TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 173, November 1972

ON TRIGONOMETRIC SERIES ASSOCIATED WITH SEPARABLE, TRANSLATION INVARIANT SUBSPACES OF $L^\infty(G)$ (1)

BY

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ABSTRACT. G denotes a compact abelian group, and Γ denotes its dual. Our main result is that every non-Sidon set $E \subset \Gamma$ contains a non-Sidon set F such that $L^\infty_F(G) = \bigoplus_{l^1} {}_{i=1}^{\infty} C_{F_i}(G)$, where the F_i's are finite, mutually disjoint, and $\bigcup_{i=1}^\infty F_i = F$.

0. **Introduction.** Let G be a locally compact abelian group with Haar measure dx, and let Γ denote its dual with Haar measure dy. $A(G)$ denotes the Gelfand representation of the convolution algebra $L^1(\Gamma) = L^1(\Gamma, dy)$. $A(G)$ is a dense subspace of $C_0(G)$, the Banach algebra of all continuous functions on G which vanish at infinity. $M(G)$ denotes the convolution algebra of all complex valued finite regular measures on G. If $\mu \in M(G)$, then the Fourier-Stieltjes transform of μ is the uniformly continuous function on Γ defined by

$$\hat{\mu}(y) = \int_G (x, -y)\, d\mu(x), \qquad y \in \Gamma.$$

We let G_d denote the abelian group G endowed with the discrete topology. $(G_d)\hat{}$, the dual group of G_d, is denoted by $\overline{\Gamma}$, the Bohr compactification of Γ. Γ is dense on $\overline{\Gamma}$, and $C(\overline{\Gamma})$ can be naturally identified with the almost periodic functions on Γ.

Let $E \subset \Gamma$ and let $I(E) = \{f \in A(\Gamma): f = 0 \text{ on } E\}$. We set $A(E) = A(\Gamma)/I(E)$, where the quotient is the usual Banach algebra quotient. An element of $A(E)$ may be viewed as the restriction to E of a function in $A(\Gamma)$. Clearly, $A(E) \subset C_0(E)$; $A(E)$ is dense in $C_0(E)$, and $\|g\|_{A(E)} \geq \|g\|_\infty$ for all $g \in A(E)$. When $A(E) = C_0(E)$, and Γ is a nondiscrete group, we say that E is a Helson set; when Γ is a discrete group, we say that E is a Sidon set (cf. [7, Chapter 5]). We define the Helson (Sidon) constant of a set $E \subset \Gamma$ to be

$$h(E) = \inf\{\|f\|_\infty / \|f\|_{A(E)}: f \in A(E), f \neq 0\}.$$

Received by the editors December 16, 1971.

AMS (MOS) subject classifications (1969). Primary 4250, 4252; Secondary 4655.

Key words and phrases. Sidon set, Helson set, sup-norm partition.

(1) The results in this paper constitute a portion of the author's doctoral dissertation at the University of California, Berkeley, written under the direction of Professor O. C. McGehee.

Clearly, E is a Helson (Sidon) set if and only if $h(E) > 0$.

When $E \subset \Gamma$, Γ a discrete group, and $B(G)$ is any subspace of $L^1(G)$, we set

$$B_E(G) = \{f \in B(G): \hat{f} = 0 \text{ outside } E\}.$$

We consider the following inclusion,

$$L_E^\infty(G) \supseteq C_E(G) \supseteq A_E(G).$$

It is easy to see that $L_E^\infty(G) = A_E(G)$ if and only if E is a Sidon set. Also, if $C_E(G) = A_E(G)$, then E is a Sidon set.

