TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 173, November 1972

ITERATED FINE LIMITS AND ITERATED NONTANGENTIAL LIMITS

BY

KOHUR GOWRISANKARAN[1]

ABSTRACT. Let Ω_k, $k = 1$ to n, be harmonic spaces of Brelot and $u_k > 0$ harmonic functions on Ω_k. For each w in a class of multiply superharmonic functions it is shown that the iterated fine limits of $[w/u_1 \cdots u_n]$ exist up to a set of measure zero for the product of the canonical measures corresponding to u_k and are independent of the order of iteration. This class contains all positive multiply harmonic functions on the product of Ω_k's. For a holomorphic function f in the Nevanlinna class of the polydisc U^n, it is shown that the nth iterated fine limits exist and equal almost everywhere on T^n the nth iterated nontangential limits of f, for any fixed order of iteration. It is then deduced that, with the exception of a set of measure zero on T^n, the absolute values of the different iterated limits of f are equal. It is also shown that the nth iterated nontangential limits are equal almost everywhere on T^n for any f in $N_1(U^n)$.

Let f be a holomorphic function belonging to the Nevanlinna class $N(U^n)$ of the polydisc U^n, i.e.

$$\int \log^+ |f(r_1 e^{i\theta_1}, \cdots, r_n e^{i\theta_n})| \, d\theta_1 \cdots d\theta_n$$

is bounded for $0 \le r_j < 1$. A well-known result states that, for such a function f, the iterated nontangential limits exist except for a set E contained in T^n ($T = \{z: |z| = 1\}$) such that the repeated integral of the characteristic function of E relative to $d\theta_1 \cdots d\theta_n$ (in the order reverse to that of taking limits) is zero. Further, if $f \in N_{n-1}(U^n)$, i.e.

$$\int \log^+ |f| \{\log^+ \log^+ |f|\}^{n-1} \, d\theta_1 \cdots d\theta_n$$

is bounded for $0 \le r_j < 1$, then the iterated nontangential limits of f are equal almost everywhere on T^n. It has been an open problem to decide whether the iterated limits of any $f \in N(U^n)$, for different orders of iteration, are equal up to a set of measure zero on T^n [4]. In this paper, we shall show that the answer is in the affirmative to one part of the problem; viz., for any $f \in N(U^n)$, the absolute values of the iterated nontangential limits are equal on T^n except for a set of measure zero (Theorem 11). We shall also show that the iterated nontangential limits are

Presented to the Society, October 30, 1971; received by the editors March 8, 1971.

AMS (MOS) subject classifications (1969). Primary 3115, 3120, 3150, 3225; Secondary 6062, 2820.

Key words and phrases. Polydisc, Nevanlinna class, holomorphic function, nontangential limit, fine limit, minimal boundary, multiply superharmonic functions, Radon measures.

[1]Part of this work was done when the author was a fellow at the Kingston Branch of the Summer Research Institute 1970, of the Canadian Mathematical Congress.

identical (almost everywhere) for all functions in a class wider than $N_{n-1}(U^n)$ (Theorem 12).

In §5, we prove that the iterated fine limits of $f \in N(U^n)$ exist as a measurable function on T^n and equal almost everywhere the iterated nontangential limits of f, for each order of iteration. We then deduce the equality of the absolute values of the various iterated fine limits as a particular case of the results of §§4 and 3, which are of independent interest in the theory of harmonic functions.

We consider harmonic spaces $\Omega_1, \cdots, \Omega_n$ of Brelot and functions w that are n-superharmonic on $\Omega_1 \times \cdots \times \Omega_n$. Let $u_k > 0$ be a harmonic function on Ω_k with 'canonical measure' μ_k charging some 'minimal boundary' Δ_1^k, for $k = 1$ to n. In §3, we show that for a n-harmonic function $w > 0$ on the product space, $w/u_1 u_2 \cdots u_n$ has iterated fine limits for $(\mu_1 \times \cdots \times \mu_n)$-almost every element of $\Delta_1^1 \times \Delta_1^2 \times \cdots \times \Delta_1^n$ and these limit functions are equal almost everywhere to the Radon-Nikodym derivative of the 'canonical measure' of w relative to $\mu_1 \times \cdots \times \mu_n$. This proves the uniqueness of the iterated limits. In §4, we consider the case of n-superharmonic functions. If a n-superharmonic function $v > 0$ has the property that the iterated fine limits exist in such a way that the kth iterated limit $(1 \le k \le n-1)$ of $(v/u_1 u_2 \cdots u_k)$ is $(n-k)$-superharmonic on the product $\Omega_{k+1} \times \cdots \times \Omega_n$, then the iterated fine limits are equal $(\mu_1 \times \mu_2 \times \cdots \times \mu_n)$-almost everywhere (once again by showing that these are identical to the Radon-Nikodym derivative of a fixed measure relative to $\mu_1 \times \cdots \times \mu_n$). In §5, we apply these results to $(-\log |f|)$ to derive the results corresponding to f.

In §2, we have a number of results on measurability of certain functions. The fine filters in general do not have countable bases; we have developed methods to prove the measurability of functions of the form $g(b, y)$, where, for every b in a minimal boundary, g is the fine limit (or lim sup, etc.) of $f(x, y)$ as x tends to b, even when f is with values in certain function spaces. These results are fundamental in our proofs.

1. **Preliminaries.** Let Ω be a locally compact (noncompact) locally connected, Hausdorff topological space with a countable base for its open sets. We shall say that Ω is a (Brelot) harmonic space if there is a system of harmonic functions defined on the open subsets of Ω satisfying the Axioms 1, 2 and 3 of Brelot [1, Part II]. All the harmonic spaces are assumed to have potentials > 0 existing on them (to avoid the trivial case). In the case of U, the unit disc, we shall consider the classical harmonic functions (viz., satisfying the Laplace equation). An open connected set δ of Ω is called regular if, for every continuous real valued function f on $\partial\delta$, there is a continuous extension of f into δ as a harmonic function H_f such that, for $f \ge 0$, $H_f \ge 0$. The harmonic measure at $x \in \delta$ for such a domain is the Radon measure on $\partial\delta$ defined by $f \mapsto H_f(x)$.

Let $\Omega_1, \cdots, \Omega_n$ be n harmonic spaces and δ an open subset of the product space. A real valued function (resp. extended real valued and $\not\equiv +\infty$) is said to be n-harmonic (resp. n-superharmonic) on δ if, for every fixed value of any $(n-1)$-variables, f is harmonic (resp. superharmonic or $+\infty$) and f is continuous (resp. lower semicontinuous) on δ.

Let $H^+(\Omega)$ (resp. $nH^+(\Omega_1 \times \cdots \times \Omega_n)$) be the cone of positive (> 0) harmonic (resp. n-harmonic) functions on Ω (resp. the product). These cones are locally compact, separable and metrisable for the topology of local uniform convergence and consequently have compact bases. Let Δ_1 be the extreme (or minimal) elements belonging to such a compact base. Then, corresponding to each $u \in H^+(\Omega)$, there is a unique Radon measure μ on this compact base, charging Δ_1, such that, for every $x \in \Omega$, $u(x) = \int h(x)\, \mu(dh)$ [2]. This measure, corresponding to u, is referred to as the canonical measure corresponding to u on Δ_1. It is possible to choose a corresponding compact base for $nH^+(\Omega_1 \times \cdots \times \Omega_n)$ such that the extreme elements of this base are precisely of the form $h^1 \cdots h^n$, $h^j \in \Delta_1^j$. And to any $w \in nH^+$, corresponds a unique 'canonical' Radon measure ν_w, carried by $\Delta_1^1 \times \cdots \times \Delta_1^n$ such that $w = \int h^1 \cdots h^n \nu_w(dh^1 \cdots dh^n)$ [8].

For any nonnegative valued function f defined on a set $E \subset \Omega$, the reduced function $R[E, f]$ is by definition $\inf\{v\colon v \geq f$ on E and v superharmonic or $+\infty$ on $\Omega\}$. (We use this rather than the standard R_f^E for the sake of notational simplicity.)

Let $h \in \Delta_1$. Then, $\mathcal{F}_h = \{E\colon R[\complement E, h] \not\equiv h\}$ is the fine filter on Ω corresponding to h. For any function f, the limit (lim sup, etc.) of the image filter $f(\mathcal{F}_h)$ is called the fine limit (fine lim sup, etc.) of $f(x)$ as x tends to h. For any $u \in H^+(\Omega)$, with the canonical measure μ on Δ_1, and any superharmonic function $v \geq 0$, the fine limit of $v(x)/u(x)$ as x tends to h exists for μ-almost every h in Δ_1 ([6], [7]).

Let X be a Hausdorff topological space. A Radon measure on X is by definition a measure $\mu \geq 0$ defined on the Borel σ-algebra of X such that (1) μ is locally finite and (2) for any Borel set $B \subset X$, $\mu(B)$ is the supremum of $\mu(K)$ for compact sets $K \subset B$ [11].

A function $f\colon X \mapsto Y$, Y any Hausdorff space, is said to be Borel measurable, μ-Borel measurable or μ-Lusin measurable according as $f^{-1}(B)$ is Borel in X for every Borel set $B \subset Y$, $f^{-1}(B)$ is μ-measurable for every Borel set $B \subset Y$ or, for any compact set $K \subset X$ and any $\epsilon > 0$, $\exists$ a compact subset C such that $\mu(C) > \mu(K) - \epsilon$ and f restricted to C is continuous. In general, μ-Lusin measurability of f implies the μ-Borel measurability, and the converse is true if Y is a separable metrisable space.

In the sequel every topological space is assumed to be Hausdorff and with a countable basis for neighborhoods at each point.

2. **Some measurability theorems.**

Lemma 1. *Let Ω be a harmonic space and Y a topological space. Let ψ be an extended real valued function $> -\infty$ on $\Omega \times Y$ such that it is lower semicontinuous in each variable, for every fixed value of the other. Then, for every x in Ω, and every $a \in \mathbf{R}$, the function $(h, y) \mapsto R[V_a(y), h](x)$ is lower semicontinuous on $\Delta_1 \times Y$, where $V_a(y) = \{\xi \in \Omega : \psi(\xi, y) > a\}$.*

Proof. Let $(h_n, y_n) \in \Delta_1 \times Y$ converge to $(h_0, y_0) \in \Delta_1 \times Y$. The set $V_a(y_n)$ is open in Ω for every n and $v_n = R[V_a(y_n), h_n]$ is superharmonic > 0 on Ω and equals h_n on $V_a(y_n)$. Let $v = \liminf v_n$ as n tends to ∞. Then, w, the lower semicontinuous regularisation of v, is > 0 and superharmonic on Ω.

