A PROOF THAT $C^2$ AND $T^2$ ARE DISTINCT MEASURES(1)

BY

LAWRENCE R. ERNST

ABSTRACT. We prove that there exists a nonempty family $X$ of subsets of $R^3$ such that the two-dimensional Carathéodory measure of each member of $X$ is less than its two-dimensional $T$ measure. Every member of $X$ is the Cartesian product of 3 copies of a suitable Cantor type subset of $R$.

1. Introduction. To any positive integers $m, n$ with $m < n$ there correspond several $m$-dimensional measures over $R^n$. These measures are studied extensively in [3]. We consider two of them, the $m$-dimensional Carathéodory measure, denoted by $C^m$, and the $m$-dimensional $T$ measure, denoted by $T^m$. It is known that $C^m(S) \leq T^m(S)$ for all $S \subseteq R^n$ [3, 2.10.34], and $C^m(S) = T^m(S)$ if $m = 1, m = n$, or $S$ is $m$ rectifiable [3, 2.10.35, 3.2.26].

In this paper we prove (Theorem 3.4) that there exists a nonempty family $X$ of subsets of $R^3$ such that $C^2(S) < T^2(S)$ for all $S \in X$. A precise definition of $X$ is given in §2, using the method of [3, 2.10.28], but roughly each member of $X$ is the Cartesian product of 3 copies of a suitable Cantor type subset of $R$.

We obtain Theorem 3.4 directly from Theorems 3.2, 3.3. A key step in the proof of Theorem 3.2 depends in turn on Lemma 3.1.

2. Preliminaries. In general we adopt in this paper the notation and terminology of [3]. Presented in this section are modifications and additional definitions that we use.

For $S \subseteq R^n$ let $S - S = \{x - y: x, y \in S\}$.

For $a, b \in R^n$ define $[a, b]$ to be the closed line segment with endpoints $a, b$.

For $\emptyset \neq S \subseteq R^n$ let

$$c^2(S) = \sup \{d^2[p(S)]: p \in O(n, 2)\},$$

$$t^2(S) = (\pi/4)\sup \{|(a_1 - b_1) \wedge (a_2 - b_2)|: a_1, b_1, a_2, b_2 \in S\}.$$
These are the gauge functions used in defining $C^2$ and $T^2$ respectively [3, 2.10.1, 2.10.3, 2.10.4].

The following series of definitions culminate in the definition of $X$ and $E$. For any sequence $\nu = (\nu_1, \nu_2, \nu_3, \ldots)$ of integers greater than 1 we denote

\[ H_0(\nu) = \{[0, 1]\}, \]

\[ H_k(\nu) = \bigcup \{\Phi(J, \nu_k) : J \in H_{k-1}(\nu)\} \quad \text{for } k \geq 1, \]

where

\[ \Phi(J, \nu_k) = \{\inf J + (i-1)p, \inf J + (i-1)p + q \} : i = 1, \ldots, \nu_k \]

with $p = (1 - \nu_k^{-3/2})(\nu_k - 1)^{-1} \text{diam } J$, $q = \nu_k^{-3/2} \text{diam } J$; then we define

\[ A(\nu) = \bigcap_{k=0}^{\infty} \bigcup H_k(\nu). \]

Finally, we let $X = \{A(\nu) \times A(\nu) \times A(\nu) : \nu \text{ is a bounded sequence}\}$ and $E = A(\nu) \times A(\nu) \times A(\nu)$ be any fixed element of $X$.

3. Principal results. We proceed to prove that $C^2(E) < T^2(E)$.

3.1. Lemma. If $B$ is a compact convex subset of $\mathbb{R}^2$, $Y = \text{Bdry } B$, $d = \text{diam } B$, and there exists a one-dimensional vector subspace $L$ of $\mathbb{R}^2$ and a positive real number $k$ such that $H^1(L \cap Y) \geq kd$, then

\[ \mathcal{Q}^2(B) \leq \mathcal{Q}^2(B)/(1 + 2^{-10k^4}). \]

Proof. We can assume that $\mathcal{Q}^2(B) > 0$, since (1) clearly holds when $\mathcal{Q}^2(B) = 0$.

