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SIMPLE GROUPS OF ORDER $2^a3^b5^c7^dp$

BY

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ABSTRACT. Let $\mathrm{PSL}(n, q)$ denote the projective special linear group of degree n over $\mathrm{GF}(q)$, the field with q elements. The following theorem is proved. **Theorem.** *Let G be a simple group of order $2^a3^b5^c7^dp$, $a > 0$, p an odd prime. If the index of a Sylow p-subgroup of G in its normalizer is two, then G is isomorphic to one of the groups,* PSL(2, 5), PSL(2, 7), PSL(2, 9), PSL((2, 8), PSL(2, 16), PSL(2, 25), PSL(2, 27), PSL(2, 81), *and* PSL(3, 4).

1. Introduction.

1.1. This work was motivated by the work of R. Brauer, D. Mutchler and the author. Brauer [4] classified the simple groups of order $5 \cdot 3^a2^b$. Mutchler [15] and the author [1] classified simple groups of orders $2^a3^b5^cp$ and $2^a3^b7^cp$ respectively, in which the index of Sylow p-subgroup in its normalizer is 2. In this paper these results are extended to groups of order $2^a3^b5^c7^dp$.

In §2 we discuss the degrees of the irreducible characters in the principal p-block, $B_0(p)$, of G. In particular, we list the possibilities for the equation relating these character degrees.

In §3 we consider each of the possible character degree equations for $B_0(p)$ listed in §2. Using class algebra coefficients and simple group classification theorems, we show that G is isomorphic to one of the groups, PSL(2, 5), PSL(2, 7), PSL(2, 9), PSL(2, 8), PSL(2, 16), PSL(2, 25), PSL(2, 27), PSL(2, 81), and PSL(3, 4).

1.2. *Notation.* In general, upper case letters denote groups, and S_p is used to denote a Sylow p-subgroup. If A is a subgroup of a group G, then $N(A)$, $C(A)$, $[G: A]$, $|A|$ denote the normalizer of A in G, the centralizer of A in G, the index of A in G, and the order of A, respectively.

The notation x_n is used for a group element of order n. Then $C(x_n)$ refers to the centralizer of the element x_n in G.

A character of degree m is denoted by χ_m.

The notation $a(x_n, x_m, x_q)$ denotes the class algebra coefficient which is the

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number of ways each element of the conjugacy class of x_q can be written as a product of an element of the class of x_n by an element of the class of x_m.

2. **The degree equation for** $B_0(p)$.

2.1. Let G be a simple group satisfying the hypotheses:

I. $|G| = 2^a 3^b 5^c 7^d p$, where $a > 0$ and p is an odd prime.

II. $[N(S_p): S_p] = 2$.

Let x_p be an element of order p, and x_q be a p-regular element. Hypothesis II and Brauer's work [5] yield the following information concerning $B_0(p)$.

$B_0(p)$ contains the identity character, 1, a character χ and $(p-1)/2$ characters $\chi^{(m)}$, $m = 1, 2, \cdots, (p-1)/2$. There are signs δ and δ' so that $\chi(x_p) = \delta$, $\Sigma\chi^{(m)}(x_p) = \delta'$, $m = 1, 2, \cdots, (p-1)/2$, $\chi(1) \equiv \delta \pmod{p}$, $\chi^{(m)}(1) \equiv -2\delta' \pmod{p}$, $m = 1, 2, \cdots, (p-1)/2$,

$$1 + \delta\chi(x_q) + \delta'\chi^{(m)}(x_q) = 0. \tag{2.1}$$

If $x_q = 1$ in (2.1) we obtain the following relation between the degrees of the irreducible characters in $B_0(p)$,

$$1 + \delta\chi(1) + \delta'\chi^{(m)}(1) = 0. \tag{2.2}$$

We call (2.2) the degree equation for $B_0(p)$.