Non-Sidon sets $E \subset \mathbf{Z}$ such that $L_E^\infty(T) = C_E(T)$ (we refer to such sets as R-sets) were first constructed by Rosenthal [5], who subsequently conjectured that every non-Sidon $E \subset \mathbf{Z}$ contained a non-Sidon set F such that $L_F^\infty(T) = C_F(T)$. In §2 we prove the above conjecture where E is any non-Sidon set in any discrete abelian group (Corollary 2.4). In what follows below, we search for non-Sidon sets $E \subset \Gamma$ for which there exists a partition, $\{F_j\}_{j=1}^\infty$, so that the F_j's are finite, $F_i \cap F_j = \varnothing$ if $i \neq j$, $\bigcup_{i=1}^\infty F_i = E$ and $C_E(G)$ is isomorphic to $\bigoplus_{l^1} {}_{i=1}^{\infty} C_{F_i}(G)$ (see Definition 1.1). Not every non-Sidon set can be so partitioned. Theorem 2.1 states that every non-Sidon set $E \subset \Gamma$ contains a non-Sidon set F, so that F can be partitioned in the above sense. To prove 2.1, we first generalize a result of Katznelson and McGehee, and prove that every countable and closed non-Helson subset of a compact abelian group contains a non-Helson set which can be partitioned (Theorem 1.2). We then, using Drury's theorem (cf. [1]), reduce the problem of constructing partitions in discrete abelian groups to that of constructing partitions in compact abelian groups. Having established 2.1, we obtain Corollary 2.4.

1. Sup-norm additivity in LCA groups.

Definition 1.1. Let G be a locally compact abelian group, and let E be a countable subset of G. E is said to be partitioned with respect to the supremum norm, if there exists a family of finite, mutually disjoint sets, $\{F_j\}$, such that the following hold:

(i) $\bigcup_j F_j = E$.

(ii) There exists a constant $K > 0$ such that if $\mu \in M(E)$ and μ_j denotes the restriction of μ to F_j, then, given any $N > 0$,

$$\sum_{j=1}^{N} \|\hat{\mu}_j\|_\infty \leq K \left\| \sum_{j=1}^{N} \hat{\mu}_j \right\|_\infty .$$

$\{F_j\}$ is said to be a sup-norm partition of E.

We note that every countable Helson (Sidon) set can be partitioned in the above sense. The following theorem is a generalization of a result by Katznelson and McGehee (cf. [3, Theorem 3.1]):

Theorem 1.2. *Let G be a locally compact abelian group, and let $\Gamma = \hat{G}$. If $E \subset \Gamma$ is a non-Helson, countable and compact set with a finite number of accumulation points, then E contains a non-Helson set F, such that F can be partitioned with respect to the sup-norm.*

The following lemma is a generalization of Lemma I of [8]:

Lemma 1.3. *Given $\epsilon > 0$, and T, a compact symmetric neighborhood of 0 in G. Then, there exists Δ, a compact neighborhood of 0 in Γ, such that if $\mu \in M(\Delta)$, then $|\hat{\mu}(g_1) - \hat{\mu}(g_2)| < \epsilon \|\hat{\mu}\|_\infty$ whenever $g_1 - g_2 \in T$.*

Proof. We first show that there is Δ, a compact neighborhood of 0 in Γ, such that

$$\|1 - (g, x)\|_{A(\Delta)} < \epsilon \quad \text{whenever } g \in T. \tag{1}$$

By a theorem of Wiener, if g is fixed, then there exists $\Delta = \Delta_g$, a compact neighborhood of 0 in Γ, such that $\|1 - (g, x)\|_{A(\Delta_g)} < \epsilon$. If $\{g_1, \cdots, g_n\}$ is a finite subset of G, then, letting $\Delta = \bigcap_{i=1}^n \Delta_{g_i}$, we obtain

$$\|1 - (g_i, x)\|_{A(\Delta)} < \epsilon \quad \text{for } i = 1, \cdots, n. \tag{2}$$

Let $g_1 \in T$ be fixed, and let Δ_{g_1} be a compact neighborhood of 0 so that $\|1 - (g_1, x)\|_{A(\Delta_{g_1})} < \epsilon$. Then, by the definition of $A(\Delta_{g_1})$, there exists $f \in L^1(G)$ such that $\hat{f}(x) = 1 - (g_1, x)$ for $x \in \Delta_{g_1}$ and $\|f\|_1 < \epsilon$. Now, choose $k \in L^1(G)$ so that $\hat{k} = 1$ on Δ_{g_1} and $\hat{k} = 0$ outside a compact set; let $h = k * \delta_{g_1}$ (δ_{g_1} denoted the pt. mass measure at g_1). Clearly, $\hat{h}(x) = (g_1, x)$ on Δ_{g_1}. Since translation is continuous in $L^1(G)$, there exists $\mathfrak{U}$, an open neighborhood of 0 in G, such that

$$\|h_{g_2 - g_1} - h\|_{L^1(G)} < \epsilon - \|f\|_{L^1(G)} \quad \text{whenever } g_2 - g_1 \in \mathfrak{U}. \tag{3}$$

