ON TRIGONOMETRIC SERIES ASSOCIATED WITH
SEPARABLE, TRANSLATION INVARIANT SUBSPACES OF $L^\infty(G)$ (1)

BY
RON C. BLEI

ABSTRACT. $G$ denotes a compact abelian group, and $\Gamma$ denotes its dual.
Our main result is that every non-Sidon set $E \subset \Gamma$ contains a non-Sidon set $F$
such that $L^\infty_F(G) = \bigoplus_{i=1}^{\infty} C^\infty_F_i(G)$, where the $F_i$'s are finite, mutually disjoint,
and $\bigcup_{i=1}^{\infty} F_i = F$.

0. Introduction. Let $G$ be a locally compact abelian group with Haar measure $dx$, and let $\Gamma$ denote its dual with Haar measure $dy$. $A(G)$ denotes the Gel-
fand representation of the convolution algebra $L^1(\Gamma) = L^1(\Gamma, dy)$. $A(G)$ is a
dense subspace of $C^\infty_0(G)$, the Banach algebra of all continuous functions on $G$
which vanish at infinity. $M(G)$ denotes the convolution algebra of all complex
valued finite regular measures on $G$. If $\mu \in M(G)$, then the Fourier-Stieltjes transform of $\mu$
is the uniformly continuous function on $\Gamma$ defined by
$$\tilde{\mu}(\gamma) = \int_G (x, -\gamma) d\mu(x), \quad \gamma \in \Gamma.$$  

We let $G_d$ denote the abelian group $G$ endowed with the discrete topology.
$(G_d)^\wedge$, the dual group of $G_d$, is denoted by $\overline{\Gamma}$, the Bohr compactification of $\Gamma$.
$\Gamma$ is dense on $\overline{\Gamma}$, and $C(\Gamma)$ can be naturally identified with the almost periodic
functions on $\Gamma$.

Let $E \subset \Gamma$ and let $I(E) = \{f \in A(\Gamma) : f = 0 \text{ on } E\}$. We set $A(E) = A(\Gamma)/I(E),$ 
where the quotient is the usual Banach algebra quotient. An element of $A(E)$ may
be viewed as the restriction to $E$ of a function in $A(\Gamma)$. Clearly, $A(E) \subset C^\infty_0(E)$;
$A(E)$ is dense in $C^\infty_0(E)$, and $\|g\|_{A(E)} \geq \|g\|_\infty$ for all $g \in A(E)$. When $A(E) = C^\infty_0(E)$,
and $\Gamma$ is a nondiscrete group, we say that $E$ is a Helson set; when $\Gamma$ is a dis-
crete group, we say that $E$ is a Sidon set (cf. [7, Chapter 5]). We define the Hel-
son (Sidon) constant of a set $E \subset \Gamma$ to be
$$b(E) = \inf \{\|f\|_\infty / \|f\|_{A(E)} : f \in A(E), f \neq 0\}.$$
Clearly, $E$ is a Helson (Sidon) set if and only if $h(E) > 0$.

When $E \subset \Gamma$, $\Gamma$ a discrete group, and $B(G)$ is any subspace of $L^1(G)$, we set

$$B^*(E) = \{ f \in B(G) : \hat{f} = 0 \text{ outside } E \}.$$ 

We consider the following inclusion,

$$L^\infty_E(G) \supseteq C_E(G) \supseteq A_E(G).$$

It is easy to see that $L^\infty_E(G) = A_E(G)$ if and only if $E$ is a Sidon set. Also, if $C_E(G) = A_E(G)$, then $E$ is a Sidon set.