Let $x \in V_a(y_0)$. Since $y \mapsto \psi(x, y)$ is lower semicontinuous on Y and $\psi(x, y_0) > a$, there is an integer $N(x)$ such that, for all $n \geq N(x)$, we have $\psi(x, y_n) > a$. Hence, $x \in V_a(y_n)$ for all $n \geq N(x)$. It follows that

$$V_a(y_0) \subset \bigcup_1^\infty \bigcap_n^\infty V_a(y_n).$$

Now, since h_n converges to h_0 (in particular pointwise), for all $x \in V_a(y_0)$,

$$v(x) = \liminf_n v_n(x) = \liminf_n h_n(x) = h_0(x).$$

Now, $V_a(y_0)$ is open and h_0 is continuous and we deduce that $w = h_0$ on $V_a(y_0)$. Hence, $w \geq R[V_a(y_0), h_0]$ on Ω. It follows that

$$\liminf R[V_a(y_n), h_n](x) \geq v(x) \geq w(x) \geq R[V_a(y_0), h_0](x),$$

for every x in Ω, proving the lemma.

Theorem 1. *Let Ω, Y and ψ be as in the previous lemma. Then, the function ψ_1, defined by $\psi_1(h, y) =$ fine lim sup $\psi(x, y)$ as $x \to h$, is Borel measurable on $\Delta_1 \times Y$.*

Proof. It is enough to show that, for every real number b, the set $A(b) = \{(h, y) : \psi_1(h, y) \leq b\}$ is Borel in $\Delta_1 \times Y$. Let, for every n,

$$E_n = \{(h, y) : R[V_{b+1/n}(y), h] \neq h\}.$$

Then, we assert that $A(b) = \bigcap_1^\infty E_n$. For, suppose $(h, y) \in A(b)$. Then since $\psi_1(h, y) \leq b < b + 1/n$, there is a set F belonging to $\mathfrak{F}_h$ such that, for every $x \in F$, $\psi(x, y) \leq b + 1/n$. Hence, $F \cap V_{b+1/n} = \emptyset$ and $h \neq R[V_{b+1/n}(y), h]$, i.e. $(h, y) \in E_n$. Conversely, if (h, y) belongs to E_n, for every n, then fine lim sup $\psi(x, y) \leq b + 1/n$ as x tends to h. We deduce that (h, y) belongs to $A(b)$.

To complete the proof of the theorem we shall show that, for every real number a, the set $E_a = \{(h, y) : R[V_a(y), h] \neq h\}$ is a Borel set in $\Delta_1 \times Y$. For this, let (δ_m) be a sequence of regular domains of Ω forming a covering of the space and, for every m, x_m an arbitrary element in δ_m. From Lemma 1 and Fatou's lemma, we deduce that, for every m,

$$(h, y) \mapsto \int R[V_a(y), h](\xi)\rho^m_{x_m}(d\xi)$$

is a lower semicontinuous function, where $\rho^m_{x_m}$ is the harmonic measure on $\partial\delta_m$ corresponding to x_m. Hence, it follows that

$$E_{a,m} = \{(h, y): \int R[V_a(y), h](\xi)\rho^m_{x_m}(d\xi) < h(x_m)\}$$

is a Borel subset of $\Delta_1 \times Y$, in fact, a countable union of closed sets. But $E_a = \bigcup_1^\infty E_{a,m}$ [6, Theorem II.1]. It follows that E_a is a Borel subset of $\Delta_1 \times Y$. The proof is complete.

The following two corollaries are immediate consequences.

Corollary 1. *Let v be a n-superharmonic function on $\Omega_1 \times \cdots \times \Omega_n$. Then, the function $(h, x^2, x^3, \cdots, x^n) \mapsto$ fine lim sup $v(x, x^2, \cdots, x^n)$ as x tends to h is a Borel measurable function on $\Delta_1 \times \Omega_2 \times \cdots \times \Omega_n$.*

Corollary 2. *Let u be a n-harmonic function on $\Omega_1 \times \cdots \times \Omega_n$. Then the set of points $(h, x^2, \cdots, x^n)$ of $\Delta_1 \times \Omega_2 \times \cdots \times \Omega_n$ for which the fine limit of $u(x, x^2, \cdots, x^n)$ exists as x tends to h is a Borel subset of this space.*

Theorem 2. *Let Ω be a harmonic space, Y any topological space and X a polish space (separable, complete metrisable). Let $f: \Omega \times Y \mapsto X$ be a separately continuous mapping. Then the set E defined by*

$$E = \{(h, y): \text{fine lim} f(x, y) \text{ exists in } X \text{ as } x \longrightarrow h\}$$

is a Borel subset of $\Delta_1 \times Y$.

Proof. Let d be a complete metric compatible with the topology of X. Let $X' = (x_n)_1^\infty$ be a countable dense subset of X. Let $(\delta_m)_1^\infty$ be a countable family of regular domains of Ω forming a base for the open sets and $\xi_m \in \delta_m$ an arbitrary element, for every m. Let, for any integer $k > 0$,

$$V_{n,k}(y) = \{\xi \in \Omega: d[x_n, f(\xi, y)] > 1/k\}.$$

We assert that

$$E = \bigcap_{k=1}^\infty \bigcup_{n=1}^\infty \bigcup_{m=1}^\infty \left\{(h, y): \int R[V_{n,k}(y), h](\xi)\rho^m_{\xi_m}(d\xi) < h(\xi_m)\right\}.$$

For, suppose (h, y) belongs to E. Let ζ be the fine limit in X of $f(\xi, y)$ as ξ tends to h. Given $k \in \mathbf{N}$, let l be a positive integer such that $2k < l$. Let $x_N \in X'$ such that $d[\zeta, x_N] < 1/l$. Let $F \subset \Omega$ be such that $F \in \mathfrak{F}_h$ and $d[\zeta, f(\xi, y)] < 1/l$, for every $\xi \in F$. Consider $V_{N,k}(y)$. Since, for every $\xi \in F$,

$$d[x_N, f(\xi, y)] \le d[x_N, \zeta] + d[\zeta, f(\xi, y)] < 1/k$$

we conclude that $F \subset \Omega - V_{N,k}(y)$, i.e. $\complement F \supset V_{N,k}(y)$. Hence, $R[V_{N,k}(y), h] \not\equiv h$ on Ω and we conclude that there exists an $M \in \mathbf{N}$ such that

$\int R[V_{N,k}(y), h](\xi)\rho^M_{\xi_M}(d\xi) < h(\xi_M)$ [6, Theorem II.1]. Thus E is contained in the set on the right. Conversely, suppose (h, y) belongs to the right side. To prove that $(h, y) \in E$, it is enough to show that the image of $\mathfrak{F}_h$ under the mapping $(\xi, y) \mapsto f(\xi, y)$ (y is fixed) is the base of a d-Cauchy filter. Given $\epsilon > 0$, choose $k \in \mathbf{N}$ such that $2/k < \epsilon$. For this k, we can find integers N and M such that

$$\int R[V_{N,k}(y), h](\xi)\rho^M_{\xi_M}(d\xi) < h(\xi_M).$$

This, in particular, implies that $F = \complement(V_{N,k}(y))$ belongs to $\mathfrak{F}_h$. Further, for every $\xi, \eta \in F$, since $d[x_N, f(\xi, y)] < 1/k$ and $d[x_N, f(\eta, y)] < 1/k$ we get by triangle inequality that $d[f(\xi, y), f(\eta, y)] < 2/k < \epsilon$. This proves the assertion.

Now, to complete the proof of the theorem, it is enough to show that, for every $\xi \in \Omega$, k and n, the mapping $(h, y) \mapsto R[V_{n,k}(y), h](\xi)$ is lower semicontinuous on $\Delta_1 \times Y$. But this follows from Lemma 1, since $(\xi, y) \mapsto d[x_n, f(\xi, y)]$ is also separately continuous for every n. The theorem is proved.

Theorem 3. *Let λ be a Radon measure on a topological space Y and Ω a harmonic space. Let f be a nonnegative real valued (resp. complex valued) function defined on $\Omega \times Y$ (resp. $U^n \times Y$) such that (1) for every $x \in \Omega$ (resp. U^n), $y \mapsto f(x, y)$ is λ-measurable and (2) for every $y \in Y$, $x \mapsto f(x, y)$ is harmonic on Ω (resp. holomorphic on U^n). Then, the mapping $y \mapsto f(\cdot, y)$ of Y into $H^+(\Omega) \cup \{0\}$ (resp. $\mathcal{H}(U^n)$) is λ-Lusin measurable, where the function spaces are provided with the topology τ of uniform convergence on compact subsets of the respective sets.*

Proof. The spaces $H^+(\Omega) \cup \{0\}$ and $\mathcal{H}(U^n)$ provided with the τ-topology are polish. Hence, the class of Borel sets for any weaker Hausdorff topology on these sets is the same as the class of τ-Borel sets. Consider one such topology τ_1 defined as follows. Let $Z \subset \Omega$ (resp. U^n) be a countable dense subset. The topology τ_1 is the topology of simple convergence on Z. We observe that there is a countable base for the τ_1-open sets: the finite intersections of sets of the form $\{u: u(z) \in V\}$ where $z \in Z$ and V belongs to a countable base for open sets of $\mathbf{R}$ (resp. $\mathbf{C}$).

Consider the mapping $Y \mapsto H^+(\Omega) \cup \{0\}$ (resp. $\mathcal{H}(U^n)$). For every τ_1-open set of the form $\{u: u(z) \in V\}$ (as above), we have

$$\{y \in Y: f(z, y) \in V\} = \{y \in Y: [f(\cdot, y) \in \{u(z) \in V\}]\}.$$

Hence, the inverse image of such τ_1-open sets and hence every τ_1-open set is μ-measurable in Y. Hence this mapping is μ-Borel measurable when $H^+(\Omega) \cup \{0\}$ (resp. $\mathcal{H}(U^n)$) is provided with τ_1. Now, by the earlier remark, we see that the same is true with τ-topology. However, the spaces involved on the right are polish; hence we get the required μ-Lusin measurability. The proof is complete.

