ON A VARIATION OF THE RAMSEY NUMBER

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ABSTRACT. Let $c(m, n)$ be the least integer $p$ such that, for any graph $G$ of order $p$, either $G$ has an $m$-cycle or its complement $\overline{G}$ has an $n$-cycle. Values of $c(m, n)$ are established for $m, n \leq 6$ and general formulas are proved for $c(3, n), c(4, n),$ and $c(5, n)$.

Introduction. It is a well-known fact that in any gathering of six people, there are three people who are mutual acquaintances or three people who are mutual strangers. This statement has the graph-theoretic formulation that either a given graph of order 6 or its complement contains a triangle. It might further be mentioned that "6" is minimum with respect to this property.

The Ramsey number $r(m, n)$ may be considered a generalization of the above observation. For integers $m, n \geq 2$, the number $r(m, n)$ is defined as the smallest positive integer $p$ such that given any graph $G$ of order $p$, either $G$ contains the complete subgraph $K_m$ of order $m$ or the complement $\overline{G}$ of $G$ contains $K_n$. Hence, the aforementioned fact states that $r(3, 3) = 6$. One may easily note that $r(m, n) = r(n, m)$ and that $r(2, n) = n$ for all $n \geq 2$.

It is a result due to Ramsey [3] that the number $r(m, n)$ exists for all $m, n \geq 2$. Despite the fact that a great deal of research has been done on Ramsey numbers, only six values $r(m, n)$ have been determined for $m, n \geq 3$ (see [1]); namely, $r(m, n)$ is known (for $m, n \geq 3$) only when $(m, n) = (3, 3), (3, 4), (3, 5), (3, 6), (3, 7), (4, 4)$.

If we denote an $n$-cycle (a cycle of length $n$) by $C_n$, the original problem may be stated as: Given a graph $G$ of order 6, either $G$ or $\overline{G}$ contains a 3-cycle (triangle). This suggests a generalization different from that which leads to the Ramsey numbers. For $m, n \geq 3$, we define the number $c(m, n)$ to be the least positive integer $p$, such that for any graph $G$ of order $p$, either $G$ contains the $m$-cycle $C_m$ or $\overline{G}$ contains $C_n$. Of course, we have $c(3, 3) = 6$. The number $c(m, n)$ always exists since $c(m, n) \leq r(m, n)$. It is the object of this paper to determine the value of $c(m, n)$ for several pairs $(m, n)$; in particular, $c(3, n), c(4, n)$, and $c(5, n)$ are determined for all $n \geq 3$. Before proceeding further, we present a few definitions and some additional notation. All terms not defined here may be found in [2].

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The complete bipartite graph $K(m, n)$, $m, n \geq 1$, is that graph $G$ of order $m + n$, whose vertex set may be partitioned as $V_1 \cup V_2$ such that $|V_1| = m$, $|V_2| = n$ and $e = uv$ is an edge of $G$ if and only if $u \in V_i$ and $v \in V_j$, $i \neq j$. For connected graphs $G_1$ and $G_2$, we define $G_1 \cup G_2$ to be the disconnected graph having the two components $G_1$ and $G_2$. Note that if $G = K(m, n)$, then $\overline{G} = K_m \cup K_n$.

The numbers $c(3, n)$. We have already mentioned that $c(3, 3)$ is the well-known Ramsey number $r(3, 3) = 6$. We consider $c(3, 4)$ next.

**Theorem 1.** $c(3, 4) = 7$.

**Proof.** Let $H = K(3, 3)$ so that $\overline{H} = K_3 \cup K_3$. The graph $H$ contains no 3-cycle and its complement $\overline{H}$ fails to contain a 4-cycle; thus, $c(3, 4) \geq 7$. To verify that $c(3, 4) = 7$, we let $G$ be an arbitrary graph of order 7 and assume $G$ contains no 3-cycle. We show that $G$ contains a 4-cycle.

Since $c(3, 3) = 6$, either $G$ or $\overline{G}$ has a 3-cycle; hence, $\overline{G}$ contains a 3-cycle, which we represent as $C: u_1, u_2, u_3, u_4$. (See Figure 1a, where the edges of $\overline{G}$ are represented by dashed lines.) Denote the remaining vertices by $v_1, v_2, v_3, v_4$. If some $v_i$ is joined in $\overline{G}$ to more than one vertex of $C$, then $\overline{G}$ contains a 4-cycle. We may assume, then, that each $v_i$ is adjacent in $G$ to at least two vertices of $C$. This implies that every two distinct $v_i$ must be joined in $G$ to a common vertex of $C$. (See Figure 1b, where the edges of $G$ are represented by solid lines.) Because $G$ contains no triangles, every two distinct $v_i$ must be adjacent in $\overline{G}$ (see Figure 1c) which implies that $\overline{G}$ contains $K_4$ and hence $C_4$ as a subgraph.

We now proceed to the general situation.