Non-Sidon sets $E \subset \mathbb{Z}$ such that $L^\infty_E(T) = C_E(T)$ (we refer to such sets as R-sets) were first constructed by Rosenthal [5], who subsequently conjectured that every non-Sidon $E \subset \mathbb{Z}$ contained a non-Sidon set $F$ such that $L^\infty_F(T) = C_F(T)$. In §2 we prove the above conjecture where $E$ is any non-Sidon set in any discrete abelian group (Corollary 2.4). In what follows below, we search for non-Sidon sets $E \subset \Gamma$ for which there exists a partition, $\{ F_j \}_{j=1}^\infty$, so that the $F_j$'s are finite, $F_i \cap F_j = \emptyset$ if $i \neq j$, $\bigcup_{i=1}^\infty F_i = E$ and $C_E(G)$ is isomorphic to $\bigoplus_{i=1}^\infty C_{F_i}(G)$ (see Definition 1.1). Not every non-Sidon set can be so partitioned. Theorem 2.1 states that every non-Sidon set $E \subset \Gamma$ contains a non-Sidon set $F$, so that $F$ can be partitioned in the above sense. To prove 2.1, we first generalize a result of Katznelson and McGehee, and prove that every countable and closed non-Helson subset of a compact abelian group contains a non-Helson set which can be partitioned (Theorem 1.2). We then, using Drury's theorem (cf. [1]), reduce the problem of constructing partitions in discrete abelian groups to that of constructing partitions in compact abelian groups. Having established 2.1, we obtain Corollary 2.4.

1. Sup-norm additivity in LCA groups.

Definition 1.1. Let $G$ be a locally compact abelian group, and let $E$ be a countable subset of $G$. $E$ is said to be partitioned with respect to the supremum norm, if there exists a family of finite, mutually disjoint sets, $\{ F_j \}_{j=1}^N$, such that the following hold:

(i) $\bigcup_{j} F_j = E$.

(ii) There exists a constant $K > 0$ such that if $\mu \in M(E)$ and $\mu_j$ denotes the restriction of $\mu$ to $F_j$, then, given any $N > 0$,

$$\sum_{j=1}^{N} \| \mu_j \|_{\infty} \leq K \sum_{j=1}^{N} \| \hat{\mu}_j \|_{\infty}.$$ 

$\{ F_j \}$ is said to be a sup-norm partition of $E$.

We note that every countable Helson (Sidon) set can be partitioned in the above sense. The following theorem is a generalization of a result by Katznelson and McGehee (cf. [3, Theorem 3.1]):
Theorem 1.2. Let $G$ be a locally compact abelian group, and let $\Gamma = \hat{G}$. If $E \subset \Gamma$ is a non-Helson, countable and compact set with a finite number of accumulation points, then $E$ contains a non-Helson set $F$, such that $F$ can be partitioned with respect to the sup-norm.

The following lemma is a generalization of Lemma I of [8]:

Lemma 1.3. Given $\epsilon > 0$, and $T$, a compact symmetric neighborhood of 0 in $G$. Then, there exists $\Delta$, a compact neighborhood of 0 in $\Gamma$, such that if $\mu \in M(\Delta)$, then $|\hat{\mu}(g_1) - \hat{\mu}(g_2)| < \epsilon \|\hat{\mu}\|_\infty$ whenever $g_1 - g_2 \in T$.

Proof. We first show that there is $\Delta$, a compact neighborhood of 0 in $\Gamma$, such that

\begin{equation}
\|1 - (g, x)\|_{\Delta} < \epsilon \quad \text{whenever } g \in T.
\end{equation}

By a theorem of Wiener, if $g$ is fixed, then there exists $\Delta = \Delta g$, a compact neighborhood of 0 in $\Gamma$, such that $\|1 - (g, x)\|_{\Delta g} < \epsilon$. If $\{g_1, \ldots, g_n\}$ is a finite subset of $G$, then, letting $\Delta = \bigcap_{i=1}^n \Delta g_i$, we obtain

\begin{equation}
\|1 - (g_i, x)\|_{\Delta} < \epsilon \quad \text{for } i = 1, \ldots, n.
\end{equation}