Choose a one-dimensional vector subspace $V$ of $\mathbb{R}^2$ perpendicular to $L$, and let $B'$ be the subset of $\mathbb{R}^2$ obtained by applying Steiner symmetrization [3, 2.10.30] to $B$ with respect to $V$. Let $Y' = \text{Bdry } B'$. Then by [3, 2.10.30, proof in 2.10.32] $B'$ is a compact convex set, $\mathcal{Q}^2(B') = \mathcal{Q}^2(B)$, $\mathcal{I}^2(B') \leq \mathcal{I}^2(B)$ and $H^1(L \cap Y') \geq kd$.

We now proceed to prove the existence of $G \subset \mathbb{R}^2$ satisfying

\[ \mathcal{Q}^2(B') \leq \mathcal{Q}^2(G)/(1 + 2^{-10k^4}), \]

\[ \mathcal{I}^2(G) \leq \mathcal{I}^2(B'). \]

This will establish the lemma, since (2) and (3) combined with the relations $\mathcal{Q}^2(B) = \mathcal{Q}^2(B')$, $\mathcal{Q}^2(G) \leq \mathcal{I}^2(G)$ [3, 2.10.32], and $\mathcal{I}^2(B') \leq \mathcal{I}^2(B)$ yield (1).

Let $a_1$ be the midpoint of $L \cap Y'$. Choose orthonormal basis vectors $e_1, e_2$ for $\mathbb{R}^2$ so that $\text{Re}_2 = L$, and $(x - a_1) \cdot e_1 \geq 0$ for all $x \in B'$. Let $m_1, m_2 \in L \cap Y'$ be such that $m_2 - a_1 = a_1 - m_1 = kd e_2/4$. Choose $b_1, b_2 \in Y'$ satisfying...
\((a_1 - b_1) \cdot e_2 = (b_2 - a_1) \cdot e_2 = \sup \{(x - a_1) \cdot e_2 : x \in B'\}\)

and \((b_2 - b_1) \cdot e_1 = 0\). Let

\[ z = \text{diam}(B' \cap \{x: (x - b_1) \cdot e_2 = 2^{-6} k^2 d\}). \]

Choose \(m_3 \in \mathbb{R}^2\) with \(a_1 - m_3 = kze_1/4\). Then let

\[ F = B' \cap \{x: (x - b_1) \cdot e_2 \geq 2^{-6} k^2 d \text{ and } (b_2 - x) \cdot e_2 \geq 2^{-6} k^2 d\} \]

and \(G\) be the convex hull of \(F \cup \{m_3\}\).

We now verify (2). Since \(B' \sim G\) is contained in the union of two rectangles with dimensions \(z\) and \(2^{-6} k^2 dz\), \(\mathcal{L}^2(G \sim B') \leq 2^{-5} k^2 dz\), while, since the interior of the convex hull of \(\{m_1, m_2, m_3\}\) is contained in \(G \sim B'\), \(\mathcal{L}^2(G \sim B') \geq 2^{-4} k^2 dz\); hence

\[ \mathcal{L}^2(G) - \mathcal{L}^2(B') \geq 2^{-5} k^2 dz. \]

Choose \(a_2 \in Y\) with \((a_2 - a_1) \cdot e_2 = 0, a_2 \neq a_1\). Then \(B'\) is contained in a rectangle of side lengths \(|a_2 - a_1|\) and \(|b_2 - b_1|\); consequently,

\[ \mathcal{L}^2(B') \leq |a_2 - a_1| \cdot |b_2 - b_1|. \]

Take \(i = 1, 2\). Let \(w_i = [a_i, b_1] \cap \{x: (x - b_1) \cdot e_2 = 2^{-6} k^2 d\}, s = (a_i - b_1) \cdot e_2/(w_i - b_1) \cdot e_2\}. \) We see from our construction that \(|s| \leq 2^{-k^2-2}\), since \(|(a_i - b_1) \cdot e_2| \leq d/2\) and \((w_i - b_1) \cdot e_2 = 2^{-6} k^2 d\). Furthermore, \((a_i - b_1) = s(w_i - b_i), \) and by subtraction \(a_2 - a_1 = s(w_2 - w_1). \) Therefore, \(|a_2 - a_1| \leq 2^{-5} k^{-2}|w_2 - w_1|\). We combine this result with the inequalities (4), \(|w_2 - w_1| \leq z, |b_2 - b_1| \leq d\) and (5) to obtain

\[ \mathcal{L}^2(G) - \mathcal{L}^2(B') \geq 2^{-5} k^2 dz \geq 2^{-10} k^4 |a_2 - a_1| \cdot |b_2 - b_1| \geq 2^{-10} k^4 \mathcal{L}^2(B') \]

and then by addition (2).