But, for $x \in \Delta_{g_1}$ we have

$$(h_{g_2 - g_1})\hat{\ }(x) = (g_2 - g_1, x)\hat{h}(x) = (g_2 - g_1, x)(g_1, x) = (g_2, x).$$

If we now let $s = f - h_{g_2 - g_1} + h$, we obtain $\hat{s}(x) = 1 - (g_2, x)$ for $x \in \Delta_{g_1}$, and by (3),

$$\|s\|_{L^1(G)} \le \|f\|_1 + \|h_{g_2 - g_1} - h\|_1 < \epsilon.$$

Therefore, $\|\hat{s}\|_{A(\Delta_{g_1})} < \epsilon$, and we finally deduce that

$$\|1 - (g_2, x)\|_{A(\Delta_{g_1})} < \epsilon \quad \text{whenever } g_2 \in \mathfrak{U} + g_1.$$

Now, for each $g \in T$ we produce $\mathfrak{U} = \mathfrak{U}(g)$ as above. But, by the compactness of

T, there exist $\{g_1, \cdots, g_n\}$ so that

$$T \subset \bigcup_{i=1}^{n} (g_i + \mathfrak{U}(g_i)). \tag{4}$$

Let $\Delta = \bigcap_{i=1}^n \Delta_{g_i}$, and, as in (2), we have $\|1 - (g, x)\|_{A(\Delta)} < \epsilon$ for all $g \in T$. Having proved (1), we easily establish the conclusion of the lemma: If $\mu \in M(\Delta)$ and $g_2 - g_1 \in T$, then

$$|\hat{\mu}(g_2) - \hat{\mu}(g_1)| \leq |(1 - (g_2 - g_1, x), \mu)|$$
$$\leq \|1 - (g_2 - g_1, x)\|_{A(\Delta)} \|\hat{\mu}\|_\infty \leq \epsilon \|\hat{\mu}\|_\infty,$$

and the lemma is proved. □

We proceed to establish generalizations of lemmas about finitely supported measures (cf. [4]).

Definition 1.4. A subset K of G is said to be relatively dense in G if finitely many translates of K cover G.

Lemma 1.5. *If U is a symmetric neighborhood of 0 in $\overline{G}$, the Bohr compactification of G, then $U \cap G$ is relatively dense in G.*

Proof. *Claim.* $\bigcup_{g \in G} (g + U) = \overline{G}$. For, if $h \in \overline{G}$, then $h + U$ is a neighborhood of h in $\overline{G}$. By the density of G in $\overline{G}$, there exists $g \in G$ such that $g \in h + U$. Therefore, by the symmetry of U, $h \in g + U$, and claim is proved.

Since $\overline{G}$ is compact, we can find $\{g_1, \cdots, g_N\}$ so that $\bigcup_{i=1}^N (g_i + U) = \overline{G}$. Therefore,

$$\bigcup_{i=1}^{N} (g_i + (U \cap G)) = G. \quad \square$$

Lemma 1.6. *Let $F = \{x_1, \cdots, x_N\} \subset \Gamma$ $(\hat{G} = \Gamma)$, then $h(F) \geq (1/N)^{1/2}$ ($h(F)$ = Helson constant of F).*

Proof. Suppose $\mu \in M(F)$. Since F is a finite set, $\mu \in C(\overline{G})$. Therefore, by Plancherel's theorem,

$$\|\hat{\mu}\|_\infty \geq \|\hat{\mu}\|_{L^2(\overline{G})} = \left(\sum_{j=1}^{N} |\mu(\{x_j\})|^2 \right)^{1/2}$$
$$\geq \left(\frac{1}{N^{1/2}} \right) \sum_{j=1}^{N} |\mu(\{x_j\})|. \quad \square$$

Lemma 1.7. *Let F be a finite set in Γ. Then, given $\epsilon > 0$, there exists U, a compact neighborhood of 0 in G, such that if $g \in G$, then every translate of U in G contains an element z, so that*

$$|\hat{\mu}(z) - \hat{\mu}(g)| \leq \epsilon \|\hat{\mu}\|_\infty \quad \textit{for all } \mu \in M(F).$$