**Theorem 2.** For $n \geq 4$, $c(3, n) = 2n - 1$.

**Proof.** First, we note that if $H = K(n - 1, n - 1)$ so that $\overline{H} = K_{n-1} \cup K_{n-1}$, then $H$ contains no 3-cycles and $\overline{H}$ contains no $n$-cycles. Thus, $c(3, n) \geq 2n - 1$. We prove that $c(3, n) = 2n - 1$, for all $n \geq 4$, by using induction on $n$. 
By Theorem 1, $c(3, 4) = 7$. Assume that $c(3, n) = 2n - 1$ for some $n \geq 4$. It follows, therefore, that if $F$ is any graph of order $2n - 1$, either $F$ contains a 3-cycle or $\overline{F}$ contains an $n$-cycle. We now show that $c(3, n + 1) = 2n + 1$. Let $G$ be a graph of order $2n + 1$, and assume $G$ has no 3-cycles. Because $c(3, n + 1) \geq 2n + 1$, it suffices to prove that $\overline{G}$ contains an $(n + 1)$-cycle. Since, by the induction hypothesis, $c(3, n) = 2n - 1$, the graph $\overline{G}$ contains an $n$-cycle, say $C: u_1, u_2, \ldots, u_n, u_1$. Designate the remaining vertices by $v_1, v_2, \ldots, v_n$, and $v_{n+1}$.

Suppose that some vertex $u_i$, $1 \leq i \leq n$, is adjacent in $G$ to all vertices $v_j$, $j = 1, 2, \ldots, n + 1$. Since $G$ contains no 3-cycles, every two distinct vertices $v_j$ are adjacent in $\overline{G}$. However, this implies that $\overline{G}$ contains $K_{n+1}$ and therefore $C_{n+1}$ as a subgraph. We henceforth assume that for $i = 1, 2, \ldots, n$, the vertex $u_i$ is adjacent in $\overline{G}$ to some $v_j$.

We now consider two cases.

Case 1. Suppose there exist two alternate vertices on $C$ which are respectively joined in $\overline{G}$ to two distinct $v_j$. Assume $u_1v_1, u_3v_3 \in E(\overline{G})$. If some vertex $u_i$ is joined in $\overline{G}$ to two consecutive vertices of $C$, then $\overline{G}$ contains an $(n + 1)$-cycle. Otherwise we have $u_2v_1, u_2v_3 \in E(G)$, which implies that $v_1v_3 \in E(\overline{G})$. However, then $\overline{G}$ contains the $(n + 1)$-cycle $u_1, v_1, v_3, u_3, u_4, \ldots, u_n, u_1$.

Case 2. Suppose no two alternate vertices on $C$ are joined in $\overline{G}$ to distinct vertices $v_j$. This implies that $u_1$ and $u_3$ are joined in $\overline{G}$ to the same $v_i$, say $v_1$. Indeed, every $u_i$, with $i$ odd, is joined in $\overline{G}$ to $v_1$. If $n$ is odd, then $v_1$ is joined in $G$ to both $u_1$ and $u_n$ which produces an $(n + 1)$-cycle in $\overline{G}$. Assume that $n$ is even. It follows here that each $u_i$, with $i$ even, is adjacent in $\overline{G}$ to the same $v_i \neq v_1$, say $v_2$.

Each $v_j$, $3 \leq j \leq n + 1$, is necessarily joined in $G$ to every vertex of $C$; otherwise, we revert back to Case 1. Since $G$ contains no triangles, $v_i$ and $v_j$, $3 \leq i < j \leq n + 1$, are adjacent in $\overline{G}$. For the same reason and because $v_1u_2, v_2u_1 \in E(G)$, all edges $v_1u_1$ and $v_2u_1$, $3 \leq i \leq n + 1$, belong to $\overline{G}$. Now $v_1, v_3, v_2, v_4, v_5, \ldots, v_{n+1}, v_1$ is a desired $(n + 1)$-cycle in $\overline{G}$.

The numbers $c(4, n)$. As $c(m, n) = c(n, m)$, it follows that $c(4, 3) = 7$ by Theorem 1. Thus we need only consider $c(4, n)$ for $n \geq 4$. Since the values of $c(4, 4)$ and $c(4, 5)$ do not follow the general formula which we present in this section, we must establish these numbers individually. We begin by doing this.

**Theorem 3.** $c(4, 4) = 6$.

**Proof.** Let $H = C_5$ so that $\overline{H} = C_5$. Since neither $H$ nor $\overline{H}$ contains a 4-cycle, $c(4, 4) \geq 6$. Let $G$ be a graph of order 6, and assume neither $G$ nor $\overline{G}$ contains a 4-cycle. Because $c(3, 3) = 6$, either $G$ or $\overline{G}$ contains a triangle. Without loss of generality, we assume $G$ contains the 3-cycle $C: u_1, u_2, u_3, u_1$. Denote the other vertices of $G$ by $v_1, v_2,$ and $v_3$. No vertex $v_i$ can be joined in $G$ to more than one
vertex of $C$, for otherwise $G$ contains a 4-cycle. Hence each $v_i$ is adjacent in $\overline{G}$ to at least two vertices of $C$. If there exist two $v_i$ which are adjacent in $\overline{G}$ to the same two vertices of $C$, then $\overline{G}$ contains a 4-cycle. Hence, we suppose $v_i$ is adjacent in $\overline{G}$ to $u_1$ and $u_2$, $v_2$ is adjacent in $\overline{G}$ to $u_2$ and $u_3$, and $v_3$ is adjacent in $\overline{G}$ to $u_1$ and $u_3$; moreover, $v_1u_3, v_2u_1, v_3u_2 \in E(G)$. No two $v_i$ are adjacent in $G$, for then $G$ has a 4-cycle. This implies that $v_1v_2, v_1v_3, v_2v_3 \in E(\overline{G})$, but then $v_1, u_2, v_2, v_3, v_1$ is a 4-cycle in $\overline{G}$ which produces a contradiction.