Let $g_1 \in T$ be fixed, and let $\Delta g_1$ be a compact neighborhood of 0 so that $\|1 - (g_1, x)\|_{\Delta g_1} < \epsilon$. Then, by the definition of $A(\Delta g_1)$, there exists $f \in L^1(G)$ such that $\hat{f}(x) = 1 - (g_1, x)$ for $x \in \Delta g_1$ and $\|f\|_1 < \epsilon$. Now, choose $k \in L^1(G)$ so that $k = 1$ on $\Delta g_1$ and $k = 0$ outside a compact set; let $h = k * \delta_{g_1}$ (denoted the pt. mass measure at $g_1$). Clearly, $\hat{h}(x) = (g_1, x)$ on $\Delta g_1$. Since translation is continuous in $L^1(G)$, there exists $U$, an open neighborhood of 0 in $G$, such that

\begin{equation}
\|b_{g_2 - g_1} - h\|_{L^1(G)} < \epsilon - \|f\|_1 \quad \text{whenever } g_2 - g_1 \in U.
\end{equation}

But, for $x \in \Delta g_1$, we have

\[ (b_{g_2 - g_1} - h)(x) = (g_2 - g_1, x)\hat{h}(x) = (g_2 - g_1, x)(g_1, x) = (g_2, x). \]

If we now let $s = f - h_{g_2 - g_1} + b$, we obtain $\hat{s}(x) = 1 - (g_2, x)$ for $x \in \Delta g_1$, and by (3),

\[ \|s\|_{L^1(G)} \leq \|f\|_1 + \|b_{g_2 - g_1} - h\|_1 < \epsilon. \]

Therefore, $\|\hat{s}\|_{A(\Delta g_1)} < \epsilon$, and we finally deduce that

\[ \|1 - (g_2, x)\|_{A(\Delta g_1)} < \epsilon \quad \text{whenever } g_2 \in U + g_1. \]

Now, for each $g \in T$ we produce $U = U(g)$ as above. But, by the compactness of
T, there exist \(g_1, \ldots, g_n\) so that

\[
T \subset \bigcup_{i=1}^{n} (g_i + \mathcal{U}(g_i)).
\]

Let \(\Delta = \bigcap_{i=1}^{n} \Delta_{g_i}\), and, as in (2), we have \(\|1 - (g, x)\|_{A(\Delta)} < \epsilon\) for all \(g \in T\).

Having proved (1), we easily establish the conclusion of the lemma: If \(\mu \in M(\Delta)\) and \(g_2 - g_1 \in T\), then

\[
|\hat{\mu}(g_2) - \hat{\mu}(g_1)| \leq \|1 - (g_2 - g_1, x)\|_{A(\Delta)} \|\hat{\mu}\|_{\infty} \leq \epsilon \|\hat{\mu}\|_{\infty},
\]

and the lemma is proved. □

We proceed to establish generalizations of lemmas about finitely supported measures (cf. [4]).

**Definition 1.4.** A subset \(K\) of \(G\) is said to be relatively dense in \(G\) if finitely many translates of \(K\) cover \(G\).

**Lemma 1.5.** If \(\mathcal{U}\) is a symmetric neighborhood of 0 in \(\widehat{G}\), the Bohr compactification of \(G\), then \(\mathcal{U} \cap G\) is relatively dense in \(G\).

**Proof.** Claim. \(\bigcup_{g \in G} (g + \mathcal{U}) = \mathcal{G}\). For, if \(h \in \mathcal{G}\), then \(h + \mathcal{U}\) is a neighborhood of \(h\) in \(\mathcal{G}\). By the density of \(G\) in \(\mathcal{G}\), there exists \(g \in G\) such that \(g \in h + \mathcal{U}\). Therefore, by the symmetry of \(\mathcal{U}\), \(h \in g + \mathcal{U}\), and claim is proved.

Since \(\mathcal{G}\) is compact, we can find \(\{g_1, \ldots, g_N\}\) so that \(\bigcup_{i=1}^{N} (g_i + \mathcal{U}) = \mathcal{G}\). Therefore,

\[
\bigcup_{i=1}^{N} (g_i + (\mathcal{U} \cap G)) = G.
\] □

**Lemma 1.6.** Let \(F = \{x_1, \ldots, x_N\} \subset \Gamma\) (\(\widehat{G} = \Gamma\)), then \(b(F) \geq (1/N)^{\frac{1}{2}}\) (\(b(F) = \) Helson constant of \(F\)).