Proof. Let $\mu \in M(F)$ be arbitrary. By Lemma 1.6, we have that

$$|\hat{\mu}(g) - \hat{\mu}(z)| = \left|\sum_{j=1}^{N} \mu(\{x_j\})((g, x_j) - (z, x_j))\right|$$

$$\leq N^{1/2}\|\hat{\mu}\|_\infty \sup_{1\leq j\leq N} (|1 - (g - z, x_j)|)$$

where N = number of elements of F. But, $V = \{w: |1 - (w, x_j)| < \epsilon/N^{1/2},\ j = 1, \cdots, N\}$ is a symmetric neighborhood of 0 in $\overline{G}$. Therefore, by Lemma 1.5, $V \cap G$ is relatively dense; i.e., there exist $g_1, \cdots, g_k \in G$ so that $\bigcup_{i=1}^k (g_i + V \cap G) = G$. Now, take any compact neighborhood of 0 in G, say C, and let $U = \bigcup_{i=1}^k (g_i + C)$. It is easy to see that U satisfies the requirements of the lemma. □

We are now ready to establish Theorem 1.2: Without loss of generality, we can assume that $0 \in \Gamma$ is the only accumulation point of E. Therefore, if $\mathfrak{U}$ is any neighborhood of 0 in Γ, then $\mathfrak{U} \cap E$ is a non-Helson set. We shall construct inductively a sup-norm partition, $\{F_j\}_{j=1}^\infty$, for a non-Helson subset of E.

Let $1/500 > \epsilon > 0$ be given, and let $\langle \epsilon_j \rangle$ be a sequence of real numbers such that

$$2 \sum_{j=1}^{\infty} \epsilon_j < \epsilon, \quad \text{and } \epsilon_j > 0.$$

Let F_1 be any finite subset of E so that $h(F_1) < \epsilon_1$; suppose that $k \geq 2$ and that $F_1, \cdots, F_{k-1}$, finite subsets of E, were chosen. For each $j \leq k - 1$, there exists U_j, a compact neighborhood of 0 in G, such that if $g \in G$, every translate of U_j in G contains an element z so that

$$|\hat{\mu}(z) - \hat{\mu}(g)| \leq \epsilon_j \|\hat{\mu}\|_\infty, \tag{1}$$

for all $\mu \in M(F_j)$ (Lemma 1.7).

By Lemma 1.3, there exists Δ_k, a compact neighborhood of 0 in Γ, such that if $\mu \in M(\Delta_k)$ and $u_1 - u_2 \in U_1 + \cdots + U_{k-1}$, then

$$|\hat{\mu}(u_1) - \hat{\mu}(u_2)| \leq \epsilon_k \|\hat{\mu}\|_\infty. \tag{2}$$

We now select F_k, a finite subset of $\Delta_k \cap E\backslash\{0\}$, such that $F_k \cap F_j = \varnothing$ for all $j \leq k - 1$, and $h(F_k) < \epsilon_k$. Having completely described our selection process, we shall now prove that if $\mu_j \in M(F_j)$, $1 \leq j \leq k$, then

$$\left(\frac{1}{6} - \epsilon\right) \sum_{j=1}^{k} \|\hat{\mu}_j\|_\infty \leq \left\|\left(\sum_{j=1}^{k} \hat{\mu}_j\right)\right\|_\infty. \tag{*}$$

Claim. The range of $(\sum_{j=1}^k \mu_j)\hat{}$ is $(\epsilon \sum_{j=1}^k \|\hat{\mu}_j\|_\infty)$-dense in range of $\hat{\mu}_1 + \cdots$ + range of $\hat{\mu}_k$. Consider $\sum_{j=1}^k \hat{\mu}_j(z_j)$, where $z_1, \cdots, z_k$ are arbitrary elements in G.

Let $y_k = z_k$, and assume that $y_k, \cdots, y_{j+1}$ were picked for $0 < j \le k-1$. We select y_j so that

$$(3)' \qquad y_j - y_{j+1} \in U_j \quad \text{and} \quad |\hat{\mu}_j(y_j) - \hat{\mu}_j(z_j)| \le \epsilon_j \|\hat{\mu}\|_\infty.$$

(The choice is possible by (1).) Having thus chosen $y_1, \cdots, y_k$, we have for $1 < j \le k$,

$$y_1 - y_j = (y_1 - y_2) + \cdots + (y_{j-1} - y_j) \in U_1 + \cdots + U_{j-1}.$$