**Theorem 4.** $c(4, 5) = 7$.

**Proof.** Let $H = K_3 \cup K_3$ so that $\overline{H} = K(3, 3)$. The graph $H$ has no 4-cycle and $\overline{H}$ has no 5-cycle; thus, $c(4, 5) \geq 7$. Let $G$ be a graph of order 7, and assume $G$ has no 4-cycle. We prove that $\overline{G}$ contains a 5-cycle.

Since $c(4, 4) = 6$ by Theorem 3, the graph $\overline{G}$ contains a 4-cycle, say $C: u_1, u_2, u_3, u_4, u_5$. Let $v_1, v_2, v_3$ denote the other vertices. If any of $v_1, v_2,$ and $v_3$ is adjacent in $\overline{G}$ to two consecutive vertices of $C$, then $\overline{G}$ contains a 5-cycle. Suppose, then, that none of $v_1, v_2,$ and $v_3$ is adjacent in $\overline{G}$ to consecutive vertices of $C$. Hence each $v_i$ is joined in $\overline{G}$ to opposite vertices of $C$. Necessarily, there exist two $v_i$, say $v_1$ and $v_2$, which are adjacent in $\overline{G}$ to the same opposite vertices of $C$, say $u_1$ and $u_2$. The graph $G$ therefore contains the 4-cycle $u_1, v_1, u_3, v_2, u_5, v_1$, which is contrary to hypothesis.

In order to determine a general formula for $c(4, n)$, we establish the number $c(4, 6)$.

**Theorem 5.** $c(4, 6) = 7$.

**Proof.** Let $H = K(1, 5)$; thus, $\overline{H} = K_1 \cup K_5$. Because $H$ has no cycles (hence no 4-cycles) and $\overline{H}$ has no 6-cycles, $c(4, 6) \geq 7$. Let $G$ be a graph of order 7, and assume $G$ has no 4-cycles. We show that $\overline{G}$ contains a 6-cycle. Since $c(4, 5) = 7$ by Theorem 4 and since $G$ has no 4-cycles, it follows that $\overline{G}$ has a 5-cycle $C: u_1, u_2, u_3, u_4, u_5, u_1$. Let the remaining vertices be denoted by $v_1$ and $v_2$.

If $v_1$ or $v_2$ is adjacent in $\overline{G}$ to two consecutive vertices of $C$, then $\overline{G}$ has a 6-cycle. Assume neither $v_1$ nor $v_2$ is adjacent in $\overline{G}$ to consecutive vertices of $C$ so that each of $v_1$ and $v_2$ is joined in $\overline{G}$ to a set of three vertices of $C$ (not all consecutive). If there exist two vertices of $C$ joined in $G$ to both $v_1$ and $v_2$, then $G$ has a 4-cycle which produces a contradiction. However, there must exist one vertex of $C$ joined in $G$ to $v_1$ and $v_2$; hence we assume, without loss of generality, that $v_1$ and $v_2$ are joined in $G$ to $u_1$, the edges $v_1u_2, v_1u_4, v_2u_3, v_2u_5 \in E(G)$, while $v_1u_3, v_1u_5, v_2u_2, v_2u_4 \in E(\overline{G})$. Now $\overline{G}$ contains the 6-cycle $v_1, u_3, u_2, v_2, u_4, u_5, v_1$.

We are now prepared to determine the remaining values of $c(4, n)$.

**Theorem 6.** For $n \geq 6$, $c(4, n) = n + 1$. 

Proof. Let \( n \geq 6 \) and let \( H = K(1, n - 1) \) so that \( \overline{H} = K_1 \cup K_{n-1} \). The graph \( H \) has no 4-cycles and its complement \( \overline{H} \) has no \( n \)-cycles; therefore, \( c(4, n) \geq n + 1 \). We proceed by induction on \( n \) (\( \geq 6 \)). That \( c(4, 6) = 7 \) is the result of Theorem 5. Assume that, for some \( n \geq 6 \), \( c(4, n) = n + 1 \); hence, for every graph \( F \) of order \( n + 1 \), either \( F \) contains a 4-cycle or \( \overline{F} \) contains an \( n \)-cycle. We consider the number \( c(4, n + 1) \). Since \( c(4, n + 1) \geq n + 2 \), it suffices to prove that if \( G \) is a graph of order \( n + 2 \), either \( G \) has a 4-cycle or \( \overline{G} \) has an \( (n + 1) \)-cycle. Suppose \( G \) does not contain a 4-cycle. Since \( c(4, n) = n + 1 \) by the induction hypothesis, it follows that \( \overline{G} \) contains an \( n \)-cycle, say \( C: u_1, u_2, \ldots, u_n, u_1 \). Designate the other two vertices by \( v_1 \) and \( v_2 \).