To establish (3) we need only show that

\[ t^2(F \cup \{m_3\}) \leq t^2(B'), \]

since \(t^2(F \cup \{m_3\}) = t^2(G)\) by [3, 2.10.3]. We now proceed to prove (6) by the following method:

Let

\[ Q = [(F \cup \{m_3\}) \setminus (F \cup \{m_3\})] \times [(F \cup \{m_3\}) \setminus (F \cup \{m_3\})]. \]

To each ordered pair \((v_1, v_2) \in Q, v_1 = p_1 e_1 + p_2 e_2, v_2 = q_1 e_1 + q_2 e_2\), we associate \((v_1^*, v_2^*) \in Q\) by means of a map \(f\) such that \((v_1^*, v_2^*) = (v_1, v_2)\) satisfies the three conditions, \(v_1^* \in F - F, v_2 \in F - F\) implies \(v_2^* \in F - F,\) and \(|v_1^* \wedge v_2^*| \geq |v_1 \wedge v_2|\). The existence of such a map \(f\) will prove (6), since \((v_2^*, v_1^*) = \ldots\)
$f(v^*_2, v^*_1)$ will then satisfy $v^* \in F - F \subset B' - B'$ and $|v^*_2 \wedge v^*_1| \geq |v^*_1 \wedge v^*_2|$. To define $f$ and show the required conditions are satisfied we will consider the following cases and subcases:

**Case I.** $v_1 \in F - F$.

Let $v^*_1 = v_1$ and $v^*_2 = v_2$.  

**Case II.** $v_1 \notin F - F$ and $p_1 \geq 0$.

We note that $v_1 = x - m_3$ for a unique $x$ in $F$. Let $r = z/d$, $u = kd/4$. We then consider four subcases:

**Case II.A.** $|q_1| > r|q_2|$ and $p_1 q_2 - p_2 q_1 \geq 0$.

Let $v^*_1 = x - m_2$, $v^*_2 = v_2$. Then, since $x - m_2 = (x - m_3) + (m_3 - m_2) = (p_1 - ru)e_1 + (p_2 - u)e_2$, we deduce that

$$|v^*_1 \wedge v^*_2| = |p_1 q_2 - p_2 q_1 + u q_1 - ru q_2| \geq |p_1 q_2 - p_2 q_1| = |v^*_1 \wedge v^*_2|.$$  

**Case II.B.** $|q_1| \geq r|q_2|$ and $p_1 q_2 - p_2 q_1 < 0$.

Let $v^*_1 = x - m_1 = (x - m_3) + (m_3 - m_1)$, $v^*_2 = v_2$, and proceed as in Case II.A.

**Case II.C.** $|q_1| \leq r|q_2|$ and $p_2 > 0$.

Let $v^*_1 = x - m_1 = (x - m_3) + (m_3 - m_1)$, $v^*_2 = v_2$, and proceed as in Case II.C.

**Case II.D.** $|q_1| < r|q_2|$ and $p_2 < 0$.

Let $v^*_1 = x - m_2$, $v^*_2 = (r_2 q_1 - q_2) e_2$, and proceed as in Case II.C.

**Case III.** $v_1 \notin F - F$ and $p_1 \leq 0$.

Let $f(v, v^*_1) = f(-v, v^*_1)$. Thus the existence of the required map $f$ has been shown, (6) has been established, and the proof of the lemma is complete.

### 3.2. Theorem

There exists $s < 1$ such that if $K$ is a closed subset of $E$ and $M$ is the convex hull of $K$, then $c^2(M) \leq s t^2(M)$.

**Proof.** Choose any $\theta \in \mathbf{0}^*(3, 2)$. Let $\theta(M) = B$, $d = \text{diam } B$. We can assume by excluding the trivial case when $B$ is a single point that $d > 0$. Denote by $Y$ the boundary of $B$ in $\theta(R^3)$. Let $S = M \cap \theta^{-1}(Y)$. Note that $S$ is a continuum.

The first main step in our proof will be to show that there exists a closed line segment $[a, b] \subset S$ such that $|a - b| = rd$, where $r$ is a number depending only on the sequence $\nu$. (The construction involved in establishing this is basically a generalization of a procedure in [4].) Choose $\alpha, \beta \in S$ with $|\alpha - \beta| \geq d$. Let $i$ be such that $|\alpha_i - \beta_i| \geq |\alpha_j - \beta_j|$ for $j = 1, 2, 3$, where $\alpha_j$ is the $j$th coordinate of $\alpha$. Then $|\alpha_i - \beta_i| \geq d/3^{1/2} > d/2$. 