Recalling that $\mu_j \in M(\Delta_j)$, we obtain by (2)

$$(4) \qquad |\hat{\mu}_j(y_1) - \hat{\mu}_j(y_j)| \le \epsilon_j \|\hat{\mu}_j\|_\infty.$$

Combining (3) and (4), we have

$$|\hat{\mu}_j(y_1) - \hat{\mu}_j(z_j)| \le 2\epsilon_j \|\hat{\mu}_j\|_\infty.$$

Finally,

$$(5) \qquad \left|\sum_{j=1}^{k} \hat{\mu}_j(y_1) - \sum_{j=1}^{k} \hat{\mu}_j(z_j)\right| \le \epsilon \sum_{j=1}^{k} \|\hat{\mu}_j\|_\infty,$$

and the claim is proved.

For each j, $\int_G \hat{\mu}_j(x)\,dx = 0$, since $\mu_j(\{0\}) = 0$. Therefore, Re $\hat{\mu}_j$ and Im $\hat{\mu}_j$ must both assume positive and negative values. It then easily follows that there exist $g_1, \cdots, g_k \in G$ so that

$$(6) \qquad \left|\sum_{j=1}^{k} \hat{\mu}_j(g_j)\right| \ge \frac{1}{6} \sum_{j=1}^{k} \|\hat{\mu}_j\|_\infty.$$

(*) now follows from (5) and (6). Since $h(F_k) \to 0$ as $k \to \infty$, $F = \bigcup_{k=1}^{\infty} F_k \subset E$ is non-Helson, and the theorem is proved. □

We note that the above technique of producing sup-norm partitions of non-Helson subsets of countable sets in compact groups cannot be applied in the obvious way to subsets of discrete groups. For example, if E is any subset of $\mathbf{Z}$, and $\epsilon > 0$, $\delta > 0$ are given, we cannot conclude that there exists K, a sufficiently large integer, for which all $\mu \in M(E \setminus [-K, K])$ are so that $|\hat{\mu}(t_1) - \hat{\mu}(t_2)| < \epsilon \|\hat{\mu}\|_\infty$ whenever $|t_1 - t_2| < \delta$.

2. Sup-norm additivity in discrete abelian groups.

Theorem 2.1. *Let Γ be an infinite discrete abelian group, and let $E \subset \Gamma$ be non-Sidon. Then, there exists $F \subset E$ such that F can be partitioned with respect to the sup-norm, and F is a non-Sidon set.*

Remark. It is clear that $E \subset \Gamma$ is a Sidon set if and only if every countable subset of E is a Sidon set. Therefore, we may assume without loss of generality that E is countable, and hence that Γ is countable.

Since Γ is countable, $\hat{\Gamma} = G$ is a compact metrizable group (cf. [7, 2.2.6]), and therefore, there exists D, a countable, dense subgroup of G. Consider D as a discrete abelian group, and let $\phi: \Gamma \longrightarrow \hat{D}$ be the natural injective map: $(\phi(\gamma), d) = (\gamma, d)$ for $\gamma \in D$ and $d \in D$. We shall say that $F \subset \hat{D}$ is a Sidon set if F is a Sidon set in $(\hat{D})_d$, $\hat{D}$ discretized.

Lemma 2.2. *Let Γ, ϕ, D be as above. Then, $E \subset \Gamma$ is Sidon if and only if $\phi(E)$ is Sidon. Furthermore, $h(E) = h(\phi(E))$.*

Proof. To prove the lemma it suffices to show that if $\{a_i\}_{i=1}^n$ is any finite set of complex numbers, then

$$\sup_{x \in G} \left| \sum_{i=1}^{n} a_i (\gamma_i, x) \right| = \sup_{y \in \bar{D}} \left| \sum_{i=1}^{n} a_i (\phi(\gamma_i), y) \right| \tag{1}$$

where $\gamma_i \in E$, $i = 1, \cdots, n$, and $\bar{D} = (\hat{D})_d\hat{}$, the Bohr compactification of D.

