U)}) \) is the tensor product of these maps.

**Proof.** Consider the diagonal map \( \Delta: \mathbb{B(U)} \rightarrow \prod_{0}^{p-1} \mathbb{B(U)} \). Let \( X \) be a monomial in \( \Omega^*(\mathbb{B(U)}) \). Then \( \Delta (X) = \sum M(X_0 \otimes \cdots \otimes X_{p-1}) \) where \( X_i \) is in
\( \Omega_*(BU(l)) \) and \( M \) is in \( \Omega_* \). Consider \( C_{a_0} \otimes \cdots \otimes C_{a_{p-1}} \) in \( \Omega^*(\Pi_0^{p-1} BU(l)) \).

\[
\langle \Delta_*(X), C_{a_0} \otimes \cdots \otimes C_{a_{p-1}} \rangle = \langle X, \Delta^*(C_{a_0}) \cdots \Delta^*(C_{a_{p-1}}) \rangle
\]

\[
= \langle X, C_{a_0} \cdots C_{a_{p-1}} \rangle = 1
\]

if \( X \) is dual to \( C_{a_0} \cdots C_{a_{p-1}} \) and zero otherwise. A term \( M(X \otimes \cdots \otimes X_{p-1}, C_{a_0} \otimes \cdots \otimes C_{a_{p-1}}) \) is one only if \( M = 1 \) and \( X \) is dual to \( C_{a_j} \). Then by definition, \( X \otimes \cdots \otimes X_{p-1} \) is dual to \( C_{a_0} \cdots C_{a_{p-1}} \) and so equals \( X \). Now the lemma follows.

**Remark.** Let \( P_i = n^{+(B\Omega)}(P_i, P) = \Omega^*(\Pi_0^{p-1} BU(l)) \), and \( \Omega_* \otimes P_1 \otimes \cdots \otimes P_{p-1} = \Omega^*(BU) \).

And the map \( \Omega_*(BU) \to \Omega_*(BU) \) is precisely the map \( 1 \otimes \Phi \) where \( \Phi \) is the map of (2.4).

The map (3.7) induces a group homomorphism \( \mathbb{Z} \times \mathbb{U} \to \mathbb{Z} \times \mathbb{U} \) and so a map

(3.9)

\[
BZ_p \times B\mathbb{U} \to B(\mathbb{Z}_p \times \mathbb{U}).
\]

**Lemma (3.10).** The maps \( H_-(BZ_p \times B\mathbb{U}) \to H_-(B(\mathbb{Z}_p \times \mathbb{U})) \) and \( H_-(BZ_p \times B\mathbb{U}) \to H_-(B(\mathbb{Z}_p \times \mathbb{U})) \) are surjective.

**Proof.** Let us consider the two Atiyah spectral sequences

\[
E^2_{i,*} = H_i(Z_p; \Omega_*(BU)) \implies \Omega_*(B(\mathbb{Z}_p \times \mathbb{U})),
\]

\[
E^2_{i,*} = H_i(Z_p; \Omega_*(BU)) \implies \Omega_*(B(\mathbb{Z}_p \times \mathbb{U})).
\]

The map (3.9) induces a map of these spectral sequences \( \bar{E}^2_{*,*} \to \bar{E}^2_{*,*} \). In the remark following (3.8) we noted that \( \Omega_*(BU) = \Omega_* \otimes \bar{P}, \Omega_*(BU) = \Omega_* \otimes P \) and the map \( H_i(Z_p; \Omega_*(BU)) = \bar{E}^2_{i,*} \to \bar{E}^2_{i,*} = H_i(Z_p; \Omega_* \otimes P) \) is the map \( 1 \otimes \Phi \). \( \bar{E}_{i,j} = 0 \) if \( i \) is even, \( j \neq 0 \) by (2.1). \( \bar{E}_{i,j} = 0 \) if \( i \) odd, \( j \) odd, since \( \Omega_*(BU) \) is concentrated in even dimensions. \( \bar{E}_{i,j} \) are \( P \)-torsion and \( \bar{E}_{i,j} \) is surjective for \( i \) odd, \( j \) even, by (2.5). \( \bar{E}_{0,j} = \Omega_j(BU) \) is free abelian for \( j \) even, and \( E_{0,i} = H_0(Z_p; \Omega_* \otimes P) \) is free abelian by (2.2). For \( j \) odd \( E_{0,j} = 0 \). Thus no differential could possibly be nonzero and so both spectral sequences collapse. Consider the filtrations \( \bar{F}_{*,*} \) on \( \Omega_*(BZ_p \times B\mathbb{U}) \) and \( \bar{F}_{*,*} \) on \( \Omega_*(B(\mathbb{Z}_p \times \mathbb{U})) \).