Remark. In particular, if, in the above theorem, λ is a finite Radon measure, then, given $\epsilon > 0$, we can find a compact set $K \subset Y$ satisfying (1) $\lambda(Y - K) < \epsilon$ and (2) $y \mapsto f(\cdot, y)$ is continuous on K into $H^+(\Omega) \cup \{0\}$ (resp. $\mathcal{H}(U^n)$).

Theorem 4. *Let Ω be a harmonic space, $u > 0$ a harmonic function on Ω with the canonical measure μ on Δ_1; λ a finite Radon measure on a topological space X. Let $w \geq 0$ be a function defined on $\Omega \times X$ such that, for every $x \in X$, $w(\cdot, x)$ is harmonic and, for every $y \in \Omega$, $x \mapsto w(y, x)$ is λ-measurable. Then the following two functions f and g on $\Delta_1 \times X$ are $(\mu \times \lambda)$-measurable, where $f(h, x) =$ fine lim inf $[w(y, x)/u(y)]$ and $g(h, x) =$ fine lim sup $[w(y, x)/u(y)]$ as y tends to h.*

Proof. Let $\epsilon > 0$. From the above theorem, we have a compact set K_ϵ satisfying $\lambda(X - K)_\epsilon < \epsilon$ and $x \mapsto w(y, x)$ of $K_\epsilon \mapsto H^+(\Omega)$ is continuous. This certainly implies the separate continuity of $(y, x) \mapsto [w(y, x)/u(y)]$. We deduce from Theorem 1 that the functions f and g restricted to $\Delta_1 \times K$ are Borel measurable. Let, for $\epsilon = 1/n$, $n = 1, 2, \cdots$, K_n be the corresponding compact subset of X. We conclude that f and g are Borel measurable on $\Delta_1 \times F$ where $F = \bigcup_{n=1}^{\infty} K_n$. Now $\Delta_1 \times (X - F)$ is of $\mu \times \lambda$ measure zero since it is a measurable (in fact, Borel) rectangle, μ is a totally finite measure and $\lambda(X - F) = 0$. This completes the proof.

Corollary. *Let w, u and λ be as in the theorem. Then, except for a set of $\mu \times \lambda$ measure zero on $\Delta_1 \times X$, for every (h, x), the fine limit of $[w(y, x)/u(y)]$ as y tends to h exists and is finite. Further, this fine limit is $(\mu \times \lambda)$-measurable.*

Proof. Let $E = \{(h, x): f(h, x) < g(h, x)\} \cup \{(h, x): g(h, x) = \infty\}$. Clearly, E is $(\mu \times \lambda)$-measurable. Further, for every $x \in X$, for almost μ-every $h \in \Delta_1$, $f(h, x) = g(h, x) < +\infty$ [7, Theorem 8]. It follows, by Fubini's theorem, that $(\mu \times \lambda)(E) = 0$. This proves the Corollary.

3. **Limits of n-harmonic functions.** We shall consider, for the sake of simplicity of notation, 3-harmonic functions. In the general case, the proofs are absolutely similar. We shall fix harmonic functions $u_k > 0$ defined on harmonic spaces Ω_k for $k = 1, 2, 3$. Let μ_k be the canonical measure corresponding to u_k, on some convenient compact base of positive harmonic functions on Ω_k, charging the extreme elements Δ_1^k in that base.

Lemma 2. *Let $w > 0$ be a 3-superharmonic function on $\Omega_1 \times \Omega_2 \times \Omega_3$ such that, for every fixed $x^1 \in \Omega_1$, w is 2-harmonic on $\Omega_2 \times \Omega_3$. Then, except for a set E of μ_1 measure zero on Δ_1^1, for every (x^2, x^3) in $\Omega_2 \times \Omega_3$, the fine limit of $[w(x, x^2, x^3)/u_1(x)]$ exists (and is finite) as x tends to h in $(\Delta_1^1 - E)$. Further, this fine limit is 2-harmonic on $\Omega_2 \times \Omega_3$ and Borel measurable in all the three variables together (on $\Delta_1^1 \times \Omega_2 \times \Omega_3$).*

Proof. Let X^2 and X^3 be respectively countable dense subsets of Ω_2 and Ω_3. Consider the positive superharmonic function $w(\cdot, x_n^2, x_m^3)$ on Ω_1, for every $x_n^2 \in X^2$ and $x_m^3 \in X^3$. We can find a set $E_{n,m}$ of μ_1 measure zero on Δ_1^1 such that, for every $h \notin E_{n,m}$, the fine limit of $[w(x, x_n^2, x_m^3)/u_1(x)]$ exists (and is finite) as x tends to h [7, Theorem 8].

Let

$$E = \bigcup \{E_{n,m} : x_n^2 \in X^2, \ x_m^3 \in X^3\}.$$

Let $h \in \Delta_1^1 - E$. Let $\mathcal{G}_h$ be the filter, on the cone of positive 2-harmonic functions on $\Omega_2 \times \Omega_3$, generated by the image of $\mathcal{F}_h$ under the mapping $x \mapsto [w(x, \cdot, \cdot)/u_1(x)]$. Since $[w(x, x_1^2, x_1^3)/u_1(x)]$ converges to a finite limit following $\mathcal{F}_h$, we deduce that there is a set $A \in \mathcal{F}_h$ such that, for $x \in A$, $0 < [w(x, x_1^2, x_1^3)/u_1(x)] \leq M$, for some positive number M. But, the subset of positive 2-harmonic functions bounded above at some point of $\Omega_2 \times \Omega_3$ is relatively compact and hence there is a filter finer that $\mathcal{G}_h$ converging to some $v \in (2-H)^+(\Omega_2 \times \Omega_3)$, in the topology of uniform convergence on the compact subsets. However, any two adherent points v_1 and v_2 of $\mathcal{G}_h$ coincide on $X^2 \times X^3$ which is a dense subset of $\Omega_2 \times \Omega_3$; hence $v_1 = v_2$. We deduce, therefore, that $\mathcal{G}_h$ is convergent to an element w^h, i.e. we get that $w^h(x^2, x^3) = \text{fine} \lim[w(x, x^2, x^3)/u_1(x)]$ as x tends to h, for every (x^2, x^3), and the convergence is uniform for compact subsets of $\Omega_2 \times \Omega_3$. The Borel measurability of the limit function is an immediate consequence of Theorem 1. The lemma is proved.

Theorem 5. *Let $w > 0$ be a 3-harmonic function on $\Omega_1 \times \Omega_2 \times \Omega_3$. Then, except for a set of $\mu_1 \times \mu_2 \times \mu_3$ measure zero, for every (h^1, h^2, h^3) in $\Delta_1^1 \times \Delta_1^2 \times \Delta_1^3$, the iterated fine limit of $(w/u_1u_2u_3)$ exists and is finite; and any iterated limit function, defined $(\mu_1 \times \mu_2 \times \mu_3)$-almost everywhere, is $(\mu_1 \times \mu_2 \times \mu_3)$-measurable.*

Proof. It is enough to prove the results for some order of iteration. Without loss of generality, let us consider the natural order. From Lemma 2, we deduce the existence of a set $E \subset \Delta_1^1$ of μ_1 measure zero such that, for every $h \notin E$, $w^h(x^2, x^3) = \text{fine} \lim [w(x, x^2, x^3)/u_1(x)]$ as $x \to h$ and 2-harmonic and Borel measurable in all the three variables. Now, from the corollary to Theorem 4, we deduce that, for every $x_k^3 \in X^3$ (X^3 as in the previous lemma), there exists a set F'_k contained in $(\Delta_1^1 - E) \times \Delta_1^2$ of $\mu_1 \times \mu_2$ measure zero such that, for (h^1, h^2) not in F'_k, the fine limit of $[w^{h^1}(x, x_k^3)/u_2(x)]$ as x tends to h^2 exists and is finite (may be zero). Further, this limit function is $(\mu_1 \times \mu_2)$-measurable. We observe that $E \times \Delta_1^2$ is of $\mu_1 \times \mu_2$ measure zero since μ_2 is a totally finite measure. Let $F_k = F'_k \cup E \times \Delta_1^2$ for every $x_k^3 \in X^3$ and $F = \bigcup F_k$. Then, F is a $(\mu_1 \times \mu_2)$-measurable set of measure zero; also, for every $x_k^3 \in X^3$, fine limit of $[w^{h^1}(x, x_k^3)/u_2(x)]$ as x tends to h^2 exists and is finite for every $(h^1, h^2) \in F$. Now, using the fact that X^3 is dense in Ω_3, we deduce (exactly as in the proof of Lemma 2) that, for every $(h^1, h^2) \in F$, the fine limit of $[w^{h^1}(x, x^3)/u_2(x)]$ as x tends to h^2 exists locally uniformly for $x^3 \in \Omega_3$. (We use the fact that the set of positive harmonic functions on Ω_2 bounded at one point is relatively compact for the local uniform convergence topology on Ω_2 [2].) This limit w^{h^1,h^2} is harmonic on Ω_3 for every $(b^1, b^2) \in F$ and $(\mu_1 \times \mu_2)$-measurable for every $x_3 \in \Omega_3$. Hence, by the corollary to Theorem 4, we get that the fine limit of $[w^{b^1,b^2}(x)/u_3(x)]$ exists and is finite except for $(b^1, b^2, b^3) \in G$, where $G \subset (\Delta_1^1 \times \Delta_1^2 - F) \times \Delta_1^3$ and is of $\mu_1 \times \mu_2 \times \mu_3$ measure zero. Further the above limit is a $(\mu_1 \times \mu_2 \times \mu_3)$-measurable function on $[(\Delta_1^1 \times \Delta_1^2 - F) \times \Delta_1^3] - G$. Once again, $F \times \Delta_1^3$ is of $\mu_1 \times \mu_2 \times \mu_3$ measure zero, since μ_3 is totally finite. Hence the iterated fine limit of $(w/u_1u_2u_3)$ exists as a $(\mu_1 \times \mu_2 \times \mu_3)$-measurable function, except for all (b^1, b^2, b^3) in $[G \cup (F \times \Delta_1^3)]$. The proof is complete.