**Proof.** Suppose \(\mu \in M(F)\). Since \(F\) is a finite set, \(\mu \in C(\widehat{G})\). Therefore, by Plancherel's theorem,

\[
\|\hat{\mu}\|_{\infty} \geq \|\hat{\mu}\|_{L^2(\widehat{G})} = \left( \sum_{j=1}^{N} |\mu(\{x_j\})|^2 \right)^{\frac{1}{2}} \geq \left( \frac{1}{N^{\frac{1}{2}}} \right) \sum_{j=1}^{N} |\mu(\{x_j\})|. \quad \Box
\]

**Lemma 1.7.** Let \(F\) be a finite set in \(\Gamma\). Then, given \(\epsilon > 0\), there exists \(U\), a compact neighborhood of 0 in \(G\), such that if \(g \in G\), then every translate of \(U\) in \(G\) contains an element \(z\), so that

\[
|\hat{\mu}(z) - \hat{\mu}(g)| \leq \epsilon \|\hat{\mu}\|_{\infty} \quad \text{for all } \mu \in M(F).
\]
Proof. Let \( \mu \in M(F) \) be arbitrary. By Lemma 1.6, we have that

\[
|\hat{\mu}(g) - \hat{\mu}(z)| = \left| \sum_{j=1}^{N} \mu((x_j, g, x_j) - (z, x_j)) \right| \\
\leq N^{1/2} \|\hat{\mu}\|_{\infty} \sup_{1 \leq j \leq N} |1 - (g - z, x_j)|
\]

where \( N \) = number of elements of \( F \). But, \( V = \{ w : |1 - (w, x_j)| < \epsilon/N^{1/2}, j = 1, \ldots, N \} \) is a symmetric neighborhood of \( 0 \) in \( \mathbb{G} \). Therefore, by Lemma 1.5, \( V \cap G \) is relatively dense; i.e., there exist \( g_1, \ldots, g_k \in G \) so that \( \bigcup_{i=1}^{k} (g_i + V \cap G) = G \). Now, take any compact neighborhood of \( 0 \) in \( G \), say \( C \), and let \( U = \bigcup_{i=1}^{k} (g_i + C) \). It is easy to see that \( U \) satisfies the requirements of the lemma. \( \square \)

We are now ready to establish Theorem 1.2: Without loss of generality, we can assume that \( 0 \in \Gamma \) is the only accumulation point of \( E \). Therefore, if \( U \) is any neighborhood of \( 0 \) in \( \Gamma \), then \( U \cap E \) is a non-Helson set. We shall construct inductively a sup-norm partition, \( \{T_{ij}\}_{i=1}^{\infty} \), for a non-Helson subset of \( E \).

Let \( 1/500 > \epsilon > 0 \) be given, and let \( \langle \epsilon_j \rangle \) be a sequence of real numbers such that

\[
2 \sum_{j=1}^{\infty} \epsilon_j < \epsilon, \quad \text{and} \quad \epsilon_j > 0.
\]

Let \( F_1 \) be any finite subset of \( E \) so that \( b(F_1) < \epsilon_1 \); suppose that \( k \geq 2 \) and that \( F_1, \ldots, F_{k-1}, \) finite subsets of \( E \), were chosen. For each \( j \leq k - 1 \), there exists \( U_j \), a compact neighborhood of \( 0 \) in \( G \), such that if \( g \in G \), every translate of \( U_j \) in \( G \) contains an element \( z \) so that

\[
|\hat{\mu}(z) - \hat{\mu}(g)| \leq \epsilon_j \|\hat{\mu}\|_{\infty}, \quad \text{for all} \quad \mu \in M(F_j) \quad \text{(Lemma 1.7)}.
\]