Since D is dense in $\bar{D}$, we have

$$\sup_{y \in \bar{D}} \left| \sum_{i=1}^{n} a_i (\phi(\gamma_i), y) \right| = \sup_{x \in D} \left| \sum_{i=1}^{n} a_i (\phi(\gamma_i), x) \right|.$$

But D was chosen to be dense in G, and hence it follows from the definition of ϕ that

$$\sup_{x \in D} \left| \sum_{i=1}^{n} a_i (\phi(\gamma_i), x) \right| = \sup_{x \in D} \left| \sum_{i=1}^{n} a_i (\gamma_i, x) \right|.$$

(1) now follows, and lemma is proved. □

Lemma 2.3. *Let $\{\gamma_i\}_{i=1}^\infty = E \subset \Gamma$ be a non-Sidon set. Let ϕ be as in Lemma 2.2. Then, there exists $\{\gamma_{j_k}\}_{k=1}^\infty = F \subset E$, F non-Sidon, and $\overline{\phi(F)}$ (closure in $\hat{D}$) is a countable set with one accumulation point.*

Proof. First, we claim that there exists $x_0 \in \overline{\phi(E)}$ such that if U is any open set containing x_0, then $\phi^{-1}(U) \cap E$ is a non-Sidon set. Suppose this were not so. Then, for each $x \in \overline{\phi(E)}$ there exists an open set, U_x, so that $x \in U_x$, and $\phi^{-1}(U_x) \cap E$ is a Sidon set. By the compactness of $\overline{\phi(E)}$, there exist $x_1, \cdots, x_n \in \overline{\phi(E)}$, so that $\bigcup_{i=1}^n U_{x_i} \supset \overline{\phi(E)}$. But, $E = \bigcup_{i=1}^n \phi^{-1}(U_{x_i}) \cap E$ is a Sidon set by Drury's theorem (cf. [1]).

Now let $\{V_n\}_{n=1}^\infty$ be a family of open sets so that $V_n \supsetneqq V_{n+1}$ and $\bigcap_{n=1}^\infty V_n = \{x_0\}$. Let $F_1 \subset \phi^{-1}(V_1) \cap E$ be any finite set. We proceed inductively to select $F_n \subset \phi^{-1}(V_n) \cap E$, such that F_n is finite, Sidon constant of $F_n < 1/n$, and $F_n \cap F_j = \emptyset$ for $j < n$. Let $F = \bigcup_{n=1}^\infty F_n$. The conclusion of the lemma easily follows. □

We are now ready to prove Theorem 2.1: By the preceding lemma, produce $E' \subset E$ such that E' is a non-Sidon set, and $\overline{\phi(E')}$ is a countable set with one limit point. By Theorem 1.2, find $S \subset \overline{\phi(E')}$ so that S is non-Helson (therefore, also non-Sidon) such that S can be partitioned with respect to the sup-norm. But,

$$\sup_{x \in G} \left| \sum_{i=1}^{n} a_i (\gamma_i, x) \right| = \sup_{y \in \overline{D}} \left| \sum_{i=1}^{n} a_i (\phi(\gamma_i), y) \right|$$

(see proof of Lemma 2.2), and it easily follows that $F = \phi^{-1}(S) \subset E$ can be partitioned with respect to the sup-norm. In fact, the inverse image under ϕ of the partition for S is a partition for F.

Corollary 2.4. *Let Γ be an infinite discrete abelian group, and let $E \subset \Gamma$ be non-Sidon. Then, there exists $F \subset E$ so that F is non-Sidon, and F is an R-set.*

Proof. Let $F \subset E$ be non-Sidon, so that F can be partitioned with respect to the sup-norm via $\{F_n\}$. For each N, $k_N \in L^1(G)$ is chosen so that $\|k_N\|_1 \leq 2$, $\hat{k}_N$ has compact support, and $\hat{k}_N(\gamma) = 1$ whenever $\gamma \in \bigcup_{n=1}^{N} F_n$. Suppose $f \in L_F^\infty(G)$. Given any N, and letting $f_n = \Sigma_{\gamma \in F_n} \hat{f}(\gamma)(\gamma, \cdot)$, we have that

$$\sum_{n=1}^{N} \|f_n\|_\infty = \sum_{n=1}^{N} \|(k_N * f)_n\|_\infty \leq C\|k_N * f\|_\infty \leq 2C\|f\|_\infty.$$

Since N is arbitrary, $\Sigma_{n=1}^\infty \|f_n\|_\infty < \infty$, and it follows that $f \in C_F(G)$. □