2.2. *Solutions to the degree equation.* Since $\chi(1)$, $\chi^{(m)}(1)$, divide $|G|$ and are relatively prime to p, hypothesis I implies that (2.2) can be written in the form

$$1 + x = y, \tag{2.3}$$

where x and y are of the form $2^r 3^s 5^t 7^u$. Lehmer [14] and the author [2] have shown independently that the solutions to (2.3) are $(x, y) = (1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 7), (7, 8), (8, 9), (9, 10), (14, 15), (15, 16), (20, 21), (24, 25), (27, 28), (35, 36), (48, 49), (49, 50), (63, 64), (80, 81), (125, 126), (224, 225), (2400, 2401)$, and $(4374, 4375)$.

3. **Proof of main theorem.**

3.1. *Preliminary results.* Before considering the possible degree equations we list some lemmas which are extremely useful in using the degree equation for $B_0(p)$ to determine the order of the group and the structures of various subgroups of G. The first lemma follows from the work [9] of Brauer and Tuan.

Lemma 3.1. *Let G be a simple group of order $p \cdot q^b \cdot r$, where p and q are primes, $(pq, r) = 1$. Suppose the degree equation for $B_0(p)$ is $\Sigma\delta_i\chi_i(1) = 0$, and G has no elements of order pq. Then for any q-block, $B(q)$,*

$$\sum\delta_i\chi_i(1) \equiv 0 \pmod{q^b},$$

where the summation is taken over all characters in $B_0(p) \cap B(q)$.

The following lemma follows from the work [9] of Brauer and Tuan and Lemma 3.1.

Lemma 3.2. *Let* G *be a group satisfying hypotheses* I *and* II *with degree equation* $q^r = w \pm 1$, *where* q *is* 2, 3, 5 *or* 7. *Then* q^r *is the full power of* q *dividing* $|G|$ *and a character of degree* q^r *vanishes on all* q-*singular elements.*

The following lemmas follow immediately from Lemmas 3.4 and 3.5 of [1].

Lemma 3.3. *Let* G *be a simple group satisfying hypotheses* I *and* II. *Let* x_r, x_s *be* p-*regular elements of* G, *and* x_p *be an element of order* p. *If* (2.3) *is the degree equation for* $B_0(p)$, *then the class algebra coefficient* $a(x_r, x_s, x_p)$ *satisfies*

$$a(x_r, x_s, x_p) = \frac{|G|[x - \chi_x(x_r)][x - \chi_x(x_s)]}{|C(x_r)|\,|C(x_s)|x(x+1)}, \tag{3.1}$$

where χ_x *is a character in* $B_0(p)$ *of degree* x.

Lemma 3.4. *If* G *is a simple group satisfying hypotheses* I *and* II *then* G *has one class of involutions and, if* x_2 *is an involution in* G *and* x_p *is an element of order* p *in* G, *then*

$$a(x_2, x_2, x_p) = p. \tag{3.2}$$

Finally we list the main results of [1] and [15].

Lemma 3.5. *Let* G *be a simple group of order* $2^a 3^b 7^c p$, $a, b > 0$, p *a prime distinct from* 2, 3 *and* 7. *If* G *satisfies hypothesis* II *then* G *is isomorphic to one of the groups,* PSL(2, 5), PSL(2, 9), PSL(2, 27) and PSL(3, 4).

Lemma 3.6. *Let* G *be a simple group of order* $2^a 3^b 5^c p$, $a, b > 0$, p *a prime distinct from* 2, 3 *and* 5. *If* G *satisfies hypothesis* II *then* G *is isomorphic to one of the groups,* PSL(2, 8), PSL(2, 16), PSL(2, 25), *and* PSL(2, 81).

3.2. *Proof of main theorem.* Next the possible degree equations are considered and the simple groups, G, which satisfy hypotheses I and II are determined.

First of all, the equation $1 + 1 = 2$ is impossible since G is a simple group, and the equation $1 + 2 = 3$ is inconsistent with the relations above (2.1).