\[
\bar{F}_{i,j}/\bar{F}_{i-1,j+1} = F_{i,j}/F_{i-1,j+1} = 0
\]

for \( i + j \) even and \( i \neq 0 \). For \( m \) odd,
\[ \Omega_\ast((BZ_p \times B\tilde{U})) = F_{m,0} \supset \cdots \supset F_{1,m-1} \supset 0, \]
\[ \Omega_\ast(B(Z_p \times_f U)) = F_{m,0} \supset F_{m-2,2} \supset \cdots \supset F_{1,m-1} \supset 0. \]

And
\[ F_{i,j}/F_{i-2,j+2} = H_i(Z_p; \Omega_j(B\tilde{U})), \quad F_{i,j}/F_{i-2,j+2} = H_i(Z_p; \Omega_j(BU)) \]
for \( i + j \) odd. We have maps
\[ 0 \to F_{i-2,j+2} \to F_{i,j} \to H_i(Z_p; \Omega_j(B\tilde{U})) \to 0 \]
\[ \alpha' \downarrow \quad \alpha \downarrow \quad \alpha'' \downarrow \]
\[ 0 \to F_{i-2,j+2} \to F_{i,j} \to H_i(Z_p; \Omega_j(BU)) \to 0 \]
For \( i = 1, j = m - 1 \), \( F_{1,m-1} = H_1(Z_p; \Omega_{m-1}(B\tilde{U})) \to H_1(Z_p; \Omega_{m-1}(BU)) = F_{1,m-1} \) is onto. By induction \( \alpha'' \) is surjective, and \( \alpha'' \) is surjective by computation.
Hence \( \alpha \) is surjective. Thus by induction \( F_{m,0} \to F_{m,0} \) is surjective. The same argument applied to \( H_\ast(BZ_p \times B\tilde{U}) \) and \( H_\ast(B(Z_p \times_f U)) \) gives the corresponding conclusion for homology.

**Lemma (3.11).** \( \Omega_\ast(B\tilde{U}) \to \Omega_\ast(B(Z_p \times_f U)) \) and \( H_\ast(BU) \to H_\ast(B(Z_p \times_f U)) \)
are surjective.

**Proof.** In the spectral sequence \( E_{i,j}^{\ast,*} \) of (3.10), \( E_{0,*}^{2} = H_0(Z_p; \Omega_\ast(B\tilde{U})) \) and \( E_{i,j}^{2} = 0 \) for \( i + j \) even, \( i \neq 0 \). Thus \( \Omega_\ast(B(Z_p \times_f U)) = H_0(Z_p; \Omega_\ast(BU)) \) and \( \Omega_\ast(B\tilde{U}) = H_0(Z_p; \Omega_\ast(BU)) \) is surjective by definition of \( H_0 \). Similarly for homology.