By Lemma 1.3, there exists \( \Delta_k \), a compact neighborhood of \( 0 \) in \( \Gamma \), such that if \( \mu \in M(\Delta_k) \) and \( u_1 - u_2 \in U_1 + \cdots + U_{k-1} \), then

\[
|\hat{\mu}(u_1) - \hat{\mu}(u_2)| \leq \epsilon_k \|\hat{\mu}\|_{\infty}.
\]

We now select \( F_k \), a finite subset of \( \Delta_k \cap E \setminus \{0\} \), such that \( F_k \cap F_j = \emptyset \) for all \( j \leq k - 1 \), and \( b(F_k) < \epsilon_k \). Having completely described our selection process, we shall now prove that if \( \mu_j \in M(F_j) \), \( 1 \leq j \leq k \), then

\[
\left( \frac{1}{6} - \epsilon \right) \sum_{j=1}^{k} \|\hat{\mu}_j\|_{\infty} \leq \left\| \sum_{j=1}^{k} \hat{\mu}_j \right\|_{\infty}.
\]

Claim. The range of \( \{\hat{\mu}_j\} \) is \( (\epsilon \sum_{j=1}^{k} \|\hat{\mu}_j\|_{\infty}) \)-dense in range of \( \hat{\mu}_1 + \cdots + \text{range of} \ \hat{\mu}_k \). Consider \( \sum_{j=1}^{k} \hat{\mu}_j (z_j) \), where \( z_1, \ldots, z_k \) are arbitrary elements in \( G \).
Let $y_k = z_k$, and assume that $y_k, \ldots, y_{j+1}$ were picked for $0 < j \leq k - 1$. We select $y_j$ so that

$$y_j - y_{j+1} \in U_j \quad \text{and} \quad |\hat{\mu}_j(y_j) - \hat{\mu}_j(z_j)| \leq \epsilon_j \|\hat{\mu}\|_{\infty}.$$  

(The choice is possible by (1).) Having thus chosen $y_1, \ldots, y_k$, we have for $1 < j \leq k$,

$$y_1 - y_j = (y_1 - y_2) + \cdots + (y_{j-1} - y_j) \in U_1 + \cdots + U_{j-1}.$$  

Recalling that $\mu_j \in M(\Delta_j)$, we obtain by (2)

$$|\hat{\mu}_j(y_1) - \hat{\mu}_j(z_j)| \leq \epsilon_j \|\hat{\mu}_j\|_{\infty}.$$  

Combining (3) and (4), we have

$$|\hat{\mu}_j(y_1) - \hat{\mu}_j(z_j)| \leq 2\epsilon_j \|\hat{\mu}_j\|_{\infty}.$$  

Finally,

$$\left| \sum_{j=1}^{k} \hat{\mu}_j(y_1) - \sum_{j=1}^{k} \hat{\mu}_j(z_j) \right| \leq \epsilon \sum_{j=1}^{k} \|\hat{\mu}_j\|_{\infty},$$

and the claim is proved.

For each $j$, $\int_{C} \hat{\mu}_j(x) dx = 0$, since $\mu_j(010) = 0$. Therefore, $\text{Re} \hat{\mu}_j$ and $\text{Im} \hat{\mu}_j$ must both assume positive and negative values. It then easily follows that there exist $g_1, \ldots, g_k \in G$ so that

$$\left| \sum_{j=1}^{k} \hat{\mu}_j(g_j) \right| \geq \frac{1}{\delta} \sum_{j=1}^{k} \|\hat{\mu}_j\|_{\infty}.$$  

(\*) now follows from (5) and (6). Since $b(F_k) \to 0$ as $k \to \infty$, $F = \bigcup_{k=1}^{\infty} F_k \subset E$ is non-Helson, and the theorem is proved. \qed

We note that the above technique of producing sup-norm partitions of non-Helson subsets of countable sets in compact groups cannot be applied in the obvious way to subsets of discrete groups. For example, if $E$ is any subset of $\mathbb{Z}$, and $\epsilon > 0$, $\delta > 0$ are given, we cannot conclude that there exists $K$, a sufficiently large integer, for which all $\mu \in M(E \setminus [-K, K])$ are so that $|\hat{\mu}(t_1) - \hat{\mu}(t_2)| < \epsilon \|\hat{\mu}\|_{\infty}$ whenever $|t_1 - t_2| < \delta$.