Lemma 3. *Let $w > 0$ be a 3-harmonic function on $\Omega_1 \times \Omega_2 \times \Omega_3$ such that $w \lneq M u_1u_2u_3$ for some positive M. Then, except for a set of $\mu_1 \times \mu_2 \times \mu_3$ measure zero, the various (3!) iterated fine limits of $(w/u_1u_2u_3)$ are identical and the common value is a Radon-Nikodym derivative of the canonical measure of w relative to $\mu_1 \times \mu_2 \times \mu_3$.*

Proof. Since the canonical measure ν_w of w is $\leq M(\mu_1 \times \mu_2 \times \mu_3)$, ν_w is absolutely continuous relative to $\mu_1 \times \mu_2 \times \mu_3$. Now, using the notation of Theorem 5, $(w^{b^1,b^2}/u_3)$ is bounded on Ω_3 and, for $(\mu_1 \times \mu_2)$-almost every element of $\Delta_1^1 \times \Delta_1^2$, as x tends to b^3, fine lim $[w^{b^1,b^2}(x)/u_3(x)] = \tilde{w}(b^1, b^2, b^3)$ μ_3-almost everywhere. Hence, we get

$$\int_{\Delta_1^3} \tilde{w}(b^1, b^2, b^3)\, b^3(x)\, \mu_3(db^3) = w^{b^1,b^2}(x) \quad \text{on } \Omega_3 \quad [7, \text{Theorem 7}].$$

Again, $(b^1, b^2) \mapsto w^{b^1,b^2}(x^3)$ is measurable and, for any $x^3 \in \Omega_3$ and $b^1 \in \Delta_1^1 - E$ (E as in Theorem 5), $[w^{b^1}(x, x^3)/u_2(x)]$ is bounded on Ω_2 and its limit following $\mathcal{F}_{b^2}$ equals $w^{b^1,b^2}(x^3)$ for μ_2-almost every $b^2 \in \Delta_1^2$. Hence, for $x \in \Omega_2$,

$$w^{b^1}(x, x^3) = \int_{\Delta_1^2} w^{b^1,b^2}(x^3)\, b^2(x)\, \mu_2(db^2).$$

Finally, going one more step backward, we deduce that, for all (x^1, x^2, x^3),

$$w(x^1, x^2, x^3) = \int_{\Delta_1^1} \mu_1(db^1) \int_{\Delta_1^2} \mu_2(db^2) \int_{\Delta_1^3} \tilde{w}(b^1, b^2, b^3)\, b^1(x^1)\, b^2(x^2)\, b^3(x^3)\, \mu_3(db^3),$$

$$= \iiint \tilde{w}(b^1, b^2, b^3)\, b^1(x^1)\, b^2(x^2)\, b^3(x^3)(\mu_1 \times \mu_2 \times \mu_3)(db^1db^2db^3)$$

[Fubini's theorem].

It follows from the uniqueness of integral representation that $d\nu_w = \tilde{w}(b^1, b^2, b^3)\, d(\mu_1 \times \mu_2 \times \mu_3)$ [8, Theorem 7]. We conclude that $\tilde{w}$ is a Radon-Nikodym derivative of ν_w relative to $\mu_1 \times \mu_2 \times \mu_3$. The Radon-Nikodym derivative of ν_w relative to $\mu_1 \times \mu_2 \times \mu_3$ is unique up to a set of $\mu_1 \times \mu_2 \times \mu_3$ measure zero; the proof is complete.

Theorem 6. *Let w and u_1, u_2, u_3, etc. be as in Theorem 5. Then, the iterated fine limits of $(w/u_1u_2u_3)$ are equal $(\mu_1 \times \mu_2 \times \mu_3)$-almost everywhere.*

Proof. Let ν be the canonical measure on $\Delta_1^1 \times \Delta_1^2 \times \Delta_1^3$ corresponding to w. Let $\nu = \nu_1 + \nu_2$ where $\nu_1 << \mu_1 \times \mu_2 \times \mu_3$ and ν_2 is singular relative to $\mu_1 \times \mu_2 \times \mu_3$. Let f be a Radon-Nikodym derivative of ν_1 relative to $\mu_1 \times \mu_2 \times \mu_3$. We note that $f \geq 0$ and is finite $(\mu_1 \times \mu_2 \times \mu_3)$-almost everywhere. Let Σ_f be the 3-harmonic function on $\Omega_1 \times \Omega_2 \times \Omega_3$ with the canonical measure ν_1 on $\Delta_1^1 \times \Delta_1^2 \times \Delta_1^3$. Consider the iterated limit f_1 (resp. w_1) of Σ_f (resp. w) for the same order of iteration say, for the natural order. Since $w \geq \Sigma_f$, we get that $w_1 \geq f_1$ everywhere. However, if $f_n = \inf(f, n)$, for any positive integer n (≥ 2), then $f_n \geq 0$ and is bounded, and we get, from Lemma 3, that the iterated fine limits of Σ_{f_n} equal f_n $(\mu_1 \times \mu_2 \times \mu_3)$-almost everywhere. ($\Sigma_{f_n}$ is the 3-harmonic function with the canonical measure $f_n(\mu_1 \times \mu_2 \times \mu_3)$.) It follows that $f_1 \geq f_n$ almost everywhere. This is true for every n and we deduce that $f_1 \geq f$ $(\mu_1 \times \mu_2 \times \mu_3)$-almost everywhere. On the other hand, (using the notation of Theorem 5) $w^{b^1, b^2} \geq 0$ harmonic and the fine $\lim_{x \to b^3} [w^{b^1,b^2}(x)/u_3(x)] = w_1(b^1, b^2, b^3)$ μ_3-almost everywhere. Hence [7, Theorem 8], $w^{b^1,b^2}(x) \geq \int w_1(b^1, b^2, b^3) b^3(x) \mu_3(db^3)$ for all $x \in \Omega_3$. All this is true for almost every (b^1, b^2) in $\Delta_1^1 \times \Delta_1^2$. Proceeding backwards and repeating the argument, we deduce that $w \geq \Sigma_{w_1}$. In view of the uniqueness of integral representation, we conclude that

$$\text{(the canonical measure of } w) \geq w_1(\mu_1 \times \mu_2 \times \mu_3) \text{ (the canonical measure of } \Sigma_w),$$

i.e., $f(\mu_1 \times \mu_2 \times \mu_3) + \nu \geq w_1(\mu_1 \times \mu_2 \times \mu_3)$. But $w_1 \geq f_1 \geq f$ $(\mu_1 \times \mu_2 \times \mu_3)$-almost everywhere and, hence, the positive measure $(w_1 - f)(\mu_1 \times \mu_2 \times \mu_3) \leq \nu_2$, a measure singular relative to $\mu_1 \times \mu_2 \times \mu_3$. It follows that $w_1 = f$ $(\mu_1 \times \mu_2 \times \mu_3)$-almost everywhere. This completes the proof of the theorem.

We deduce immediately

Corollary 1. *If ϕ is the difference of two positive 3-harmonic functions on $\Omega_1 \times \Omega_2 \times \Omega_3$, then the iterated fine limits of $(\phi/u_1u_2u_3)$ exist and are equal (and finite) $(\mu_1 \times \mu_2 \times \mu_3)$-almost everywhere.*

In the course of the proof of the theorem it is shown that the iterated fine limit of w is a Radon-Nikodym derivative of the absolutely continuous part of ν relative to $\mu_1 \times \mu_2 \times \mu_3$. In particular, we have

Corollary 2. *Let $w \geq 0$, a 3-harmonic function, having canonical measure ν absolutely continuous relative to $\mu_1 \times \mu_2 \times \mu_3$. Let f_w be the iterated fine limit of $w/u_1u_2u_3$ (for some order of iteration). Then, $d\nu = f_w d(\mu_1 \times \mu_2 \times \mu_3)$; equivalently, for every (x, y, z),*

$$w(x, y, z) = \iiint b^1(x)\, b^2(y)\, b^3(z) f_w(b^1, b^2, b^3) (\mu_1 \times \mu_2 \times \mu_3) (db^1\, db^2 db^3).$$

Another consequence is the following result.

Theorem 7. *Let w, u_1, u_2 and u_3 be as in Theorem 5. Then the following are* *equivalent.*

(a) *The greatest* 3-*harmonic minorant of* $\inf(w, u_1u_2u_3)$ *is* 0.

(b) *The canonical measure* ν *of* w *is singular relative to* $\mu_1 \times \mu_2 \times \mu_3$.

(c) *The iterated fine limit of* $(w/u_1u_2u_3)$ *equals zero* $(\mu_1 \times \mu_2 \times \mu_3)$-*almost everywhere.*

Proof. We shall show that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a).

Suppose (a) is true. Let f be the Radon-Nikodym derivative of the absolutely continuous part of ν relative to $\mu_1 \times \mu_2 \times \mu_3$. Note that $f \geq 0$. The 3-harmonic functions with canonical measures $f_n/n\ (\mu_1 \times \mu_2 \times \mu_3)$, where $n \in \mathbf{N}$, $f_n = \inf(f, n)$ minorise both w and $u_1u_2u_3$. Hence $f_n/n = 0$, i.e. $f_n = 0\ (\mu_1 \times \mu_2 \times \mu_3)$-almost everywhere. This implies (b). The fact that (b) $\Rightarrow$ (c) is an immediate consequence of Theorem 6. Now, assume that (c) holds. Let λ be the canonical measure on $\Delta_1^1 \times \Delta_1^2 \times \Delta_1^3$ corresponding to the greatest 3-harmonic minorant w_1 of $\inf(w, u_1u_2u_3)$ and g a Radon-Nikodym derivative of λ relative to $\mu_1 \times \mu_2 \times \mu_3$. From Lemma 3 and (c) we deduce that $0 =$ iterated fine limits of $w/u_1u_2u_3 \geq$ those of $w_1/u_1u_2u_3 = g \geq 0\ (\mu_1 \times \mu_2 \times \mu_3)$-almost everywhere. Hence $g = 0$ almost everywhere and $w_1 \equiv 0$. The proof is complete.

4. **The case of n-superharmonic functions.** We consider 3-superharmonic functions like in the earlier section. Let Ω_k, u_k, Δ_1^k and μ_k be as in §3, for $k = 1, 2, 3$. First we need the following result (Lemma 4). We state and prove it for functions of two variables, but obviously the proof is valid for functions of several variables.