Lemma 3.7. *If* G *is a simple group satisfying* I *and* II, *then*

(1) *If* G *has degree equation* $1 + 3 = 4$ *or* $1 + 4 = 5$, *then* G *is isomorphic to* PSL(2, 5).

(2) G *cannot have degree equation* $1 + 5 = 6$ *or* $1 + 6 = 7$.

(3) *If G has degree equation* $1 + 7 = 8$, *then G is isomorphic to* PSL(2, 7).

(4) *If G has degree equation* $1 + 8 = 9$, *then G is isomorphic to* PSL(2, 8) *or* PSL(2, 9).

(5) *G cannot have degree equation* $1 + 9 = 10$ *or* $1 + 14 = 15$.

(6) *If G has degree equation* $1 + 15 = 16$, *then G is isomorphic to* PSL(2, 16).

Proof. Statement (1) follows from the works of Blichfeldt [3] and Brauer [6]. Statement (2) is an immediate consequence of Brauer [6] and Wales [17]. Statement (3) follows from the work [17] of Wales.

When the degree equation is $1 + 8 = 9$, Lemma 3.2 and the relations above (2.1) imply that $|G| = 2^3 3^2 5^c 7^d p$, where $p = 5$ or 7. Then since a Sylow 2-subgroup of G is of order 8, statement (4) follows from the works of Brauer-Suzuki [8], Gorenstein [12], and Gorenstein-Walter [13].

When the degree equation is $1 + 9 = 10$, the relations above (2.1) imply that $p = 11$, and then the work [11] of Feit implies that G is isomorphic to PSL(2, 11). But PSL(2, 11) has no character of degree 9, and so we reach a contradiction.

When the degree equation is $1 + 14 = 15$, the relations above (2.1) and the class algebra coefficient $a(x_7, x_7, x_{13})$, $x_7 \in C(S_7)$, yield $|G| = 2^a 3^b 5^c 7 \cdot 13$. Then Lemma 3.1 applied to $B_0(7) \cap B_0(13)$ implies that the identity character and the 6 characters of degree 15 are in $B_0(7)$. This is inconsistent with the relations above (2.1) with $p = 7$. This proves statement (5).

When the degree equation is $1 + 15 = 16$, Lemma 3.2 and the relations above (2.1) yield $|G| = 2^4 3^b 5^c 7^d p$, where $p = 7$ or 17. When $p = 7$, $d = 0$ and then Lemma 3.6 leads to a contradiction. When $p = 17$, the work [11] of Feit implies that G is isomorphic to PSL(2, 16). This proves statement (6).

It is an easy matter to verify that the groups PSL(2, 5), PSL(2, 7), PSL(2, 8), PSL(2, 9) and PSL(2, 16) satisfy hypotheses I and II.

We next consider the degree equation $1 + 20 = 21$.

Lemma 3.8. *There is no simple group G satisfying hypotheses* I *and* II *with degree equation* $1 + 20 = 21$.

Proof. Let $x_2 \in C(S_2)$, $x_3 \in C(S_3)$, $x_5 \in C(S_5)$, and $x_7 \in C(S_7)$. Then the relations above (2.1) and the class algebra coefficients $a(x_i, x_i, x_p)$, $i = 2, 3, 5, 7$, yield $|G| = 2^a 3^b 5^c 7 \cdot p$ where $a \leq 8$ and even, $b \leq 5$ and odd, $c \leq 3$ and odd, and $p = 11$ or 19. When $a = 8$, the coefficient $a(x_2, x_2, x_p)$ yields $\chi_{20}(x_2) = -12$. Thus G has no elementary Abelian subgroup of order 8 and then [10, Theorem 4.2.7] leads to a contradiction.