Let \( Q(k) = \bigcap_{j \in [1, k]} \nu_j^{-3/2} \). Then \( H_k(\nu) \) is a disjointed family consisting of \( Q(k)^{-2/3} \) closed intervals of length \( Q(k) \), and \([0, 1] \sim \bigcup H_k(\nu) \) is the union of \( Q(k)^{-2/3} - 1 \) open intervals of length \([1 - \nu_j^{-1/2}]/(\nu_j - 1) \) for \( j \) ranging from 1 to \( k \). Furthermore, since \( 1 - \nu_j^{-1/2} \geq 1 - 2^{-1/2} > 1/4 \), \( \nu_j - 1 < \nu_j^{3/2} \), it follows that

\[
[(1 - \nu_j^{-1/2})/(\nu_j - 1)]Q(j - 1) > \nu_j^{-3/2}Q(j - 1)/4 = Q(j)/4 \geq Q(k)/4.
\]

Therefore, if \( J \subset [0, 1] \) is a closed interval such that \( \text{diam} J > 3Q(k)/2 \), then there exists an open interval \( U \subset J \) with

\[
U \subset [0, 1] \sim \bigcup H_k(\nu) \subset [0, 1] \sim A(\nu)
\]

and \( \text{diam} U > Q(k)/4 \). Consequently, if we choose \( k \) satisfying \( d > 3Q(k) \) and \( d \leq 3Q(k - 1) \), then, since \( |\alpha_i - \beta_i| > d/2 \), there exists an open interval \( I \subset [\alpha_i, \beta_i] \) such that \( I \subset [0, 1] \sim A(\nu) \) and

\[
\text{diam} I > Q(k)/4 = \nu_k^{-3/2}Q(k - 1)/4 \geq \nu_k^{-3/2}d/12 > \xi^{-3/2}d/12,
\]

where \( \xi \) is the least upper bound of the sequence \( \nu \). Let \( r = \xi^{-3/2}/24 \) and let \( G \) be the set of all \( x \) for which \( x_i \) is the midpoint of \( I \). Observe that \( S \cap G \neq \emptyset \), since \( \alpha, \beta \) are on opposite sides of \( G \), and \( S \) is connected. Choose \( a \in S \cap G \). Then distance \( (a, K) > rd \), since \( K \cap \{x: x_i \in I\} = \emptyset \). Let \( N \) be a supporting line of \( \theta(a) \) and \( D = \theta^{-1}(N) \). Then \( D \) is a supporting plane of \( M \) at \( a \). Since \( M \) is the convex hull of \( K \), \( D \cap S \) is convex and distance \( (a, K) > rd \), it follows that there exists \( b \in D \cap S \) with \( |a - b| = rd \).

At this point we will divide the proof into cases and subcases in each of which it will be shown that there exists a number less than 1, depending only on \( \nu \), which multiplied by \( t^2(M) \) is greater than or equal to \( \mathcal{L}^2(B) \). We will then let \( s \) be the largest of these numbers among all cases.

We first divide the remainder of the proof into two cases:

**Case I.** \( |\theta(a) - \theta(b)| > 2^{-9}r^3d \).

We use Lemma 3.1 with \( k, L \cap Y \) replaced by \( 2^{-9}r^3, [\theta(a), \theta(b)] \) to obtain that \( \mathcal{L}^2(B) \leq t^2(B)/(1 + 2^{-46}r_{12}) \). Furthermore, we note that \( t^2(B) \leq t^2(M) \), since \( \|\Lambda_2 \theta\| = 1 \). We then conclude that

(7) Case I implies \( \mathcal{L}^2(B) \leq t^2(M)/(1 + 2^{-46}r_{12}) \).

**Case II.** \( \|\theta(a) - \theta(b)\| \leq 2^{-9}r^3d \).

Let \( \lambda = b - a \). Choose orthonormal basis vectors \( e_1, e_2, e_3 \) for \( \mathbb{R}^3 \) so that kernel \( (\theta) = Re_3 \) and \( \lambda \cdot e_1 = 0 \). As a result of this choice and the fact that \( r < 1 \) it follows that \( \lambda = p_2e_2 + p_3e_3 \) with \( p_2, p_3 \) satisfying \( |p_2| \leq 2^{-9}r^3d \), \( |p_3| > (1 - 2^{-18})1/2rd > rd/2 \), \( |p_2/p_3| < 2^{-8} \). Let \( m \) be the midpoint of \( [\theta(a), \theta(b)] \). Choose \( w \in S - S \), \( w = q_1e_1 + q_2e_2 + q_3e_3 \), satisfying \( |\theta(w)| = d \).
We consider now four subcases of Case II:

Case II.A. \( \|q_3\| > d \).