If \( v_1 \) or \( v_2 \) is adjacent in \( \overline{G} \) to consecutive vertices of \( C \), then \( \overline{G} \) contains an \((n + 1)\)-cycle, completing the proof. Assume, therefore, that neither \( v_1 \) nor \( v_2 \) is adjacent in \( \overline{G} \) to consecutive vertices of \( C \), which implies that each of \( v_1 \) and \( v_2 \) is adjacent in \( G \) to some set of \( \{n/2\} \) vertices of \( C \) such that the set contains at least one of every two consecutive vertices of \( C \). If \( v_1 \) and \( v_2 \) are adjacent in \( G \) to the same two (or more) vertices of \( C \), then \( G \) contains a 4-cycle, which is contradictory. Thus we assume that \( v_1 \) and \( v_2 \) are mutually adjacent in \( G \) to one or no vertices of \( C \). We consider these two cases.

Case 1. Assume that \( v_1 \) and \( v_2 \) are mutually adjacent in \( G \) to no vertices of \( C \). In this case, it necessarily follows that each of \( v_1 \) and \( v_2 \) is joined in \( G \) to exactly \( n/2 \) vertices of \( C \) such that neither \( v_1 \) nor \( v_2 \) is adjacent in \( G \) to two consecutive vertices of \( C \). Hence, \( n \) is even and, without loss of generality, we assume \( v_1 \) is joined in \( G \) to the vertices of \( S_1 = \{u_i \mid i \text{ is odd}\} \) and \( v_2 \) is joined in \( G \) to the vertices of \( S_2 = \{u_i \mid i \text{ is even}\} \). Therefore, \( v_1 \) is joined in \( \overline{G} \) to the elements of \( S_2 \), and \( v_2 \) is adjacent in \( \overline{G} \) to the elements of \( S_1 \). If all edges \( u_i u_j \) with \( i \) and \( j \) odd, belong to \( G \), then since \( n \geq 6 \), \( G \) contains the 4-cycle \( u_1, v_1, u_3, u_5, u_1 \), which is contrary to hypothesis. Therefore, \( \overline{G} \) contains an edge \( u_i u_j \) with \( i \) and \( j \) odd such that \( 1 \leq i < j < n \), say. The graph \( \overline{G} \) thus contains the \((n + 1)\)-cycle \( u_i, u_j, u_{j-1}, \ldots, u_{i+1}, v_1, u_{j+1}, u_{j+2}, \ldots, u_{i-1}, u_i \).

Case 2. Assume that \( v_1 \) and \( v_2 \) are mutually adjacent in \( G \) to exactly one vertex of \( C \), say \( u_1 \). Exactly one of \( v_1 \) and \( v_2 \) is adjacent in \( G \) to \( u_2 \), for if this were not the case, then \( v_1 \) and \( v_2 \) must be joined in \( G \) to \( u_3 \), which is contrary to our assumption. Without loss of generality, we suppose that \( v_1 u_2 \in E(G) \). Necessarily, \( v_2 u_3 \in E(G) \) or else \( v_2 \) is joined in \( \overline{G} \) to the consecutive vertices \( u_2 \) and \( u_3 \) of \( C \), which we have previously ruled out. By the same reasoning, \( v_1 u_4 \in E(G) \), \( v_2 u_5 \in E(G) \), etc. Hence, if we let \( S_1 \) and \( S_2 \) be defined as in Case 1, then \( v_1 \) is joined in \( G \) to the elements of \( \{u_1\} \cup S_2 \) and joined in \( \overline{G} \) to the vertices of \( S_1 - \{u_1\} \), while \( v_2 \) is joined in \( G \) to the vertices of \( S_1 \) and joined in \( \overline{G} \) to the vertices of \( S_2 \).

If \( G \) contains all edges \( u_i u_j \) with \( i \) and \( j \) even, then since \( n \geq 6 \), \( G \) contains
the 4-cycle \( v_1, u_2, u_4, u_6, v_1 \) which produces a contradiction. Therefore, \( \overline{G} \) contains some edge \( u_iu_j \), where \( i \) and \( j \) are even and \( 1 < i < j \leq n \), say. The graph \( G \) then has the \((n + 1)\)-cycle \( u_j, u_i, u_{i+1}, \ldots, u_{j-1}, v_1, u_{i-1}, u_{i-2}, \ldots, u_{j+1}, u_j \), which yields the desired result.

The numbers \( c(5, n) \). We have already established the value of \( c(5, n) \) for \( n = 3 \) and \( n = 4 \). In order to present a formula for \( c(5, n) \), \( n \geq 5 \), we shall first determine \( c(5, 5) \).

**Theorem 7.** \( c(5, 5) = 9 \).

**Proof.** Let \( H = K(4, 4) \) so that \( \overline{H} = K_4 \cup K_4 \). Neither \( H \) nor \( \overline{H} \) contains a 5-cycle; thus, \( c(5, 5) \geq 9 \). Let \( G \) be a graph of order 9, and assume that neither \( G \) nor \( \overline{G} \) has a 5-cycle.