2. Sup-norm additivity in discrete abelian groups.

\textbf{Theorem 2.1.} Let $\Gamma$ be an infinite discrete abelian group, and let $E \subset \Gamma$ be non-Sidon. Then, there exists $F \subset E$ such that $F$ can be partitioned with respect to the sup-norm, and $F$ is a non-Sidon set.

\textbf{Remark.} It is clear that $E \subset \Gamma$ is a Sidon set if and only if every countable subset of $E$ is a Sidon set. Therefore, we may assume without loss of generality
that $E$ is countable, and hence that $\Gamma$ is countable.

Since $\Gamma$ is countable, $\hat{\Gamma} = \hat{G}$ is a compact metrizable group (cf. [7, 2.2.6]), and therefore, there exists $D$, a countable, dense subgroup of $G$. Consider $D$ as a discrete abelian group, and let $\phi: \Gamma \rightarrow \hat{D}$ be the natural injective map:

$$\phi(\gamma) = (\gamma, d)$$

for $\gamma \in D$ and $d \in D$. We shall say that $F \subseteq \hat{D}$ is a Sidon set if $F$ is a Sidon set in $(\hat{D})_d$, $\hat{D}$ discretized.

**Lemma 2.2.** Let $\Gamma, \phi, D$ be as above. Then, $E \subseteq \Gamma$ is Sidon if and only if $\phi(E)$ is Sidon. Furthermore, $b(E) = b(\phi(E))$.

**Proof.** To prove the lemma it suffices to show that if $\{a_i\}_{i=1}^n$ is any finite set of complex numbers, then

$$\sup_{x \in G} \left| \sum_{i=1}^n a_i (\gamma_i, x) \right| = \sup_{y \in D} \left| \sum_{i=1}^n a_i (\phi(\gamma_i), y) \right|$$

where $\gamma_i \in E$, $i = 1, \ldots, n$, and $\hat{D} = (\hat{D})_d$, the Bohr compactification of $D$.

Since $D$ is dense in $\hat{D}$, we have

$$\sup_{y \in \hat{D}} \left| \sum_{i=1}^n a_i (\phi(\gamma_i), y) \right| = \sup_{x \in \hat{D}} \left| \sum_{i=1}^n a_i (\phi(\gamma_i), x) \right|.$$ 

But $D$ was chosen to be dense in $\hat{D}$, and hence it follows from the definition of $\phi$ that

$$\sup_{x \in \hat{D}} \left| \sum_{i=1}^n a_i (\phi(\gamma_i), x) \right| = \sup_{x \in D} \left| \sum_{i=1}^n a_i (\gamma_i, x) \right|.$$ 

(1) now follows, and lemma is proved. $\square$

**Lemma 2.3.** Let $\{\gamma_i\}_{i=1}^\infty = E \subseteq \Gamma$ be a non-Sidon set. Let $\phi$ be as in Lemma 2.2. Then, there exists $\{\gamma_{i_k}\}_{k=1}^\infty = F \subseteq E$, $F$ non-Sidon, and $\phi(F)$ (closure in $\hat{D}$) is a countable set with one accumulation point.

**Proof.** First, we claim that there exists $x_0 \in \phi(E)$ such that if $U$ is any open set containing $x_0$, then $\phi^{-1}(U) \cap E$ is a non-Sidon set. Suppose this were not so. Then, for each $x \in \phi(E)$ there exists an open set, $U_x$, so that $x \in U_x$, and $\phi^{-1}(U_x) \cap E$ is a Sidon set. By the compactness of $\phi(E)$, there exist $x_1, \ldots, x_n \in \phi(E)$, so that $\bigcup_{i=1}^n U_{x_i} \supset \phi(E)$. But, $E = \bigcup_{i=1}^n \phi^{-1}(U_{x_i}) \cap E$ is a Sidon set by Drury's theorem (cf. [1]).