**Theorem (3.12).** \( \Omega_\ast(B(Z_p \times_f U)) \) has projective dimension one over \( \Omega_\ast \).

**Proof.** First consider
\[ \Omega_\ast(B\tilde{U}) \to \Omega_\ast(B(Z_p \times_f U)) \]
\[ \downarrow \bar{\mu} \quad \downarrow \mu \]
\[ H_\ast(B\tilde{U}) \to H_\ast(B(Z_p \times_f U)) \]
where \( \bar{\mu} \) and \( \mu \) are the Thom maps. \( \bar{\mu} \) is surjective since \( \Omega_\ast(B\tilde{U}) \) is a free \( \Omega_\ast \) module, and the two horizontal maps are surjective. Hence \( \mu \) is surjective.
Now consider
\[ \Omega_\ast(BZ_p \times BU) \to \Omega_\ast(B(Z_p \times_f U)) \]
\[ \downarrow \bar{\mu} \quad \downarrow \mu \]
\[ H_\ast(BZ_p \times BU) \to H_\ast(B(Z_p \times_f U)). \]
Now $\Omega_*(BZ_p \times B\tilde{U}) \cong \bigoplus_{i} \Omega_*(BZ_p^i)$ and so has projective dimension one. Then from [4], [7], $\tilde{p}$ is onto, and since the horizontal maps are onto, $\mu$ is onto. Thus $\mu: \Omega_*(B(Z_p \times f_U)) \to H_*(B(Z_p \times f_U))$ is onto and so again by [4], [7], $\Omega_*(B(Z_p \times f_U))$ and $\Omega_*(B(Z_p \times f_U))$ have projective dimension one over $\Omega_*$. 

**Corollary (3.12).** $\Omega_*(G; F(Z_q), F_0(Z_q))$ has projective dimension one over $\Omega_*$. 

**Proof.** From (1.5), $\Omega_*(G; F(Z_q), F_0(Z_q)) \cong \bigoplus \Omega_*(B(N(\sigma)/T(\sigma)))$. From (3.3), $\Omega_*(B(N(\sigma)/T(\sigma))) \cong \Omega_*(B(U) = 0 or \Omega_*(B(Z_p \times f_U))$ which has dimension one over $\Omega_*$. 

As an indication of a technique necessary in the case of the more general metacyclic group we give more information on the module $\Omega_*(B(Z_p \times f_U))$ of (3.12). As in the remark following (3.8), $\Omega_*(B\tilde{U}) = \Omega_* \otimes P$, and as in (2.5) let $P_1$ be the free Z-module with basis consisting of $X_1 \otimes \cdots \otimes X_l$ where each $X_j$ is a $p$th power in $\Omega_*(B(U(1)))$. 

**Theorem (3.13).** $\Omega_*(BZ_p) \otimes \Omega_* \otimes P_1 \cong \Omega_*(B(Z_p \times f_U))$. 

**Proof.** Let $\Lambda = \Omega_* \otimes P_1$. Let $b_*^{(X)}$ be the homology theory $\Omega_*(X) \otimes \Lambda$, $b_*^{(X)}$ the homology theory $\Omega_*(X) \otimes \Omega_*(B\tilde{U})$, and $b_*^{(X)} = \Omega_*(X \times B\tilde{U})$. We have a natural transformation $b_*^{(X)} \to b_*^{(X)}$ induced by $\Lambda \subset \Omega_*(B\tilde{U})$, and $b_*^{(X)} \to b_*^{(X)}$ induced by $[M, f] \otimes [W, g] \to [M \times W, f \times g]$. So we have induced maps on the skeleton filtrations

$$(Fb')_{*,*}(X) \to (Fb^{(X)})_{*,*}(X) \to (Fb^{(X)})_{*,*}(X).$$

The second natural transformation is a natural isomorphism. Now consider the fibration $B\tilde{U} \to X \times B\tilde{U} \to X$. The spectral sequence of this fibration is $E_2(X) = H_*(X; \Omega_*(B\tilde{U})) \Rightarrow \Omega_*(X \times B\tilde{U})$ and the associated filtration on $\Omega_*(X \times B\tilde{U})$ is precisely $(Fb')_{*,*}(X)$. Now let $X = BZ_p$ and consider the map of fibrations of (3.9). 