Since \( c(4, 4) = 6 \), at least one of \( G \) and \( \overline{G} \) contains a 4-cycle. Without loss of generality, we assume that \( G \) contains the 4-cycle \( C: u_1, u_2, u_3, u_4, u_1 \). Denote the other vertices of \( G \) by \( v_1, v_2, v_3, v_4, \) and \( v_5 \). No \( v_i \) is adjacent in \( G \) to two consecutive vertices of \( C \) since \( G \) contains no 5-cycle. Hence each \( v_i \) is joined in \( G \) to two opposite vertices of \( C \). We now consider two cases, assuming throughout that \( v_1v_2 \in E(\overline{G}) \).

**Case 1.** Assume \( v_1 \) and \( v_2 \) are adjacent in \( \overline{G} \) to the same pair of opposite vertices of \( C \), say \( u_1 \) and \( u_3 \). None of \( v_3, v_4, \) and \( v_5 \) is joined in \( \overline{G} \) to both \( u_1 \) and \( u_3 \), for then we have conditions which produce a 5-cycle, as described above. Therefore, each of \( v_3, v_4, \) and \( v_5 \) is joined in \( \overline{G} \) to \( u_2 \) and \( u_4 \). Now \( v_3v_4 \in E(G) \), for otherwise \( v_3, v_4, u_2, v_5, u_4, v_3 \) is a 5-cycle in \( G \). Similarly, \( v_3v_5, v_4v_5 \in E(G) \).

If \( v_1 \) is joined in \( \overline{G} \) to one of \( v_3, v_4, \) and \( v_5 \), and \( v_2 \) is joined in \( \overline{G} \) to some other vertex of \( v_3, v_4, \) and \( v_5 \), then \( \overline{G} \) contains a 5-cycle. Since \( G \) has no 5-cycles, it follows that one of \( v_1 \) and \( v_2 \) is joined in \( G \) to all of \( v_3, v_4, \) and \( v_5 \), say \( v_2v_3, v_2v_4, v_2v_5 \in E(G) \). If there are two edges of \( G \) from the vertices of \( C \) to two distinct vertices of \( v_3, v_4, \) and \( v_5 \), then \( G \) contains a 5-cycle. Because \( G \) cannot contain a 5-cycle, there must exist a vertex among \( v_3, v_4, \) and \( v_5 \), say \( v_5 \), which is joined in \( \overline{G} \) to all vertices of \( C \). Thus, \( \overline{G} \) contains the 5-cycle \( v_5, u_3, v_1, v_2, u_1, v_5 \), which produces a contradiction.

**Case 2.** Assume \( v_1 \) is adjacent in \( \overline{G} \) to \( u_1 \) and \( u_3 \), and \( v_2 \) is adjacent in \( \overline{G} \) to \( u_2 \) and \( u_4 \). Suppose \( v_3 \) is adjacent in \( \overline{G} \) to \( u_1 \) and \( u_3 \). Necessarily, \( v_2 \) is adjacent in \( \overline{G} \) to \( u_2 \) and \( u_4 \), for otherwise we have conditions sufficient to produce a
5-cycle in \( G \), as mentioned earlier. For the same reason, \( v_2u_1, v_2u_3 \in E(G) \). The vertex \( v_1 \) is joined in \( G \) to \( u_2 \) or \( u_4 \), for otherwise we return to Case 1. Thus, we assume \( v_1u_2 \in E(G) \). Now \( v_1v_3 \in E(G) \), or else \( v_1, v_3, u_4, u_3, u_2, v_1 \) is a 5-cycle in \( G \). However, this places us in Case 1 again, where \( v_1 \) and \( v_3 \) are playing the roles of \( v_1 \) and \( v_2 \), respectively.

This completes the proof.

We conclude this section by presenting a formula for \( c(5, n) \) for all \( n \geq 5 \).

**Theorem 8.** For \( n \geq 5 \), \( c(5, n) = 2n - 1 \).

**Proof.** Let \( H = K(n - 1, n - 1) \) so that \( \overline{H} = K_{n-1} \cup K_{n-1} \). The graph \( H \) contains no 5-cycle, and \( \overline{H} \) has no \( n \)-cycle; therefore, \( c(5, n) \geq 2n - 1 \). We verify that \( c(5, n) = 2n - 1 \) by induction on \( n \) (\( \geq 5 \)), the result following for \( n = 5 \) by Theorem 7.

Assume \( c(5, n) = 2n - 1 \) for some \( n \geq 5 \), and let \( G \) be a graph of order \( 2n + 1 \). Since \( c(5, n + 1) \geq 2n + 1 \), it suffices to show that \( G \) contains a 5-cycle or \( \overline{G} \) contains an \( (n + 1) \)-cycle. Assume that \( G \) has no 5-cycle. Since \( c(5, n) = 2n - 1 \), the graph \( \overline{G} \) contains an \( n \)-cycle \( C: u_1, u_2, \ldots, u_n, u_1 \). Designate the remaining vertices by \( v_1, v_2, \ldots, v_n, v_{n+1} \).