Remark. When $\Gamma = \mathbf{Z}$, the mapping ϕ as defined in the remark preceding Lemma 2.2 can be realized as follows: Let α be a given irrational number in T. Let $\phi_\alpha = \phi$ be the map from $\mathbf{Z}$ into T so that $\phi(n) = n\alpha \mod 2\pi$. As in Lemma 2.3, if $E \subset \mathbf{Z}$ is a non-Sidon set, then there exists $x_0 \in \phi(E)^-$ (where closure is taken in the usual topology of T) such that if U is any neighborhood of x_0, then $\phi^{-1}(U) \cap E$ is non-Sidon. We then proceed, as in Theorem 2.1 to construct $F \subset E$ non-Sidon, so that F can be partitioned with respect to the sup-norm. It seems natural to ask whether the above technique of "wrapping" subsets of integers in the circle group can be used to explore other structural properties of $\mathbf{Z}$ by investigating their analogues on T.

It is clear that not every closed subset of T can be realized in the form of $\{n_j\alpha\}^-$, where $\{n_j\}_{j=1}^\infty \subset \mathbf{Z}$, and α is an irrational number (e.g., closed independent sets in T). It is an open question whether whenever $E \subset \mathbf{Z}$ is Sidon, $\overline{E}$ is a Helson set in $\overline{\mathbf{Z}}$ ($\overline{E}$ = closure of E in the Bohr compactification of $\mathbf{Z}$). It turns out that when we "close" E in T, we are in a somewhat less complicated situation than the one where we form $\overline{E}$ in $\overline{\mathbf{Z}}$. If α is a fixed irrational number, then there exist Sidon sets, $\{n_j\} \subset \mathbf{Z}$, so that $\{n_j\alpha\}^- = T$: Arrange the rationals in T in a sequence, $\{r_j\}_{j=1}^\infty$ in such a way that every rational occurs infinitely many times in the arrangement. Now suppose $n_1, \cdots, n_J$ were picked for $J > 1$. It is clear that $E_J = \{\alpha n\colon n > 3n_J\}$ is dense in T. Select $n_{J+1} > 3n_J$ so that $\|\alpha n_{J+1} - r_{J+1}\| < 1/J$. It is now clear that $\{\alpha n_j\}_{j=1}^{\infty}$ is dense in T. Going in the other direction, we produce a Sidon set, $E \subset \mathbf{Z}$, so that $\{\alpha E\}^- = T$ for almost all α in T: Let $S = \{(\epsilon_1, \cdots, \epsilon_N)\colon N \in \mathbf{Z}, \epsilon_i = 0, 1\}$. Fix $\gamma \in S$, and set $F_\gamma = \{\alpha \subset T\colon \gamma$ does not occur anywhere in the binary expansion of $\alpha\}$. It is easy to check that $m(F_\gamma) = 0$, and hence $m(\bigcup_{\gamma \in S} F_\gamma) = 0$. But if we let $E = \{2^j\}_{j=1}^{\infty}$, then for all $\alpha \in \sim \bigcup_{\gamma \in S} F_\gamma$, $(\alpha E)^- = T$.

3. **Open questions.** 1. Let $U(T) = \{f \in C(T)\colon S_n(f) = \Sigma_{-n}^{n} \hat{f}(n)e^{int}$ converge uniformly to $f\}$. Figà-Talamanca constructed in [2] a non-Sidon set $E \subset \mathbf{Z}$ such that $U_E(T) = C_E(T)$. Therefore, by Corollary 2.4 we can produce a non-Sidon set such that $L_E^\infty(T) = U_E(T)$. In fact, out methods show that every non-Sidon set $E \subset \mathbf{Z}$ contains a non-Sidon set F for which there exists $\langle n_k \rangle$ such that $S_{n_k}(f)$ converge uniformly to f, for all $f \in L_F^\infty(T)$. Given a non-Sidon set $E \subset \mathbf{Z}$, does there exist $F \subset E$, F non-Sidon, and $U_F(T) = L_F^\infty(T)$?

2. We note that $L_E^\infty(G) = C_E(G)$ if and only if $L_E^\infty(G)$ is separable. Can an R-set $E \subset \mathbf{Z}$ be constructed so that E cannot be partitioned with respect to the sup-norm?

I would like to thank Professor O. C. McGehee, my adviser, and Professor H. P. Rosenthal for their patient guidance and good counsel. Also, I thank Professors S. Ebenstein and M. Zippin for teaching me much of the little mathematics that I know.

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