Definition 1. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be respectively a base of regular domains of Ω_1 and Ω_2. An extended real valued function f on $\Omega_1 \times \Omega_2$ is said to be an MS-$(\mathcal{B}_1, \mathcal{B}_2)$ function if (1) f is locally lower bounded at every point and (2) for every $\delta \in \mathcal{B}_1$, $\omega \in \mathcal{B}_2$ and all $(x, y) \in \delta \times \omega$,

$$f(x, y) \geq \iint^* f(\xi, \eta)(\rho_x^\delta \times \rho_y^\omega)(d\xi\, d\eta).$$

Lemma 4. *Let* $v \not\equiv +\infty$ *be an* MS-$(\mathcal{B}_1, \mathcal{B}_2)$ *function on* $\Omega_1 \times \Omega_2$. *Then,* w *the lower semicontinuous regularisation of* v *is a* 2-*superharmonic function on* $\Omega_1 \times \Omega_2$. *Further, for every* $(x, y) \in \Omega_1 \times \Omega_2$,

$$w(x, y) = \sup \left\{ \iint^* v(\xi, \eta)(\rho_x^\delta \times \rho_y^\omega)\,(d\xi\, d\eta):\ x \in \delta,\ y \in \omega;\ \delta \in \mathcal{B}_1,\ \omega \in \mathcal{B}_2 \right\}.$$

In particular, a lower semicontinuous MS-$(\mathcal{B}_1, \mathcal{B}_2)$ *function is* 2-*superharmonic.*

Proof. Let us first consider a lower semicontinuous MS-$(\mathcal{B}_1, \mathcal{B}_2)$ function g. Fix $x \in \Omega_1$ and let $\omega \in \mathcal{B}_2$ contain $y \in \Omega_2$. Let $(\delta_n)_1^\infty$ be a sequence of regular domains in $\mathcal{B}_1$ such that $\bar{\delta}_{n+1} \subset \delta_n$ and $\bigcap_1^\infty \delta_n = \{x\}$. It is easily seen that the harmonic measures ρ_x^n (on $\partial\delta_n \subset \bar{\delta}_1$) converge weakly to the Dirac measure ϵ_x at

x as n tends to ∞. Now,

$$w(x, y) \geq \int \rho_x^n(d\xi) \int w(\xi, \eta)\, \rho_y^\omega(d\eta)$$

for every n. In deducing the above inequality the use of Fubini's theorem is justified since w is locally lower bounded, measurable and ρ_x^n and ρ_y^δ are totally finite measures. But, $\xi \mapsto \int w(\xi, \eta)\, \rho_y^\omega(d\eta)$ is lower semicontinuous; hence,

$$\begin{aligned} w(x, y) &\geq \liminf \int \rho_x^n(d\xi) \left[\int w(\xi, \eta)\, \rho_y^\omega(d\eta)\right] \\ &\geq \int \epsilon_x(d\xi) \left[\int w(\xi, \eta)\, \rho_y^\omega(d\eta)\right] \\ &= \int w(x, \eta)\, \rho_y^\omega(d\eta). \end{aligned}$$

Now, from the local criterion for the superharmonicity of lower semicontinuous functions of one variable, we deduce that $y \mapsto w(x, y)$ is superharmonic (or identically $+\infty$) on Ω_2. The symmetry of the argument in the variables involved shows that w is a 2-superharmonic function.

Let v be as in the hypothesis of the lemma. Now, for any $(x, y) \in \delta \times \omega$ with $\delta \in \mathfrak{B}_1$, $\omega \in \mathfrak{B}_2$, $(a, b) \mapsto \iint^* v(\xi, \eta)(\rho_a^\delta \times \rho_b^\omega)(d\xi\, d\eta)$ is a 2-harmonic function on $\delta \times \omega$, hence, in particular, continuous [8, Theorem 2]. Hence w, the lower semicontinuous regularisation of v, satisfies

$$w(x, y) \geq \iint^* v(\xi, \eta)(\rho_x^\delta \times \rho_y^\omega)(d\xi\, d\eta).$$

But, given $\alpha < w(x, y)$, there is a neighbourhood V of (x, y) such that $v > \alpha$ on V. We deduce that

$$\begin{aligned} \sup\left\{\iint^* v(\xi, \eta)(\rho_x^\delta \times \rho_y^\omega)(d\xi\, d\eta) : \delta \in \mathfrak{B}_1,\ \omega \in \mathfrak{B}_2,\ \delta \times \omega \subset V\right\} \\ \geq \lim \alpha \iint \rho_x^\delta(d\xi)\, \rho_y^\omega(d\eta) = \alpha. \end{aligned}$$

Thus, w is the stated supremum. Further, for any $\delta \in \mathfrak{B}_1$, $\omega \in \mathfrak{B}_2$, $(x, y) \in \delta \times \omega$,

$$\begin{aligned} w(x, y) &\geq \liminf v(a, b) \quad [\text{as } (a, b) \text{ tends to } (x, y)] \\ &\geq \liminf \iint^* v(\xi, \eta)(\rho_a^\delta \times \rho_b^\omega)(d\xi\, d\eta) \qquad [(a, b) \in \delta \times \omega \to (x, y)] \\ &\geq \liminf \iint^* w(\xi, \eta)\, \rho_a^\delta(d\xi)\, \rho_b^\omega(d\eta) \\ &= \lim \iint w(\xi, \eta)\, \rho_a^\delta(d\xi)\, \rho_b^\omega(d\eta) = \iint w(\xi, \eta)\, \rho_x^\delta(d\xi)\, \rho_y^\omega(d\eta). \end{aligned}$$

Thus w is also an MS-$(\mathfrak{B}_1, \mathfrak{B}_2)$ function and we deduce from the first part that w is 2-superharmonic. The proof is complete.

Remark. It is immediate from the above lemma that the lower semicontinuous regularisation of the lower envelope of an arbitrary family of locally lower bounded n-superharmonic functions is again n-superharmonic.

We recall that $\{0\} \cup S^+(\Omega)$ (the set of positive superharmonic functions on Ω)

is locally compact, separable and metrisable in the T-topology of Mme. Hervé [9, Theorems 21.1, 21.2]. Further, for any regular domain ω, $x \in \omega$ and $\alpha > 0$,

$$\Lambda_\alpha = \left\{ w \in S^+ \cup \{0\} : \int w(\xi)\, \rho_x^\omega(d\xi) \le \alpha \right\}$$

is compact [9, Theorem 21.2]. The Cartan-Brelot topology (we shall write C-B-topology) on $S^+ \cup \{0\}$ is the coarsest topology which makes the functions $w \mapsto \int w(\xi)\, \rho_x^\delta(d\xi)$ continuous, where δ belongs to a countable base $\mathfrak{B}$ of regular domains of Ω and x belongs to a countable dense subset A of Ω (naturally $x \in \delta$ for every δ). We now have

Proposition 1. *The σ-algebras of T-Borel sets and σ-algebras of C-B-Borel sets are identical.*

Proof. The T-topology is coarser than the C-B-topology [9, Proposition 24.6]. Hence, every T-Borel set is necessarily a C-B-Borel set. However, the mapping $w \mapsto \int w(\xi)\, \rho_x^\delta(d\xi)$ is T-lower semicontinuous, for every δ and all $x \in \delta$ [9, Proposition 24.1]. Hence, every C-B-open set belonging to the following subbase is a T-Borel set: $\forall x \in A$, $\delta \in \mathfrak{B}$, V any open interval with rational endpoints, $\{w: \int w\, d\rho_x^\delta \in V\}$. This subbase is clearly countable and we deduce, by standard measure theoretic arguments, that every C-B-Borel set is also a T-Borel set. The proof is complete.

Theorem 8. *Let Ω be a harmonic space, $u > 0$ a harmonic function on Ω and μ the corresponding canonical measure carried by Δ_1. Let λ be a finite positive Radon measure on a topological space Y. Let $v \ge 0$ be an extended real valued function on $\Omega \times Y$, measurable with respect to the product σ-algebra of the Borel sets of Ω and λ-measurable sets of Y. Further, suppose that, for every $y \in Y$, $x \mapsto v(x, y)$ is superharmonic on Ω. Then, the function*

$$(h, y) \mapsto \text{fine lim sup } [v(x, y)/u(x)],$$

as x tends to h, is $(\mu \times \lambda)$-measurable on $\Delta_1 \times Y$.

Proof. Let $\mathfrak{B}$ be a countable base for the open sets of Ω consisting of regular domains and A a countable dense subset of Ω. Let us assume to start with that $\exists$ a $z \in \Omega$ and an $\omega \in \mathfrak{B}$, $z \in \omega$, such that $\int v(x, y) \rho_z^\omega(dx) \le 1$ for every $y \in Y$. Let $\Lambda = \{w: \int w\, d\rho_z^\omega \le 1\}$.

Now, consider the mapping $\phi: Y \mapsto \Lambda$ defined by $(\phi(y))(x) = v(x, y)$. For every $\delta \in \mathfrak{B}$, $x \in A$, $y \mapsto \int (\phi(y))(\xi) \rho_x^\delta(d\xi)$ is λ-measurable (Fubini). From this, it is easy to deduce by standard measure theoretic arguments, that the mapping $\phi: Y \mapsto \Lambda$ is λ-Borel measurable, when Λ is provided with the C-B-topology. This certainly implies that $\phi: Y \mapsto \Lambda$ is λ-Borel measurable when Λ is provided with the T-topology. But Λ with the T-topology is compact and metrisable, hence polish. Hence, ϕ is λ-Lusin measurable. Given $\epsilon > 0$, let C be compact $\subset Y$ such that

$\lambda(Y) < \lambda(C) + \epsilon/2$. By the Lusin measurability we can find a compact subset K of C satisfying $\lambda(K) > \lambda(C) - \epsilon/2$ and ϕ restricted to K is continuous. Now, the mapping $w \mapsto w(x)$ is T-lower semicontinuous from $S^+ \to \mathbf{R}$, for every $x \in \Omega$. For every $x \in \Omega$, therefore, $y \mapsto \phi(y)(x) = v(x, y)$ is lower semicontinuous on Y. Now we conclude, from Theorem 1, that f is Borel measurable on $\Delta_1 \times C$.