If some \( v_i \) (\( 1 \leq i \leq n + 1 \)) is adjacent in \( \overline{G} \) to two consecutive vertices of \( C \), then \( \overline{G} \) contains an \( (n + 1) \)-cycle, completing the proof. Assume, therefore, that no \( v_i \) is adjacent in \( \overline{G} \) to two consecutive vertices of \( C \). This implies that each \( v_i \) is adjacent in \( G \) to some set of \( \lceil n/2 \rceil \) vertices of \( C \), where at least one vertex in any pair of consecutive vertices of \( C \) belongs to the set. If every two distinct \( v_i \) are adjacent in \( \overline{G} \), then \( \overline{G} \) contains \( K_{n+1} \) and hence \( C_{n+1} \) as a subgraph. Suppose, then, there are no two distinct \( v_i \), say \( v_1 \) and \( v_2 \), which are adjacent in \( G \).

We now consider three cases, assuming throughout that \( v_1v_2 \in E(G) \).

**Case 1.** Assume there is a vertex \( v_k \) (\( k \neq 1, 2 \)) such that \( v_1 \) and \( v_k \) are joined in \( G \) to a vertex \( u_i \) on \( C \), and \( v_2 \) and \( v_k \) are joined in \( G \) to a vertex \( u_j \) on \( C \). If it is possible to select \( u_i \) and \( u_j \) such that \( u_i \neq u_j \), then \( G \) contains the 5-cycle \( v_k, u_i, v_1, v_2, u_j, v_k \), which is contradictory. Hence we may suppose that \( v_1 \) and \( v_k \) are joined in \( G \) to only one vertex \( u_i \) on \( C \), and \( v_2 \) and \( v_k \) are joined on \( G \) to only one vertex on \( C \), namely \( u_j \). Since at least one vertex in every pair of consecutive vertices of \( C \) is joined in \( G \) to \( v_1 \) (respectively \( v_2 \)), it follows that every vertex of \( C \) different from \( u_i \) is joined in \( G \) to exactly one of \( v_1 \) and \( v_2 \). The vertex \( v_k \) is adjacent in \( G \) to at least \( \lceil n/2 \rceil \) vertices of \( C \); therefore, \( v_k \) must be adjacent in \( G \) to a vertex \( u \) which is joined in \( G \) to \( v_1 \), and, furthermore, \( v_k \) is adjacent in \( G \) to a vertex \( u_s \) (different from \( u \)) which is joined in \( G \) to \( v_2 \). Hence \( G \) contains a 5-cycle, which is contrary to hypothesis.

We note that if \( n \) is odd, Case 1 necessarily applies. We may henceforth assume \( n \) to be even.

**Case 2.** Assume Case 1 does not hold and there exists some vertex \( v_k \) (\( k \neq 1, 2 \)) which is adjacent in \( G \) to no vertex of \( C \) which is joined in \( G \) to \( v_1 \) or \( v_2 \).
This implies that whenever $v_1u_i \in E(G)$, $1 \leq i \leq n$, then $v_ku_i \in E(\overline{G})$, and whenever $v_2u_j \in E(G)$, $1 \leq j \leq n$, then $v_ku_j \in E(\overline{G})$. Since $v_k$ is joined in $G$ to at least \( \lfloor n/2 \rfloor \) vertices of $C$, and $v_k$ is joined in $\overline{G}$ to at least \( \lfloor n/2 \rfloor \) vertices of $C$, it follows that $v_k$ is adjacent in $G$ to exactly $n/2$ vertices of $C$ and is adjacent in $\overline{G}$ to exactly $n/2$ vertices of $C$. Therefore, we may assume here that $v_1$ and $v_2$ are joined in $G$ to the vertices of $S_1 = \{u_i \mid i \text{ is odd}\}$ and joined in $\overline{G}$ to the vertices of $S_2 = \{u_i \mid i \text{ is even}\}$, while $v_k$ is joined in $G$ to the vertices of $S_2$ and joined in $\overline{G}$ to the elements of $S_1$.

If $i$ and $j$ are both even, then $u_iu_j \in E(\overline{G})$; for otherwise, we may select an even $t \neq i, j$ (since $n \geq 6$ here) to obtain the 5-cycle $u_t, u_j, v_2, u_t, v_1, u_i, v_2, u_t, u_j, v_1, v_2, u_t, u_j, v_1, u_i, v_2, u_t, u_j, v_1, v_2$, which gives the desired result.

Case 3. Assume that Case 1 and Case 2 do not hold. Hence each $v_k$, $k \geq 3$, has the properties that whenever $v_1u_i, v_ku_i \in E(G)$, $1 \leq i \leq n$, then $v_2u_i \in E(\overline{G})$, and whenever $v_2u_j, v_ku_j \in E(G)$, $1 \leq j \leq n$, then $v_1u_j \in E(\overline{G})$. Let $S_1$ and $S_2$ be defined as in Case 2. We may assume in this case that $v_1$ and $v_3$, say, are joined in $G$ to the elements of $S_1$ and joined in $\overline{G}$ to the elements of $S_2$, while $v_2$ is joined in $G$ to the vertices of $S_2$ and joined in $\overline{G}$ to the vertices of $S_1$.