When $p = 11$, a count of Sylow 11-subgroups yields $|G| = 2^2 3 \cdot 5 \cdot 7 \cdot 11$, $2^2 3^5 5^3 7 \cdot 11$, $2^4 3 \cdot 5^3 7 \cdot 11$, or $2^6 3^3 5 \cdot 7 \cdot 11$. In the first two cases [13] leads to a contradiction. When $|G| = 2^4 3 \cdot 5^3 7 \cdot 11$, Lemma 3.1 applied to

$B_0(11) \cap B_0(3)$ yields a contradiction. If $|G| = 2^6 3^3 5 \cdot 7 \cdot 11$, Lemma 3.1 applied to $B_0(11) \cap B_0(7)$, the relations above (2.1) with $p = 7$ and the coefficient $a(x_2, x_2, x_{11})$ yield

$$[N(S_7): C(S_7)] = 6 \text{ and } |C(x_2)| = 2^6 3.$$

But then a count of Sylow 7-subgroups leads to a contradiction.

When $p = 19$, a count of Sylow 19-subgroups yields $|G| = 2^2 3 \cdot 5 \cdot 7 \cdot 19$, $2^4 3^5 5 \cdot 7 \cdot 19$ or $2^6 3 \cdot 5^3 7 \cdot 19$. In the first two cases a contradiction is obtained by the application of Lemma 3.1 to $B_0(5) \cap B_0(19)$. When $|G| = 2^6 3 \cdot 5^3 7 \cdot 19$, the coefficients $a(x_2, x_2, x_{19})$ and $a(x_5, x_5, x_{19})$ yield

$$|C(x_2)| = 2^6 5, \ |C(x_5)|/5^3 2^2 \text{ and } \chi_{20}(x_5) = -5. \tag{3.3}$$

Clearly $\exists y_5 \in S_5$ so that $\chi_{20}(y_5) \geq 0$. Since y_5 cannot be in $C(S_5)$, the class algebra coefficient $a(y_5, y_5, x_{19})$ yields

$$|C(y_5)|/2^4 5^2 \text{ and } \chi_{20}(y_5) = 0. \tag{3.4}$$

Furthermore, every noncentral element y_5 in S_5 satisfies (3.4). Now since $\chi_{20}(x_5) = -5$, equation (2.1) implies that $\chi_{21}(x_5) = -4$. Then $\Sigma \chi^2(x_5) \geq 170$ where the summation is taken over all characters, χ, of G. Thus $|C(x_5)| \geq 170$ and then (3.3) implies that $2/|C(x_5)|$. Then Lemma 3.4 and equations (3.3) and (3.4) imply that $|C(y_5)| = 5^2$. Now it is clear that the Sylow 5-subgroups of $C(x_5)$ and $C(y_5)$ are normal which in turn implies that the set, S, of Sylow 5-subgroups of G is a trivial intersection, T.I., set. Next we represent a single S_7 subgroup, A, as a permutation group on the set S. Since the S_5's are a T.I. set, the subgroup fixing any other S_5, B, is the identity. The following congruence is immediate, $r \equiv 1 \pmod{125}$, where $r = |S|$. This congruence is incompatible with $|G|$. Thus $|G| \neq 2^6 3 \cdot 5^3 7 \cdot 19$ and Lemma 3.8 is proved.

Lemma 3.9. *Let G be a simple group satisfying hypotheses* I *and* II, *then*

(1) *If G has degree equation* $1 + 24 = 25$, *then G is isomorphic to* PSL(2, 25).

(2) *If G has degree equation* $1 + 27 = 28$, *then G is isomorphic to* PSL(2, 27).

(3) *G cannot have degree equation* $1 + 35 = 36$.

(4) *G cannot have degree equation* $1 + 48 = 49$.