Choose \( z \in S - S \) satisfying \( \theta(z) \cdot \theta(w) = 0 \) and

\[
|\theta(z)| = \sup \{|v| : v \in Y - Y \text{ and } v \cdot \theta(w) = 0\}.
\]

Using [5, 1.15(7)] we obtain that

\[
4t^2(M)/\pi \geq |w \wedge z| \geq \left| \frac{|w|}{|\theta(w)|} \right| |\theta(w) \wedge \theta(z)|
\]

\[
> 2^{1/2}|\theta(w) \wedge \theta(z)| = 2^{1/2}|\theta(w)||\theta(z)| \geq 2^{1/2}Q^2(B),
\]

hence

\[
(8) \quad \text{Case II.A implies } Q^2(B) \leq 4t^2(M)/(2^{1/2}\pi).
\]

Case II.B. \( \|q_3\| \leq d \) and \( Q^2(B) < rd^2/4 \).

We deduce that

\[
4t^2(M)/\pi \geq |\wedge w| = \left( (p_2q_1)^2 + (p_3q_2)^2 + (p_2q_3 - p_3q_2)^2 \right)^{1/2}
\]

\[
\geq \left( (p_3q_1)^2 + (p_3q_2)^2 - 2p_2p_3q_2q_3 \right)^{1/2} = |p_3|^2 + q_2^2 - 2p_2/p_3q_2q_3 \right)^{1/2}
\]

\[
> \frac{rd}{2} - 2^{-7}d^2)^{1/2} > 3rd^2/8 > 3Q^2(B)/2,
\]

where the fifth relation in this chain follows from the conditions \( |p_3| > rd/2, \)

\( q_1^2 + q_2^2 = d^2, \) \( \|p_2/p_3\| < 2^{-8}, \) \( |q_2| \leq d, \) \( |q_3| \leq d. \) Therefore,

\[
(9) \quad \text{Case II.B implies } Q^2(B) \leq 8t^2(M)/(3\pi).
\]

Case II.C. \( Q^2(B) \geq rd^2/4, \) and \( |(x - m) \wedge v| \leq 4(1 - 2^{-8}r^2)t^2(B)/\pi \) for all \( x \in B, \) \( v \in B - B. \)

Let \( \rho = 2^{-12}r^3d, \) \( W = B \cup B(m, \rho). \) We take any \( u_1, u_2 \in W - W \) and consider two possibilities:

First, if \( u_1, u_2 \in B - B, \) then clearly \( (\pi/4)|u_1 \wedge u_2| \leq t^2(B). \)

On the other hand, suppose at least one of \( u_1, u_2, \) say \( u_1, \) for the sake of argument, is not in \( B - B. \) Then \( u_1 = u_3 + u_4, u_2 = u_5 + u_6, \) where \( u_3 = x - m \)

for some \( x \in B, \) \( |u_4| \leq \rho, u_5 \in B - B, \) \( |u_6| \leq 2\rho. \) We also note that \( r < 1, \)

\( |u_3 \wedge u_5| \leq 4(1 - 2^{-8}r^2)t^2(B)/\pi, \) \( rd^2/4 \leq Q^2(B) \leq t^2(B) \) by [3, 2.10.32], and then obtain

\[
|u_1 \wedge u_2| \leq |u_3 \wedge u_5| + |u_3| \cdot |u_6| + |u_4| \cdot |u_5| + |u_4| \cdot |u_6|
\]

\[
\leq |u_3 \wedge u_5| + 3\rho d + 2\rho^2 < |u_3 \wedge u_5| + 4\rho d < 4t^2(B)/\pi.
\]

Consequently, \( t^2(W) \leq t^2(B). \) Furthermore,

\[
Q^2(W) \geq Q^2(B) + \pi \rho^2/2 \geq (1 + 2^{-25}r^6)Q^2(B),
\]

since \( Q^2(B) \leq rd^2 \) by [3, 2.10.33]. In addition, \( Q^2(W) \leq t^2(W), t^2(B) \leq t^2(M). \)

We combine all these inequalities and conclude that
Case II.D. \( \mathcal{L}^2(B) \geq rd/4 \), and there exists \( \eta \in Y \), \( v_1, v_2 \in Y - Y \) with \( v_1 = y - m \), such that \( |v_1 \wedge v_2| > 4(1 - 2^{-8r^2})t^2(B)/\pi \).