$$
\begin{array}{ccc}
B\tilde{U} & \longrightarrow & BU \\
\downarrow & & \downarrow \\
BZ_p \times B\tilde{U} & \longrightarrow & B(Z_p \times f_U) \\
\downarrow & & \downarrow \\
BZ_p & \longrightarrow & BZ_p \\
\end{array}
$$

gives rise to a map $E_2(BZ_p) = E_2 \Rightarrow E_2 = H_*(Z_p; \Omega_*(B(U))$ and we know from (3.10) that both spectral sequences collapse. Let
All of these spectral sequences collapse. We have a map of spectral sequences
$E^{i,2} \rightarrow E^{m,2} \rightarrow E^{n,2} = E_2 \rightarrow E_2$ where the first two maps are induced by the natural transformations. Thus we have a map $E^{i,2} \rightarrow E^{2,2}$. Now $E^{i,2} = E^{2,2}$ if $i$ is even and $i \neq 0$, or if $j$ is odd. If $i$ is odd and $j$ is even, this map is an isomorphism (2.5). Thus $E^{i,2}_j = E^{2,2}_j$ unless $i = 0$ and $j$ is even. Let $F^k_*$ be the filtration on $H^i_*(BZ_p)$ and $F^k_*$ on $\Omega_*(B(Z_p \times_f U))$. Then for $m$ odd,

$$0 \rightarrow F_{m-(2k+2)_i} \rightarrow F_{m-2k_i} \rightarrow H_{m-2k_i}(BZ_p; b^*_i) = E^{2,2}_{m-2k_i} \rightarrow 0$$

$$0 \rightarrow F_{m-(2k+2)_i} \rightarrow F_{m-2k_i} \rightarrow H_{m-2k_i}(Z_p; \Omega_*(BU)) = E^{2,2}_{m-2k_i} \rightarrow 0$$

The left most map is an isomorphism by induction, and the right most by computation. Thus, the middle map is an isomorphism and so

$b^*_i(BZ_p) \cong \Omega_*(B(Z_p \times_f U))$.

We need a description of the image of $\Omega_*(B(Z_p \times_f U))$ in $\Omega_*(G; F(Z_q), F_0(Z_q))$. In view of (3.10) we need only describe the image of $\Omega_*(BZ_p \times BU(l_j) \times \cdots \times BU(l_j))$.

**Lemma (3.14).** Let $P_0$ be a left principal $Z_p$ bundle, and $P_j$ a right principal $U(l_j)$ bundle. Let $E_j = P_j \times U(l_j)$. Then $P_0 \times P_1 \times \cdots \times P_t$ in $\Omega_*(BZ_p \times BU)$ goes under the map

$$\Omega_*(BZ_p \times BU) \rightarrow \Omega_*(B(Z_p \times_f U)) \rightarrow \Omega_*(G; F(Z_q), F_0(Z_q))$$

to the $G$ vector bundle $P_0 \times \bigoplus_{i=1}^{p-1} E_i \rightarrow P_0$ where $G$ acts on $P_0$ by $G \rightarrow G/Z_q = Z_p$ acting on $P_0$, and $G$ acts on $\bigoplus_{i=1}^{p-1} E_i$ by $b(x_0, \ldots, x_{p-1}) = (x_1, x_2, \ldots, x_{p-1}, x_0), a(x_0, \ldots, x_{p-1}) = (\xi x_0, \ldots, \xi^{p-1} x_{p-1})$.

**Proof.** Let $P = P_0 \times P_1 \times \cdots \times P_t$. $P$ is a right $Z_p \times U$ bundle. Let $Q = P \times Z_p \times U$. It is easy to see that $Q \cong P_{1} \times U(l_j) \times \cdots \times U(l_j)$, $Z_p$ acts diagonally on the right and on $P_j \times U(l_j)$, $Z_p$ acts by $[e; a_0, \ldots, a_{p-1}]b = [e; a_{p-1}, a_0, \ldots, a_{p-2}]$. Now $N(\sigma) / \Gamma(\sigma) \cong Z_p \times_f U$, the isomorphism takes $(b, B) \Gamma(\sigma) \mapsto b$ and $(1 \times U) \Gamma(\sigma) \mapsto 1 \times U$. Thus $Q$ becomes a principal $N(\sigma) / \Gamma(\sigma)$ bundle. Then $Q \times N(\sigma) / \Gamma(\sigma) \cong Z_p \times_f U$ is a left $G$ vector bundle, and from (1.5) is the element in $\Omega_*(G; F(Z_q), F_0(Z_q))$ coming from $P$. By a series of obvious isomorphisms,