Now let $\{V_n\}_{n=1}^\infty$ be a family of open sets so that $V_n \supsetneq V_{n+1}$ and $\bigcap_{n=1}^\infty V_n = \{x_0\}$. Let $F_1 \subseteq \phi^{-1}(V_1) \cap E$ be any finite set. We proceed inductively to select $F_n \subseteq \phi^{-1}(V_n) \cap E$, such that $F_n$ is finite, Sidon constant of $F_n < 1/n$, and $F_n \cap F_j = \emptyset$ for $j < n$. Let $F = \bigcup_{n=1}^\infty F_n$. The conclusion of the lemma easily follows. $\square$
We are now ready to prove Theorem 2.1: By the preceding lemma, produce $E' \subset E$ such that $E'$ is a non-Sidon set, and $\phi(E')$ is a countable set with one limit point. By Theorem 1.2, find $S \subset \phi(E')$ so that $S$ is non-Helson (therefore, also non-Sidon) such that $S$ can be partitioned with respect to the sup-norm. But,

$$\sup_{x \in G} \left| \sum_{i=1}^{n} a_i(y_i, x) \right| = \sup_{y \in D} \left| \sum_{i=1}^{n} a_i(\phi(y)_i, y) \right|$$

(see proof of Lemma 2.2), and it easily follows that $F = \phi^{-1}(S) \subset E$ can be partitioned with respect to the sup-norm. In fact, the inverse image under $\phi$ of the partition for $S$ is a partition for $F$.

**Corollary 2.4.** Let $\Gamma$ be an infinite discrete abelian group, and let $E \subset \Gamma$ be non-Sidon. Then, there exists $F \subset E$ so that $F$ is non-Sidon, and $F$ is an R-set.

**Proof.** Let $F \subset E$ be non-Sidon, so that $F$ can be partitioned with respect to the sup-norm via $\{F_n\}$. For each $N$, $k_N \in L^1(G)$ is chosen so that $\|k_N\|_1 \leq 2$, $\hat{k}_N$ has compact support, and $\hat{k}_N(y) = 1$ whenever $y \in \bigcup_{n=1}^{N} F_n$. Suppose $f \in L^\infty(G)$. Given any $N$, and letting $f_n = \sum_{y \in F_n} \hat{f}(y)(y, \cdot)$, we have that

$$\sum_{n=1}^{N} \|f_n\|_\infty = \sum_{n=1}^{N} \|k_N \ast f \|_\infty \leq C \|k_N \ast f \|_\infty \leq 2C \|f\|_\infty.$$ 

Since $N$ is arbitrary, $\sum_{n=1}^{\infty} \|f_n\|_\infty < \infty$, and it follows that $f \in C_F(G)$. \(\square\)

**Remark.** When $\Gamma = \mathbb{Z}$, the mapping $\phi$ as defined in the remark preceding Lemma 2.2 can be realized as follows: Let $\alpha$ be a given irrational number in $T$. Let $\phi_{\alpha} = \phi$ be the map from $\mathbb{Z}$ into $T$ so that $\phi(n) = n\alpha$ mod $2\pi$. As in Lemma 2.3, if $E \subset \mathbb{Z}$ is a non-Sidon set, then there exists $x_0 \in \phi(E)^-$ (where closure is taken in the usual topology of $T$) such that if $U$ is any neighborhood of $x_0$, then $\phi^{-1}(U) \cap E$ is non-Sidon. We then proceed, as in Theorem 2.1 to construct $F \subset E$ non-Sidon, so that $F$ can be partitioned with respect to the sup-norm. It seems natural to ask whether the above technique of "wrapping" subsets of integers in the circle group can be used to explore other structural properties of $\mathbb{Z}$ by investigating their analogues on $T$.