Choose a compact set $K_n \subset Y$ as above, for the choice of $\epsilon = 1/n$, for $n = 1, 2, 3, \cdots$. Let $B = \bigcup K_n$. Then B is Borel on Y and $\lambda(Y - B) = 0$. The function f restricted to each $\Delta_1 \times K_n$ is Borel measurable and, hence, f restricted to $\Delta_1 \times B$ is Borel measurable. Further, $(\mu \times \lambda)(\Delta_1 \times Y - B) = 0$ since μ is a finite measure. We conclude that $f^{-1}(E)$ is $(\mu \times \lambda)$-measurable, for every Borel set $E \subset \mathbf{R}$. This completes the proof for the v's satisfying the condition stated at the beginning.

Now, for any general w satisfying the hypothesis of the theorem, let $v(x, y) = [w(x, y)/\alpha(y)]$ where $\alpha(y) = \sup [1, \int w(x, y)\rho_z^\omega(dx)]$. Then v satisfies the additional condition that $\int v(x, y)\rho_z^\omega(dx) \le 1$. Now, for $(h, y) \in \Delta_1 \times Y$, the fine lim sup $[w(x, y)/u(x)]$ as x tends to h equals the fine lim sup $[w(x, y)/u(x)]$ divided by $\alpha(y)$. And, $y \mapsto \alpha(y)$ is clearly λ-measurable. The required measurability is now easily seen to be true, completing the proof.

Now, we proceed to consider the iterated fine limit operations on a 3-superharmonic function $v > 0$ defined on $\Omega_1 \times \Omega_2 \times \Omega_3$. As an immediate consequence of Theorem 1, we deduce that (h, y, z), going to fine lim sup of $[v(x, y, z)/u_1(x)]$ as x tends to h, is a Borel measurable function. The next lemma shows that except for a set of μ_1 measure zero this is an MS-$(\mathfrak{B}_2, \mathfrak{B}_3)$ function.

Lemma 5. *Let $\mathfrak{B}_2$ and $\mathfrak{B}_3$ be countable bases of regular domains of the spaces Ω_2 and Ω_3 respectively. Then, a set $E \subset \Delta_1^1$ of μ_1 measure zero can be chosen such that, for every $h \notin E$, $v(h, \cdot, \cdot)$ is an MS-$(\mathfrak{B}_2, \mathfrak{B}_3)$ function.*

Proof. Consider the function $w(x, y, z) = \iint v(x, \eta, \zeta)\rho_y^\delta(d\eta)\rho_z^\omega(d\zeta)$, corresponding to $\delta \in \mathfrak{B}_2$ and $\omega \in \mathfrak{B}_3$. Clearly, for every $x \in \Omega$, $w(x, \cdot, \cdot)$ is a 2-harmonic function on $\delta \times \omega$. Any easy application of Fubini's theorem and Fatou's lemma lets us conclude that, for every fixed $(y, z) \in \delta \times \omega$, $w(\cdot, y, z)$ is superharmonic. Hence w is a 3-superharmonic function on $\Omega \times \delta \times \omega$ [8, Theorem 2]; also $v \ge w$. Let us consider the three harmonic functions on $\Omega_1 \times \delta \times \omega$. w_1 is defined by

$$w_1(x, y, z) = \int \mu_1(dh) \iint h(x)\, v^h(\eta, \zeta)\, \rho_y^\delta(d\eta)\, \rho_z^\omega(d\zeta)$$

where $v^h(\eta, \zeta) =$ fine lim sup $[v(x, \eta, \zeta)/u_1(x)]$ as x tends to h. For every fixed η and ζ in δ and ω, $v(x, \eta, \zeta) \ge \int h(x)v^h(\eta, \zeta)\mu_1(dh)$ [7, Theorems 6, 7]. Hence, by Fubini's theorem, we get that $w \ge w_1$ on $\Omega_1 \times \delta \times \omega$. Now, from Lemma 2, we deduce the existence of a set G (depending on δ and ω) of μ_1 measure zero such that, for every $h \in \Delta_1^1 - G$, as x tends to h the fine limit of $[w_1(x, y, z)/u_1(x)] =$

$w_1^h(y, z)$ exists for all $(y, z) \in \delta \times \omega$ and is 2-harmonic on $\delta \times \omega$; also $(h, y, z) \mapsto w_1^h(y, z)$ is a Borel measurable function. In view of the regularity of the solution of Dirichlet problems by the Perron method [7, Theorem 7] we may assume that this set G was chosen such that, for all $h \in \Delta_1^1 - G$ and every (y, z) belonging to a countable dense subset of $\delta \times \omega$, the equality $w_1^h(y, z) = \int\int v^h(\eta, \zeta)\rho_y^\delta(d\eta)\rho_z^\omega(d\zeta)$ holds to be good. Now, $w_1^h(y, z)$ and $\int\int v^h(\eta, \zeta)\rho_y^\delta(d\eta)\rho_z^\omega(d\zeta)$ are two 2-harmonic functions on $\delta \times \omega$ and they coincide on a dense set, hence the two functions are identical. Hence, for all $h \in \Delta_1^1 - G$, $y \in \delta$, $z \in \omega$,

$$\begin{aligned} v^h(y, z) &= \text{fine lim sup } [v(x, y, z)/u_1(x)] \\ &\geq \text{fine lim } [w_1(x, y, z)/u_1(x)] \\ &= \iint v^h(\eta, \zeta)\, \rho_y^\delta(d\eta)\, \rho_z^\omega(d\zeta). \end{aligned}$$

Let $F_{n,m}$ be the exceptional set corresponding to $\delta_n \in \mathfrak{B}_2$, $\omega_m \in \mathfrak{B}_3$ and F, the union of these countably many sets. Then F is of μ_1 measure zero and, for all $h \in \Delta_1^1 - F$ and any $\delta_n \in \mathfrak{B}_2$, $\omega_m \in \mathfrak{B}_3$,

$$v^h(y, z) \geq \iint v^h(\eta, \zeta)\, \rho_y^n(d\eta)\, \rho_z^m(d\zeta), \qquad \forall\, y \in \delta_n,\ z \in \omega_m.$$

Since v' is, in addition, ≥ 0, we conclude that v' is an MS-$(\mathfrak{B}_2, \mathfrak{B}_3)$ function, completing the proof.

Lemma 6. *The lower semicontinuous regularisation (in y, z) $w^h(y, z)$ of $v^h(y, z)$, for every $h \in \Delta_1^1 - F$, is 2-superharmonic; further $(h, y, z) \mapsto w^h(y, z)$ is a measurable function on $(\Delta_1^1 - F) \times \Omega_2 \times \Omega_3$.*

Proof. The first part is a consequence of Lemma 4 and Lemma 5. To prove the measurability, consider $\delta \in \mathfrak{B}_2$ and $\omega \in \mathfrak{B}_3$. The function $(h, y, z) \mapsto \int\int v^h(\eta, \zeta)\rho_y^\delta(d\eta)\rho_z^\omega(d\zeta)$ is measurable in the three variables together (since it is μ_1-measurable on $\Delta_1^1 - F$, for every fixed y, z and 2-harmonic (≥ 0) on $\delta \times \omega$ for every fixed h) (Theorem 3). Again, from Lemma 4, we get, for every $a \in \mathbf{R}$,

$$\begin{aligned} &\{(h, y, z): w^h(y, z) > a\} \\ &= \bigcup \left\{ (h, y, z) \in (\Delta_1^1 - F) \times \delta \times \omega : \iint v^h(\eta, \zeta)\, \rho_y^\delta(d\eta)\, \rho_z^\omega(d\zeta) > a \right\} \end{aligned}$$

where the union on the right is taken over all $\delta \in \mathfrak{B}_2$ and $\omega \in \mathfrak{B}_3$. This clearly proves the measurability of $w^h(y, z)$. The lemma is proved.

Lemma 7. *There exists a set $G \subset \Delta_1^1 \times \Delta_1^2$ of $\mu_1 \times \mu_2$ measure zero such that, for every $(h^1, h^2) \notin G$, $\theta(h^1, h^2, z) =$ fine lim sup $[v^{h^1}(y, z)/u_2(y)]$ as y tends to h^2 is a $\mathfrak{B}_3$-nearly superharmonic function on Ω_3. Let, for every $(h^1, h^2) \in (\Delta_1^1 \times \Delta_1^2 - G)$, $\phi(h^1, h^2, z)$ be the lower semicontinuous regularisation of $\theta(h^1, h^2, z)$ in the z variable. Then, ϕ is a measurable function.*

Proof. The proof of measurability is exactly similar to the proof of Lemma 6, and we omit it. To prove the first part, we observe that θ is a $(\mu_1 \times \mu_2 \times \lambda)$-measurable function on $(\Delta_1^1 - F) \times \Delta_1^2 \times \Omega_3$ (Theorem 8), for any finite Radon measure λ on Ω_3. Consider a $\delta \in \mathfrak{B}_3$. The function ψ defined by

$$\psi(x, y, z) = \iiint h^1(x) h^2(y)\, \theta(h^1, h^2, \zeta) \mu_1(dh^1) \mu_2(dh^2) \rho_z^\delta(d\zeta)$$

is a 3-harmonic function on $\Omega_1 \times \Omega_2 \times \delta$, and $\psi \leq v$ (Theorem 6). Imitating the first part of the proof of Theorem 5, we can find a set G^δ of $\mu_1 \times \mu_2$ measure zero on $\Delta_1^1 \times \Delta_1^2$ such that the second iterated fine limit (as $x \mapsto h^1$, $y \to h^2$) of $[\psi(x, y, z)/u_1(x)u_2(y)]$ is harmonic on δ and further this function coincides with $\int \theta(h^1, h^2, \zeta)\rho_z^\delta(d\zeta)$, for every $(h^1, h^2) \in G^\delta$. (The latter assertion is deduced using Corollary 2, Theorem 6 and a countable dense subset of δ, as in the proof of Lemma 5.) Now, let $G = F \times \Delta_1^2 \cup_\delta G^\delta$ where $\delta \in \mathfrak{B}_3$ and F is as in Lemma 5. Then G is of $\mu_1 \times \mu_2$ measure zero and, for every $(h^1, h^2) \in (\Delta_1^1 \times \Delta_1^2 - G)$,

$$\begin{aligned}\theta(h^1, h^2, z) &\geq \operatorname*{fine\,lim}_{y \to h^2} \left[\operatorname*{fine\,lim}_{x \to h^1} (\psi(x, y, z)/u_1(x)u_2(y))\right] \\ &= \int \theta(h^1, h^2, \zeta) \rho_z^\delta(d\zeta), \qquad \delta \in \mathfrak{B}_3,\ z \in \delta.\end{aligned}$$

This together with the fact that θ is lower bounded, in fact ≥ 0, we conclude that θ is an S-$(\mathfrak{B}_3)$ function. The lemma is proved.