If $v_1v_3 \in E(G)$, then we have the conditions specified in Case 2, where $v_1$ and $v_3$ play the roles of $v_1$ and $v_2$, respectively. Hence, $v_1v_3 \in E(\overline{G})$, and $\overline{G}$ contains the $(n+1)$-cycle $v_1, v_3, u_2, u_3, \ldots, u_n, v_1$.

The number $c(6, 6)$. We next determine the value of $c(6, 6)$.

**Theorem 9.** $c(6, 6) = 8$.

**Proof.** Let $H = K(2, 5)$ so that $\overline{H} = K_2 \cup K_5$. Since neither $H$ nor $\overline{H}$ has a 6-cycle, $c(6, 6) \geq 8$. Let $G$ be a graph of order 8, and suppose neither $G$ nor $\overline{G}$ contains a 6-cycle. We distinguish two cases.

**Case 1.** Assume neither $G$ nor $\overline{G}$ has a 5-cycle. Since $c(4, 4) = 6$ by Theorem 3, at least one of $G$ and $\overline{G}$ has a 4-cycle; say $G$ contains the 4-cycle $C: u_1, u_2, u_3, u_4, u_1$. Denote the remaining vertices of $G$ by $v_1, v_2, v_3, v_4$. Since $G$ has no 5-cycle, no $v_i$ is joined in $G$ to two consecutive vertices of $C$. This implies that every $v_i$ is joined in $\overline{G}$ to some pair of opposite vertices of $C$. We consider two subcases.

**Subcase 1a.** Suppose three or more $v_i$ are joined in $\overline{G}$ to the same pair of opposite vertices of $C$; say $v_1, v_2$, and $v_3$ are joined in $\overline{G}$ to $u_1$ and $u_2$. Every two distinct vertices in $\{v_1, v_2, v_3\}$ are adjacent in $G$, for otherwise $\overline{G}$ contains a 5-cycle. Also, the vertex $v_4$ cannot be joined in $\overline{G}$ to two other $v_i$; otherwise, a 6-cycle exists in $\overline{G}$. Thus, we assume $v_2v_4$ and $v_3v_4$ are edges of $G$. Not both $u_2v_2$ and $u_4v_3$ are edges of $G$, for then $G$ contains a 6-cycle. Without loss of generality, we may assume $u_2v_2$ is an edge of $\overline{G}$. If $u_2v_1$ is an edge of $\overline{G}$, then $\overline{G}$
contains a 6-cycle. Furthermore, if \( u_2v_3 \in E(\overline{G}) \), then \( \overline{G} \) contains a 6-cycle.
Therefore, \( u_2v_1, u_2v_3 \in E(G) \) and \( G \) contains the 5-cycle \( u_2, v_1, v_2, u_4, v_3, u_2 \). This produces a contradiction.

Subcase 1b. Suppose exactly two \( v_i \) are joined in \( \overline{G} \) to the same pair of opposite vertices of \( C \); say \( v_1 \) and \( v_2 \) are joined in \( \overline{G} \) to \( u_2 \) and \( u_4 \) while \( v_3 \) and \( v_4 \) are joined in \( \overline{G} \) to \( u_1 \) and \( u_3 \). Assume further that \( v_1v_3 \) is an edge of \( \overline{G} \). The edge \( v_3u_2 \) belongs to \( G \), for otherwise \( v_3, u_2, v_2, u_4, v_1, v_3 \) is a 5-cycle in \( \overline{G} \). Similarly, 5-cycles result in \( \overline{G} \) unless \( v_4u_2 \) and \( v_4u_4 \) are edges of \( G \). Next, \( v_4v_3 \in E(\overline{G}) \), or else \( v_4, v_3, u_2, u_3, u_4, v_4 \) is a 5-cycle in \( G \). In a like manner, it follows that \( v_2u_1 \in E(G) \) and \( v_2v_3 \in E(\overline{G}) \). However, \( v_2, u_3, v_4, v_3, u_1, v_2 \) is a 6-cycle of \( \overline{G} \) which is contradictory. Hence, we must have \( v_1v_3 \in E(G) \). By symmetry, we may also conclude that \( v_2v_3, v_1v_4, v_2v_4 \in E(G) \).

We observe that not both \( u_1v_2 \) and \( u_3v_1 \) are edges of \( G \), for otherwise \( u_1, v_2, v_4, v_1, u_3, u_4, u_1 \) is a 6-cycle of \( G \). However, not both \( u_1v_2 \) and \( u_3v_1 \) are edges of \( \overline{G} \) either, since then \( u_1, v_2, u_4, v_1, u_3, v_4, u_1 \) is a 6-cycle of \( \overline{G} \). Thus, we may assume that \( u_1v_2 \in E(G) \) and \( u_3v_1 \in E(\overline{G}) \). If the edge \( u_4v_4 \) is in \( G \), then \( G \) contains the contradictory 6-cycle \( u_4, v_4, v_3, v_2, u_1, u_4 \). On the other hand, if \( u_4v_4 \) is in \( \overline{G} \), then \( \overline{G} \) contains the 6-cycle \( u_4, v_4, u_3, v_1, u_2, v_2, u_4 \). We, therefore, have a contradiction in this subcase, also.