It is clear that not every closed subset of $T$ can be realized in the form of $\{n_{j_{\alpha}}\}^\infty_{j=1}$, where $\{n_{j}\}^\infty_{j=1} \subset \mathbb{Z}$, and $\alpha$ is an irrational number (e.g., closed independent sets in $T$). It is an open question whether whenever $E \subset \mathbb{Z}$ is Sidon, $\overline{E}$ is a Helson set in $\overline{\mathbb{Z}}$ ($\overline{E}$ is closure of $E$ in the Bohr compactification of $\mathbb{Z}$). It turns out that when we "close" $E$ in $T$, we are in a somewhat less complicated situation than the one where we form $\overline{E}$ in $\mathbb{Z}$. If $\alpha$ is a fixed irrational number, then there exist Sidon sets, $\{n_{j}\}^\infty_{j=1} \subset \mathbb{Z}$, so that $\{n_{j}\}^\infty_{j=1} \subset T$: Arrange the rationals in $T$ in a sequence, $\{r_{j}\}^\infty_{j=1}$ in such a way that every rational occurs infinitely many times in
the arrangement. Now suppose $n_1, \ldots, n_J$ were picked for $J > 1$. It is clear that
$E_J = \{an: n > 3n_J\}$ is dense in $T$. Select $n_{j+1} > 3n_J$ so that $\|an_{j+1} - r_{j+1}\| < 1/J$. It is now clear that $\{an_j\}_{j=1}^\infty$ is dense in $T$. Going in the other direction, we produce a Sidon set, $E \subset \mathbb{Z}$, so that $\{aE\}^\omega = T$ for almost all $a$ in $T$: Let $S = \{(\epsilon_1, \ldots, \epsilon_N): N \in \mathbb{Z}, \epsilon_i = 0, 1\}$. Fix $\gamma \in S$, and set $F_\gamma = \{a \subset T: \gamma \text{ does not occur anywhere in the binary expansion of } a\}$. It is easy to check that $m(F_\gamma) = 0$, and hence $m(\bigcup_{\gamma \in S} F_\gamma) = 0$. But if we let $E = \{2j\}_{j=1}^\infty$, then for all $a \in \bigcup_{\gamma \in S} F_\gamma$, $(aE)^\omega = T$.

3. Open questions. 1. Let $U(T) = \{f \in C(T): \hat{S}_n(f) = \Sigma_{-n}^n \hat{f}(n)e^{int} \text{ converge uniformly to } f\}$. Figà-Talamanca constructed in [2] a non-Sidon set $E \subset \mathbb{Z}$ such that $U_E(T) = C_E(T)$. Therefore, by Corollary 2.4 we can produce a non-Sidon set such that $L^\infty_E(T) = U_E(T)$. In fact, our methods show that every non-Sidon set $E \subset \mathbb{Z}$ contains a non-Sidon set $F$ for which there exists $\langle n_k \rangle$ such that $S_{n_k}f$ converge uniformly to $f$, for all $f \in L^\infty_E(T)$. Given a non-Sidon set $E \subset \mathbb{Z}$, does there exist $F \subset E$, $F$ non-Sidon, and $U_F(T) = L^\infty_F(T)$?

2. We note that $L^\infty_E(G) = C_E(G)$ if and only if $L^\infty_E(G)$ is separable. Can an $R$-set $E \subset \mathbb{Z}$ be constructed so that $E$ cannot be partitioned with respect to the sup-norm?

I would like to thank Professor O. C. McGehee, my adviser, and Professor H. P. Rosenthal for their patient guidance and good counsel. Also, I thank Professors S. Ebenstein and M. Zippin for teaching me much of the little mathematics that I know.

REFERENCES

3. Y. Katznelson and O. C. McGehee, Measures and pseudomeasures on compact subsets of the line, Math. Scand. 23 (1968), 57–68. MR 40 #4688.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CONNECTICUT, STORRS, CONNECTICUT 06268

Current address: Istituto di Matematica, Università di Genova, Genova, Italy