Theorem 9. *Let $f_v(h^1, h^2, h^3) =$ fine lim sup $[\phi(h^1, h^2, z)/u_3(z)]$ as z tends to h^3, for $(h^1, h^2, h^3) \in (\Delta_1^1 \times \Delta_1^2 - G) \times \Delta_1^3$ (G and ϕ as in the previous lemma). Then, f_v is $(\mu_1 \times \mu_2 \times \mu_3)$-measurable and is the Radon-Nikodym derivative of (the absolutely continuous part of the) canonical measure ν on $\Delta_1^1 \times \Delta_1^2 \times \Delta_1^3$ of the greatest 3-harmonic minorant of v relative to $\mu_1 \times \mu_2 \times \mu_3$.*

Proof. The $(\mu_1 \times \mu_2 \times \mu_3)$-measurability of f_v is an immediate consequence of Theorem 8. Now, $v(x, y, z) \geq \iint \phi(h^1, h^2, z) h^1(x) h^2(y) \mu_1(dh^1) \mu_2(dh^2)$ and, since $\phi \geq 0$ is superharmonic in the z-variable, $\phi(h^1, h^2, z) \geq \int f_v(h^1, h^2, h^3) h^3(z) \mu_3(dh^3)$ [7, Theorems 6, 7]. Hence, if u is the greatest 3-harmonic minorant of v, then $u \geq \Sigma(f, v)$ where

$$\Sigma(f, v) = \int f_v(h^1, h^2, h^3)\, h^1 h^2 h^3 (\mu_1 \times \mu_2 \times \mu_3)(dh^1\, dh^2\, dh^3).$$

By the uniqueness of the integral representation of 3-harmonic functions ≥ 0, we get that the canonical measure ν of u (on $\Delta_1^1 \times \Delta_1^2 \times \Delta_1^3$) majorises the canonical measure of $\Sigma(f, v)$. But, since f_v is measurable ≥ 0 and the measure $f_v d(\mu_1 \times \mu_2 \times \mu_3)$ is on $\Delta_1^1 \times \Delta_1^2 \times \Delta_1^3$, it is clear that this is indeed the canonical measure of $\Sigma(f, v)$. Hence, $d\nu \geq f_v d(\mu_1 \times \mu_2 \times \mu_3)$. It follows that the Radon-Nikodym derivative g of (the absolutely continuous part of) ν relative to $\mu_1 \times \mu_2 \times \mu_3$ is $\geq f_v$ at $(\mu_1 \times \mu_2 \times \mu_3)$-almost every element of $\Delta_1^1 \times \Delta_1^2 \times \Delta_1^3$.

On the other hand, if we do the same operation on u (as on v) and arrive at the function f_u, then it is clear that f_u is nothing but the iterated fine limit of u; and by Theorem 6, Corollary 2, $f_u = g$ almost everywhere. However, $v \geq u$ and hence $f_v \geq f_u$ almost everywhere, i.e. $g = f_v$ $(\mu_1 \times \mu_2 \times \mu_3)$-almost everywhere. The proof is complete.

We deduce immediately

Corollary 1. *If v is 3-superharmonic > 0 and has 0 as the greatest 3-harmonic majorant, then $f_v = 0$ $(\mu_1 \times \mu_2 \times \mu_3)$-almost everywhere.*

Corollary 2. *If v is 3-superharmonic > 0 and moreover if the fine limit of v/u, etc. exists as a lower semicontinuous function in the rest of the variable(s) (as usual a set of measure zero excepted), then the iterated fine limit of $v/u_1u_2u_3$ exists almost everywhere and is the Radon-Nikodym derivative as in the above theorem. If, further, the same holds for different orders of iteration then the various iterated limits are equal $(\mu_1 \times \mu_2 \times \mu_3)$-almost everywhere.*

Remark. If v is a 3-superharmonic function with a 3-harmonic minorant w, $w \leq 0$ everywhere, then similar results are true for v. We note that $f_v = f_{(v-w)} + f$; here we have equality (and not '$\leq$') since at every stage the limits exist for w/u_1, etc.

5. Functions on the polydisc. Let us now consider a polydisc U^n, $n \geq 2$, and functions belonging to the Nevanlinna class $N(U^n)$. For any $f \in N(U^n)$, the function $\log^+|f|$ is a positive n-subharmonic function and has a positive n-harmonic majorant; see, for instance, [10]. We shall apply our earlier results on the existence and equality of iterated fine limits to the function $-\log|f|$ and deduce the corresponding conclusions for f. To do this, first of all we observe that T, the unit circle, is precisely the Martin boundary of U (for the system of harmonic functions satisfying the Laplace equation); and the Martin boundary consists entirely of minimal functions; viz., T is also the minimal boundary (belonging to the compact base of positive harmonic functions taking the value 1 at the origin). Let l denote the linear normalized (i.e., $l(T) = 1$) Lebesgue measure on T. Then, l is the canonical measure corresponding to the harmonic function 1. For the sequel, l_k will stand for the completion of the product measure $l \times l \times \cdots \times l$ k-times on T^k. Before proceeding to give our proof, we recall below three results which we use repeatedly.

Theorem Z [12, Theorem 5.16, Chapter II]. *Let $f \in N(U^n)$, $n \geq 2$. Then, there exists a set $E \subset T$ such that $l(T - E) = 0$ and, for every $t \in E$, as z tends to t nontangentially, $|z| < 1$, the limit of $f(z, z^2, \cdots, z^n)$ exists for every $(z^2, \cdots, z^n) \in U^{n-1}$; and the convergence is uniform on every compact subset of U^{n-1}. Further the function $f(t, \cdot)$ belongs to $N(U^{n-1})$ for every $t \in E$.*

Theorem B and D [3, Theorem 9]. *Let X be a metric space and f: $U \mapsto X$. Let, for every t belonging to an l-measurable set E, the limit of $f(z)$ exists in X as z, $|z| < 1$, tends to t nontangentially. Then, for l-almost every $t \in E$, the fine limit of $f(z)$ as z tends to t exists and equals the nontangential limit at this point.*

Theorem C and C [5, p. 62]. *Let f be a holomorphic function on U. Let $t \in T$ be such that the fine limit of $f(z)$ as z tends to t equals a. Then, as z tends to t nontangentially, $f(z)$ tends to the limit a.*

We shall now consider the iterated limits of f. As before, we consider the natural order of iteration.

Theorem 10. *Let $f \in N(U^n)$, $n \geq 2$. Then, for every integer k, $1 \leq k \leq n-1$, the kth iterated fine limit of f exists for every $t \in E_k \subset T^k$ and every $(z^{k+1}, z^{k+2}, \cdots, z^n) \in U^{n-k}$ such that*

(1) *the complement of E_k is of l_k measure zero,*

(2) *for every fixed $(z^{k+1}, \cdots, z^n) \in U^{n-k}$, the iterated limit is a l_k-measurable function,*

(3) *for every $t \in E_k$, the limit is holomorphic belonging to the Nevanlinna class of U^{n-k} and*

(4) *the (above) iterated fine limit equals the kth iterated nontangential limits (independent of $(z^{k+1}, \cdots, z^n)$ in U^{n-k}).*

Proof. Let us first prove the result for the case $k = 1$. We consider the mapping ϕ: $U \mapsto \mathcal{H}(U^{n-1})$, the space of holomorphic functions on U^{n-1}, defined by $z \mapsto f(z, \cdot)$. The space $\mathcal{H}(U^{n-1})$ is separable and complete metrisable. The Theorem Z states precisely that $\phi(z)$ tends to a limit in this space (in fact in $N(U^{n-1})$), as z approaches nontangentially almost every point of T. Hence, from Theorem B and D, we deduce the existence of the fine limit for almost every $t \in T$, satisfying (3) and (4). Now, (2) is an immediate consequence of Theorem 1.

Let us now assume that the result is true for all integers $1, 2, \cdots, k$, $k < n-1$; we shall show the validity for $k+1$. From Theorem 3, for $\epsilon = 1/m$, $m = 1, 2, 3, \cdots$, we can find a compact set $C_m \subset E_k$ such that (i) $l_k(E_k - C_m) < 1/m$ and, restricted to C_m, the mapping $t \mapsto f_t$ of $C_m \mapsto \mathcal{H}(U^{n-k})$ is continuous, where $f_t(z^{k+1}, \cdots, z^n)$ is the kth iterated fine limit (or nontangential limit) of f at t. This, in particular, implies the separate continuity of the mapping $(\tau, z^{k+1}) \mapsto f_\tau(z^{k+1}, \cdot)$ of $C_m \times U \mapsto \mathcal{H}(U^{n-k-1})$. (The spaces of holomorphic functions are provided with the topology of uniform convergence on the compact subsets of the corresponding sets.) If $E_{k,m}$ is the set of all (τ, t) belonging to $C_m \times T$ such that fine limit of $f_\tau(z^k, \cdot)$ exists in $\mathcal{H}(U^{n-k-1})$ as z^k tends to t, then, from Theorem 2, we deduce that $E_{k,m}$ is l_k-measurable. However, for every $\tau \in C_m$, since $f_\tau \in N(U^{n-k})$, we deduce from Theorem Z and Theorem B and D, as in the case

$k = 1$, that the section through τ of $E_{k,m}$ is of full l-measure. Now, from Fubini's theorem, we get that $l_{k+1}(E_{k,m}) = l_k(C_m) > l_k(E_k) - 1/m = 1 - 1/m$. Now, let $E_{k+1} = \bigcup E_{k,m}$. Then, $l_{k+1}(E_{k+1}) = 1 = l_{k+1}(T^{k+1})$ and for every $(\tau, t) \in E_{k+1}$, the fine limit of $f(z^{k+1}, \cdot)$ exists in $N(U^{n-k-1})$ as z^{k+1} tends to t. Once again (2) is an immediate consequence of Theorem 1. Also, property (4) of the above limit is a consequence of Theorem C and C.

The proof is complete.