$$Q \times N(\sigma) / \Gamma(\sigma) \cong Q \times U \cong P_{1} \times U(l_j) \times \cdots \times U(l_j)$$
where, following the left action of $G$ along, we find $G$ acting on $P_0 \times \Pi P_j$ by acting diagonally. $G$ acts on $P_0$ by $G \rightarrow G/Z_q = Z_p$ and on $P_j \times U(l_j)$ by acting diagonally. $G$ acts on $P^n$ by $G \rightarrow G/Z = Z$ and on $P$ by $G \rightarrow G/Z = Z$. Finally $P_0 \times U(l_j) \rightarrow \bigoplus G$ with the indicated action.

**Corollary (3.15).** $\Omega_-(G; F(Z_q), F_0(Z_q))$ is generated by the $G$ vector bundles $S^{2k-1} \times \bigoplus_{E_i} E_i \rightarrow S^{2k-1}$ where $G$ acts on $S^{2k-1}$ by $G \rightarrow Z_p$ and $Z_p$ acts by $b(x_1, \ldots, x_k) = (\eta x_1, \ldots, \eta x_k)$ and $G$ acts on $\bigoplus_{E_i}$ as in (3.14).

**Proof.** This follows from the fact that $S^{2k-1}$ with the indicated action of $Z_p$ for $k = 1, 2, \ldots$ are generators for $\Omega_-(BZ_p)$.

**Remark (3.16).** If $G$ is any finite group, $F$ the family of all subgroups, $F_0$ the family of all proper subgroups, then it is well known that $\Omega_*(G; F, F_0)$ is the bordism group of $G$ vector bundles $E$ over trivial $G$ manifolds. $E \cong \bigoplus G \cdot V$ where $V$ runs over the irreducible representation of $G$. So $\Omega_* G; F, F_0)$ $\cong \bigoplus \Omega_* (BU(k_1) \times \cdots \times BU(k_l))$ is a free $\Omega_*$-module.

4. **Proofs of Theorems A and B.** We start with the families $F_p$ and $F_1$ which are adjacent:

$$\Omega_+(G; F_1) = \Omega_+, \quad \Omega_-(G; F_p, F_1) = 0.$$ 

The long exact sequence

$$\Omega_+(G; F_1) \to \Omega_+(G; F') \to \Omega_+(G; F) \to \Omega_+(G; F', F) \to \Omega_+(G; F') \to$$

for any pair of families breaks up as

$$0 \to \Omega_+ \to \Omega_+(G; F_p) \to \Omega_+(G; F_1) \to \Omega_-(G; F_1) \to \Omega_-(G; F_p) \to 0.$$ 

Now we have, from (1.3) and (3.1),

$$\Omega_+(G; F_p, F_1) \to \Omega_-(G; F_1) = \Omega_-(Z_p; F_0) \oplus \text{Image}(\Omega_-(Z_q; F_0))$$

$$\Omega_+(Z_p; F(Z_p), F_0(Z_p)) \to \Omega_-(Z_p; F_0).$$

The map $\Omega_+(Z_p; F(Z_p), F_0(Z_p)) \to \Omega_-(Z_p; F_0)$ is well known to be onto. Thus we get short exact sequences.
\[ 0 \to \Omega_+ \to \Omega_4(G; F_p) \to \Omega_4(G; F_p; F_1) \to \Omega_-(Z_p; F_0) = \Omega_-(BZ_p) \to 0, \]
\[ \text{Image (} \Omega_-(Z_q; F_0)) \cong \Omega_-(G; F_p). \]

Now \( \text{Image (} \Omega_-(Z_q; F_0)) \) being a summand of \( \Omega_-(BG) \) has projective dimension one. Hence so does \( \Omega_-(G; F_p) \). Now \( \Omega_-(BZ_p) \) has dimension one and \( \Omega_4(G; F_p, F_1) \) is free by (3.1), hence \( \text{Image (} \gamma \text{)} \) is projective over \( \Omega_4 \) and hence free [4, 3.2]. Thus \( 0 \to \Omega_+ \to \Omega_4(G; F_p) \to \text{Image (} \gamma \text{)} \to 0 \) and so \( \Omega_4(G; F_p) \) is free.