Take any \( r \in \mathcal{S} \cap \theta^{-1}\{\eta\} \). Then choose \( \xi \in \{a - r, b - r\}, \xi = k_1e_1 + k_2e_2 + k_3e_3 \), satisfying \( |k_3| \geq rd/4 \), and \( \eta \in \mathcal{S} - \mathcal{S} \) satisfying \( \theta(\eta) = v_2 \). Let \( v_3 = \theta(\xi) \).

We observe that \( |v_1 - v_3| = |p_2|/2 \leq 2^{-10r^2}d \), \( t^2(B) \geq rd^2/4 \), and then deduce

\[
|v_1 \wedge v_2| \geq |v_1 \wedge v_2| - |v_1 \wedge v_3| \cdot |v_2|
\]

\[
> 4(1 - 2^{-8r^2})t^2(B)/\pi - 2^{-10r^2}d^2 > 4(1 - 2^{-7r^2})t^2(B)/\pi.
\]

Furthermore, \( |\xi'/v_3| > 1 + 2^{-6}r^2 \), since \( |k_3| \geq rd/4 \). These last two results combined with \[5, 1.15(7)] and the inequalities \( r < 1 \), \( \mathcal{L}^2(B) \leq t^2(B) \) yield

\[
t^2(M) \geq (\pi/4)|\xi' \wedge \eta|
\]

\[
\geq (\pi/4)(|\xi'/v_3|)|v_3 \wedge v_2| > (1 + 2^{-8r^2})\mathcal{L}^2(B).
\]

Therefore,

\[
(11) \quad \text{Case II.D implies } \mathcal{L}^2(B) \leq t^2(M)/(1 + 2^{-8r^2}).
\]

We now finish the proof of Theorem 3.2 by letting \( s = 1/(1 + 2^{-46r^2}) \) and then using (7), (8), (9), (10), (11) to conclude that \( c^2(M) \leq s t^2(M) \).

3.3. Theorem. \( 0 < \mathcal{F}^2(E) < \infty \).

Proof. From [3, 2.10.28] we see that \( \mathcal{H}^{2/3}[A(\nu)] = \alpha(2/3)^2 \), consequently, repeated application of [3, 2.10.27] yields

\[
\mathcal{H}^{2}(E) \geq \alpha(2)[\alpha(4/3)\alpha(2/3)]^{-1}\mathcal{H}^{2/3}[A(\nu) \times A(\nu) \times A(\nu)]
\]

\[
\geq \alpha(2)\alpha(2/3)^{-3}\mathcal{H}^{2/3}[A(\nu)] \times \mathcal{H}^{2/3}[A(\nu)] \times \mathcal{H}^{2/3}[A(\nu)] = \pi/4.
\]

Furthermore, \( \mathcal{F}^2(E) \geq \mathcal{H}^2(E)/6 \) by [3, 2.10.39, 2.10.6]. Therefore, \( \mathcal{F}^2(E) \geq \pi/24 \).

Let \( P(j) = \Pi_{i=1}^{j} v_i^3 \). Given any \( \delta > 0 \) choose \( k \) so that \( 3^{1/2}P(k)^{-1/2} < \delta \).

\[H_k(\nu) \times H_k(\nu) \times H_k(\nu) \text{ covers } E \text{ and consists of } P(k) \text{ cubes } D_j \text{ of diameter } 3^{1/2}P(k)^{-1/2}.\]

Therefore,

\[
\sum_{j=1}^{P(k)} t^2(D_j) \leq (\pi/4) \sum_{j=1}^{P(k)} (\text{diam } D_j)^2 = (\pi/4) \sum_{j=1}^{P(k)} 3P(k)^{-1} = 3\pi/4
\]

and hence \( \mathcal{F}^2(E) \leq 3\pi/4. \)

3.4. Main Theorem. \( \mathcal{C}^2(S) < \mathcal{F}^2(S) \text{ for all } S \text{ in } X. \)

Proof. This follows directly from Theorems 3.2, 3.3 and the definitions of \( \mathcal{C}^2 \) and \( \mathcal{F}^2 \).
REFERENCES