Theorem 11. *Let $f \in N(U^n)$. The nth iterated fine limit of f exists for l_n-almost every element of T^n; and this iterated limit is an l_n-measurable function. Further, the nth iterated nontangential limit of f equals the iterated fine limit l_n-almost everywhere.*

Proof. Let E_{n-1} be the set furnished by the above theorem and $f_\tau(z^n)$ the $(n-1)$th iterated fine (= nontangential) limit of f at $\tau \in E_{n-1}$. As before, by Theorem 3, we can find a compact set $K_m \subset E_{n-1}$, for every $m = 2, 3, \cdots$, such that $l_{n-1}(K_m) > l_{n-1}(E_{n-1}) - 1/m = 1 - 1/m$, and $\tau \mapsto f_\tau(\cdot)$ is a continuous map from $K_m \mapsto \mathcal{H}(U)$. Once again, by Theorem 2, the sets F_m,

$$F_m = \{(\tau, t) \in K_m \times T\colon \text{fine lim } f_\tau(z^n) \text{ exists as } z^n \to t\}$$

are l_n-measurable. But, for every $\tau \in E_{n-1}$, since $f_\tau(\cdot) \in N(U)$, for almost every $t \in T$, both the fine and the nontangential limits of $f_\tau(z)$ exist and are equal as z tends to t. In particular $l_n(F_m) = l_{n-1}(K_m)\cdot l(T) > 1 - 1/m$ (Fubini's theorem). Let F be the union of F_m, for $m = 2$ to ∞. Then $l_n(F) = 1$. Further, for every $\tau \in F$ the nth iterated fine and nontangential limits of f exist and are equal. Also we deduce from Theorem 4 that the iterated limit is l_n-measurable. The theorem is proved.

Theorem 12. *Let $f \in N(U^n)$. Then, there is a set $E \subset T^n$ of l_n measure zero such that, if g_1 and g_2 are two iterated fine (or nontangential) limit functions, for two different orders of iteration, then $|g_1| = |g_2|$ outside E.*

Proof. The function $-\log|f|$ is n-superharmonic on U^n and let w be the greatest n-harmonic minorant of $-\log|f|$. Since w majorises the negative (or zero) n-harmonic function which is the greatest minorant of $-\log^+|f|$, we conclude that w is the difference of two positive n-harmonic functions. By Corollary 1 to Theorem 6, we know that the nth iterated fine limits of w exist and are all identically l_n-almost everywhere. Further, the kth iterated limit of w is $(n-k)$-harmonic, in particular, continuous on U^{n-k}. We know that the kth iterated fine limit of $-\log|f|$ exists for all $k = 1$ to n and it is easy to see that, for every k between 1 and $n-1$, the kth iterated limit is a $(n-k)$-superharmonic function on U^{n-k}. Hence the same is true for $(-\log|f| - w)$. However, $(-\log|f| - w)$ is a positive

n-superharmonic function with greatest n-harmonic minorant zero and hence by Corollary 1 to Theorem 9, we deduce that the iterated fine limit of $(-\log|f| - w)$ is zero l_n-almost everywhere, whatever be the iteration order. We deduce that, with the exception of a set of l_n measure zero, the different iterated fine limits of $\log|f|$ are identical. The proof is complete.

Corollary. *For a $f \in N(U^n)$, if the nth iterated fine or nontangential limit is zero on a set of positive l_n measure, then $f \equiv 0$.*

Proof. From the above theorem we deduce that if $f \not\equiv 0$, then the iterated fine or nontangential limit of $\log|f|$ is finite l_n-almost everywhere. The Corollary follows.

Remark. The equality of the iterated limits could be proved for functions $f \in N(U^n)$ such that f has no zeroes in U^n.

Theorem 13. *Let $f \in N_1(U^n)$; viz., $\int \log^+|f| \{\log^+\log^+|f|\}\, d\theta_1 \cdots d\theta_n$ is bounded independent of $r_1, r_2, \cdots, r_n$ between 0 and 1. Then, the different iterated (fine or) nontangential limits of f are equal l_n-almost everywhere on T^n.*

Proof. For every function $g \in N_1(U^2)$, the two iterated nontangential limits of g are equal l_2-almost everywhere [12, p. 328]. We shall prove the theorem by induction on n.

Now, $x \mapsto x \log^+ x$ is an increasing convex function on $[0, \infty)$ (in fact, it is a strongly convex function). Hence, $(\log^+|f|)(\log^+\log^+|f|)$ is a n-subharmonic function on U^n. Hence, $\int \log^+|f|\, (\log^+\log^+|f|)\, d\theta_1 \cdots d\theta_n$ is increasing in each r_j. Since f is in $N_1(U^n)$ there is an upper bound $M > 0$ for these integrals. Suppose that except for a set E_k of l_k measure zero, some kth iterated nontangential limit of f exists for every $\tau \in T^k - E_k$. Then, using the fact the radial limit exists at every stage and by repeated application of Fatou's lemma, we deduce that

$$\int l_k(d\tau) \int \cdots \int \log^+|f^\tau(r_{k+1}e^{i\theta_{k+1}}, \cdots, r_n e^{i\theta_n})| \log^+\log^+|f^\tau(r_{k+1}e^{i\theta_{k+1}}, \cdots, r_n e^{i\theta_n})|\, d\theta_{k+1} \cdots d\theta_n \leq M.$$

Using the increasing nature of $\int \cdots \int \log^+|f^\tau|\, (\log^+\log^+|f^\tau|)\, d\theta_{k+1} \cdots d\theta_n$, we deduce easily that, for l_k almost every element τ of $T^k - E_k$, $f \in N_1(U^{n-k})$. (Observe that f may be assumed to belong to $N(U^{n-k})$ by Theorem 10.) This is true whatever be $k < n$.

Now, assume that, for all $k = 1, 2, \cdots, n$ $(n \geq 2)$, whatever be $g \in N_1(U^k)$, the kth iterated fine limits of g are all equal l_k-almost everywhere. We shall show that, for any $g \in N_1(U^{n+1})$, a similar result holds to be good. To prove this consider the iterated limits g_1 of g for the natural order and g_2 for an order σ which ends with say the kth variable $k < n + 1$. We can choose a countable dense subset, say (z_m^k) contained in U (for the kth variable) such that, for every z_m^k, the nth iterated fine (and nontangential) limits of f exist except for a set of measure zero on T^n and independent of the order of iteration. But then, by Theorem 10, this limit could be so chosen that it is holomorphic in the kth variable on U. Hence, the limit could be chosen, independent of the order of iteration and $z^k \in U$, except for a set of l_n measure zero on T^n. Hence we may assume that, in the above σ, the all but the last one limit is taken in the $(n+1)$th variable, i.e. g_2 is the iterated limit where all but the last two variables are in that order z^{n+1} and z^k. Similarly, we can suppose that g_1 is obtained by the iterated limit order τ, where all but the last two variables (in that order) in which the limits are taken are precisely z^k and z^{n+1}. Again, by an argument similar to the above and the induction hypothesis, we get that with a set of measure zero excepted on T^{n-1}, for every $t \in T^{n-1}$, the $(n-1)$th iterated limit functions $h^t(\tau, z^k, z^{n+1})$ and $h^t(\sigma, z^k, z^{n+1})$ following respectively τ and σ satisfy:

$$h^t(\tau, z^k, z^{n+1}) = h^t(\sigma, z^k, z^{n+1}) \quad \text{for every } (z^k, z^{n+1}) \in U^2$$

and this common function belongs to $N_1(U^2)$. Now, the two iterated limits of h^t (as z^{n+1} tends to the boundary nontangentially and then z^k tends to the boundary nontangentially and vice versa) are equal l_2-almost everywhere. Now, since the $(n+1)$th iterated nontangential limits are measurable, we deduce that $g_1 = g_2$ l_{n+1}-almost everywhere. The proof is complete.

Remark. If, for every $f \in N(U^2)$, the two iterated nontangential limits are equal l_2-almost everywhere, then the result is true in general for any function in $N(U^n)$. An exactly similar proof based on induction holds to be good. Probably the result is true for functions belonging to $N(U^2)$.

BIBLIOGRAPHY

1. M. Brelot, *Lectures on potential theory*, Lectures on Math., no. 19, Tata Institute of Fundamental Research, Bombay, 1960. MR 22 #9749.

2. ———, *Séminaire de théorie du potentiel.* II, Institut Henri Poincaré, Paris, 1958.

3. M. Brelot and J. L. Doob, *Limites angulaires et limites fines*, Ann. Inst. Fourier (Grenoble) 13 (1963), fasc. 2, 395–415. MR 33 #4299.

4. A. P. Caledrón and A. Zygmund, *Contributions to Fourier analysis*, Ann. of Math. Studies, no. 25, Princeton Univ. Press, Princeton, N. J., 1950, pp. 145–165. MR 12, 255.

5. C. Constantinescu and A. Cornea, *Über das Verhalten der analytischen Abbildungen Riemannscher Flächen auf dem idealen Rand von Martin*, Nagoya Math. J. 17 (1960), 1–87. MR 23 #A1025.

6. K. Gowrisankaran, *Extreme harmonic functions and boundary value problems*, Ann. Inst. Fourier (Grenoble) 13 (1963), fasc. 2, 307–356. MR 29 #1350.

7. ———, *Fatou-Naïm-Doob limit theorems in the axiomatic system of Brelot*, Ann. Inst. Fourier (Grenoble) 16 (1966), fasc. 2, 455–467. MR 35 #1802.

8. ———, *Multiply harmonic functions*, Nagoya Math. J. 28 (1966), 27–48. MR 35 #410.

9. R.-M. Hervé, *Recherches axiomatiques sur la théorie des fonctions surharmoniques et du potentiel*, Ann. Inst. Fourier (Grenoble) 12 (1962), 415–571. MR 25 #3186.

10. W. Rudin, *Function theory in polydiscs*, Benjamin, New York, 1969. MR 41 #501.

11. L. Schwartz, *Radon measures on general topological spaces*, Tata Institute of Fundamental Research Monographs (to appear).

12. A. Zygmund, *Trigonometrical series*. Vol. 2, 2nd rev. ed., Cambridge Univ. Press, New York, 1959. MR 21 #6498.

DEPARTMENT OF MATHEMATICS, McGILL UNIVERSITY, P.O. BOX 6070, MONTREAL 101, QUEBEC, CANADA