Next consider the families \( F_q, F_1 \) and apply (4.1). \( \Omega_-(G; F_q, F_1) \) is all torsion by (3.10). Thus
\[ 0 \to \Omega_+ \to \Omega_4(G; F_q) \to \Omega_4(G; F_q, F_1) \to \Omega_-(G; F_q) \to \Omega_-(G; F_q, F_1) \to 0. \]

Consider
\[ \begin{array}{ccc}
\Omega_4(G; F_q, F_1) & \to & \Omega_-(G; F_1) = \text{Image (} \Omega_-(Z_q; F_0) \text{)} + \Omega_-(Z_p; F_0) \\
\uparrow & & \uparrow \\
\Omega_4(Z_q; F_q, F_0) & \to & \Omega_-(Z_q; F_0) \to 0.
\end{array} \]

From (3.1) and (3.11) it follows that the map \( E \) is onto, and we also know, from (3.11), that \( \Omega_-(G; F_q, F_1) \) is free. Thus we get
\[ 0 \to \Omega_+ \to \Omega_4(G; F_q) \to \Omega_4(G; F_q, F_1) \to \text{Image (} \Omega_-(Z_q; F_0) \text{)} \to 0. \]

Now \( \Omega_-(Z_p; F_0) \) has dimension one and, by (3.12), \( \Omega_-(G; F_q, F_1) \) has dimension one and thus \( \Omega_-(G; F_q) \) has dimension one. By the same argument as for \( F_p, F_1 \), \( \text{Image (} \gamma \text{)} \) is free and hence \( \Omega_4(G; F_q) \) is free.

Now consider the families \( F_0, F_p \). By (1.4), \( \Omega_*(G; F_0, F_p) \cong \Omega_*(G; F_q, F_1) \). Thus we get the sequence of (4.1)
\[ 0 \to \Omega_4(G; F_p) \to \Omega_4(G; F_0) \to \Omega_4(G; F_q, F_1) \to \Omega_-(G; F_0) \to \Omega_-(G; F_q, F_1) \to 0. \]

Since we have previously shown \( \Omega_-(G; F_p) \cong \text{Image (} \Omega_-(Z_q; F_0) \text{)} \) we get
\( \Omega_-(G; F_0) \cong \Omega_-(G; F_q, F_1) \) and
\[ 0 \to \Omega_4(G; F_p) \to \Omega_4(G; F_0) \to \Omega_4(G; F_q, F_1) \to \text{Image (} \Omega_4(BZ_q) \text{)} \to 0. \]
From the first isomorphism and (3.12), \( \Omega_-(G; F_0) \) has projective dimension one. The isomorphism takes \( \mathfrak{w} \in \Omega_-(G; F_0) \) to the \( G \) vector bundle \( E \to M \) where \( M \) is the set of points in \( W \) with isotropy group \( \mathbb{Z}_q \), and \( E \) is the normal bundle of \( M \) in \( W \). From the second sequence we conclude, since \( \Omega_+(G; F_q; F_1) \) is free and \( \text{Image}(\Omega_-(B\mathbb{Z}_q)) \) has dimension one, that \( \text{Image}(\gamma) \) is free. Now

\[
0 \to \Omega_+(G; F_p) \to \Omega_+(G; F_0) \to \text{Image}(\gamma) \to 0.
\]

Since \( \Omega_+(G; F_p) \) and \( \text{Image}(\gamma) \) are free, it follows that \( \Omega_+(G; F_0) \) is free.