Case 2. Assume that at least one of \( G \) and \( \overline{G} \) contains a 5-cycle. Without loss of generality, we assume that \( G \) has the 5-cycle \( C: u_1, u_2, u_3, u_4, u_1 \) with the remaining vertices denoted by \( v_1, v_2, \) and \( v_3 \). Since \( G \) has no 6-cycles, no \( v_i \) (\( 1 \leq i \leq 3 \)) is adjacent in \( G \) to two consecutive vertices of \( C \). Thus, each \( v_i \) is joined in \( \overline{G} \) to three nonconsecutive vertices of \( C \).

We now make use of the following fact: If \( S_1, S_2, S_3 \) are 3-element subsets of a 5-element set, then there exist \( i, j \) (\( i \neq j \)) such that \( |S_i \cap S_j| \geq 2 \). Hence, if \( S_i \) (\( i = 1, 2, 3 \)) denotes the set of three nonconsecutive vertices of \( C \) which are joined in \( \overline{G} \) to \( v_i \), then there exist two vertices \( v_i \), say \( v_1 \) and \( v_2 \), which are mutually adjacent in \( \overline{G} \) to at least two vertices of \( C \). This suggests a breakdown into two subcases.

Subcase 2a. Assume \( v_1 \) and \( v_2 \) are joined in \( \overline{G} \) to the same three vertices of \( C \); say \( v_1 \) and \( v_2 \) are adjacent in \( \overline{G} \) to \( u_1, u_3, \) and \( u_4 \). If \( v_1 \) is joined in \( \overline{G} \) to any two of the vertices \( u_1, u_3, \) and \( u_4 \), then it follows directly that \( \overline{G} \) contains a 6-cycle, which is contrary to hypothesis. Thus, we may assume that \( v_1 \) is joined in \( \overline{G} \) to exactly one of \( u_1, u_3, \) and \( u_4 \). If \( v_3u_1 \in E(\overline{G}) \), then we must have at least one of the edges \( v_3u_3 \) and \( v_3u_4 \) in \( \overline{G} \) also; therefore, without loss of generality, we assume that \( v_1u_4 \in E(\overline{G}) \). This further implies that \( v_3u_2, v_3u_5 \in E(\overline{G}) \) and \( v_3u_1, v_3u_3 \in E(G) \). The edge \( v_2u_5 \) belongs to \( G \), for otherwise \( v_2, u_5, v_3, u_4, v_1, u_3 \) is a 6-cycle of \( \overline{G} \). In a like manner, it follows that \( v_2u_2, v_1u_5, v_1u_2 \in E(G) \). However, then, \( v_2, u_2, u_3, v_3, u_1, u_5, v_2 \) is a 6-cycle of \( G \), which is impossible.
Subcase 2b. No two $v_i$ are joined in $\overline{G}$ to the same three vertices of $C$, but $v_1$ and $v_2$ are joined in $\overline{G}$ to two common vertices of $C$. Assume $v_1$ is adjacent in $\overline{G}$ with each of the vertices $u_1, u_3, \text{and } u_4$; thus, $v_2$ is adjacent in $\overline{G}$ with exactly two of the three vertices $u_1, u_3, \text{and } u_4$. The vertex $v_2$ cannot be joined in $\overline{G}$ to $u_3$ and $u_4$, for then $v_2$ must be joined in $\overline{G}$ to $u_1$ as well. Hence, without loss of generality, we assume $v_2$ is joined in $\overline{G}$ to $u_1$ and $u_4$. Necessarily, then, $v_2u_2 \in E(\overline{G})$. We now consider the location of the edges $v_3u_1$ and $v_3u_4$, observing that not both $v_3u_1$ and $v_3u_4$ are in $\overline{G}$ (for this is the situation discussed in Subcase 2a).

(i) If $v_3$ is joined in $\overline{G}$ to neither $u_1$ nor $u_4$, then, of course, $v_3$ is adjacent in $\overline{G}$ to $u_2, u_3, \text{and } u_5$. However, $\overline{G}$ contains the 6-cycle $v_1, u_3, v_3, u_2, v_2, u_4, v_1$, which is impossible.

(ii) Suppose $v_3$ is joined in $\overline{G}$ to only one of $u_1$ and $u_4$. Without loss of generality, we assume that $v_3u_1 \in E(\overline{G})$. Unless $v_3u_3, v_3u_5 \in E(\overline{G})$, we are returned to previously treated cases. However, $v_1, u_3, v_3, u_1, v_2, u_4, v_1$ is now a 6-cycle of $\overline{G}$ which is a contradiction.

We summarize the values established for $c(m, n)$ in the following table.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$n$</th>
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| 3   | 6 7 9 11 13 15  
| 4   | 7 6 7 7 8 9    |
| 5   | 9 7 9 11 13 15  |
| 6   | 11 7 11 8      |
| 7   | 13 8 13        |
| 8   | 15 9 15        |

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