Finally consider the families \( F_\alpha \) and \( F_0 \) of \( G \):

\[
0 \to \Omega_+(G; F_0) \to \Omega_+(G; F_\alpha) \xrightarrow{\gamma} \Omega_+(G; F_\alpha, F_0) \to \Omega_+(G; F_0) \to \Omega_-(G; F_\alpha) \to 0.
\]

We want to show \( \Omega_+(G; F_\alpha, F_0) \to \Omega_-(G; F_0) \cong \Omega_-(G; F_q; F_1) \) is onto. (3.15) tells us generators \( S^{2k-1} \times \bigoplus E_1 \times \cdots \times \bigoplus E_t \to S^{2k-1} \) for \( \Omega_-(G; F_q; F_1) \). Recall \( \{s_j\} \) were representatives for \( \mathbb{Z}^*/[r] \), \( j = 1, \ldots, t \). Let \( V_j \) be the representation of \( G \) induced from the one dimensional representation \( \mathbb{Z} \to \mathbb{C}^* \) sending \( \mathbb{Z} \) to \( \zeta \). So

\[
V_j = \mathbb{Z}[G] \otimes_{\mathbb{Z}[\mathbb{Z}_q]} \mathbb{C}^* \quad \text{(see [5, p. 333]).}
\]

Let \( G \) act on \( C^k \) by \( G \to G/Z_q = \mathbb{Z}_p \) and \( \mathbb{Z} \) acts by \( b(x_1, \ldots, x_k) = (\eta x_1, \ldots, \eta x_k) \). Then the \( G \) vector bundle \( C^k \times V_1 \otimes E_1 \times \cdots \times V_t \otimes E_t = E \) represents an element in \( \Omega_+(G; F_\alpha, F_0) \). \( S(E) \) represents an element in \( \Omega_-(G; F_0) \). The set of points with isotropy group \( \mathbb{Z}_q \) is precisely \( S(C^k) = S^{2k-1} \times \bigoplus V_1 \otimes E_1 \times \cdots \times V_t \otimes E_t \). To check that this is the generator \( S^{2k-1} \times \bigoplus V_1 \otimes E_1 \times \cdots \times V_t \otimes E_t \), consider \( V_j \otimes E_j \).

We can take as a basis of \( V_j \cdot 1 \otimes 1, b^{-1} \otimes 1, \ldots, b^{-(p-1)} \otimes 1 \) where \( d(b^{-i} \otimes 1) = \xi^j \cdot b^{-i} \otimes 1 \). The general element of \( V_j \otimes E_j = (x_0, \ldots, x_{p-1}) \) is \( 1 \otimes 1 \otimes x_0 + \cdots + 1 \otimes b^{-(p-1)} \otimes x_{p-1} \), \( U(x_0, \ldots, x_{p-1}) = 1 \otimes 1 \otimes x_1 + b^{-1} \otimes 1 \otimes x_2 + \cdots + 1 \otimes b^{-(p-2)} \otimes x_{p-1} + 1 \otimes b^{-(p-1)} \otimes x_0 = (x_1, x_2, \ldots, x_{p-1}) \). Thus \( \Omega_+(G; F_\alpha, F_0) \to \Omega_-(G; F_0) \) is onto and so \( \Omega_-(G; F_\alpha) = 0 \). Our exact sequence becomes

\[
0 \to \Omega_+(G; F_0) \to \Omega_+(G; F_\alpha) \xrightarrow{\gamma} \Omega_+(G; F_\alpha, F_0) \to \Omega_-(G; F_0) \to 0.
\]

We know \( \Omega_-(G; F_0) \) has projective dimension one and so \( \text{Image}(\gamma) \) is free. We get the exact sequence

\[
0 \to \Omega_+(G; F_0) \to \Omega_+(G; F_\alpha) \to \text{Image}(\gamma) \to 0.
\]

\( \Omega_+(G; F_0) \) and \( \text{Image}(\gamma) \) are free, hence \( \Omega_+(G; F_\alpha) \) is free. This concludes the proof of Theorems A and B.
BIBLIOGRAPHY