Proof. Let $x_2 \in C(S_2)$, $x_3 \in C(S_3)$, and $x_7 \in C(S_7)$. If the degree equation is $1 + 24 = 25$, Lemma 3.1, the relations above (2.1), and the class algebra coefficients $a(x_i, x_i, x_p)$, $i = 2, 3, 7$, yield $|G| = 2^a 3^b 5^2 7^c p$ where $a \leq 7$ and odd, $b \leq 5$ and odd, $d \leq 2$ and even and $p = 13$ or 23. When $a = 7$, $\chi_{24}(x_2) = -8$ and then [10, Theorem 4.2.7] leads to a contradiction. When $c = 0$, Lemma 3.5 implies that G is isomorphic to PSL(2, 25). Otherwise, a count of Sylow p-subgroups yields $|G| = 2^5 3 \cdot 5^2 7^2 \cdot 13$. Then the coefficient $a(x_7, x_7, x_{13})$ yields

$$|C(x_7)|/2 \cdot 7^2. \tag{3.5}$$

Thus the S_7 subgroup of $C(x_7)$ is normal and then the set of S_7's is a T.I. set. Then as in the proof of Lemma 3.8 we see that the number of S_7's is congruent to 1 modulo 49. This is incompatible with (3.5) and $|G|$. This proves (1).

When the degree equation is $1 + 27 = 28$, Lemma 3.1, the relations above (2.1) and the class algebra coefficients $a(x_i, x_i, x_p)$, $i = 2, 5, 7$, yield $|G| = 2^a 3^3 5^c 7^d p$ where $a \leq 8$ and even, $c \leq 4$ and even, $d \leq 3$ and odd and $p = 13$ or 29. If $d = 3$, $\chi_{27}(x_7) = -22$ whence $(\chi_{27} | \langle x_7 \rangle, 1\langle x_7 \rangle) < 0$, a contradiction. When $c = 0$, Lemma 3.6 implies that G is isomorphic to PSL(2, 27). Otherwise, a count of Sylow p-subgroups yields $|G| = 2^2 3^3 5^4 7 \cdot 13$. Then the work [13] leads to a contradiction. This proves (2).

Next, if the degree equation is $1 + 35 = 36$, the relations above (2.1) and the coefficients $a(x_i, x_i, x_p)$, $i = 2, 3, 5, 7$, yield $|G| = 2^a 3^b 5^c 7^d p$ where $a \leq 10$ and even, $b \leq 4$ and even, $c \leq 3$ and odd, $d \leq 3$ and odd and $p = 37$ or 17. Now if $d = 3$, $\chi_{35}(x_7) = -14$ so that

$$(\chi_{35} | \langle x_7 \rangle, 1\langle x_7 \rangle) < 0,$$

a contradiction. Thus $d = 1$. When $p = 37$, $B_0(37)$ contains a character of degree less than 37 and then the work [11] of Feit leads to a contradiction. Next, when $p = 17$, Lemma 3.1 applied to $B_0(7) \cap B_0(17)$ yields a contradiction. Thus (3) is proved.

Finally, if the degree equation is $1 + 48 = 49$, the relations above (2.1), Lemma 3.1, and the coefficients $a(x_i, x_i, x_p)$, $i = 2, 3, 5$, yield $|G| = 2^a 3^b 5^c 7^2 p$, where $a \leq 8$ and even, $b \leq 7$ and odd, $c \leq 4$ and even, and $p = 5$ or 47. If $p = 5$ or $c = 0$, Lemma 3.5 yields a contradiction. Otherwise, a count of Sylow 47-subgroups yields a contradiction. This proves (4).

It is an easy matter to verify that PSL(2, 25) and PSL(2, 27) satisfy hypotheses I and II.

Lemma 3.10. *There is no simple group, G, satisfying hypotheses* I *and* II *with degree equation* $1 + 49 = 50$.

Proof. As usual, let $x_2 \in C(S_2)$, $x_3 \in C(S_3)$, $x_5 \in C(S_5)$ and $x_7 \in C(S_7)$. The relations above (2.1), Lemma 3.2 and the class algebra coefficients $a(x_i, x_i, x_p)$, $i = 2, 3, 5$, yield $|G| = 2^a 3^b 5^2 7^2 p$, where $a \leq 11$ and odd, $b \leq 8$ and even, and $p = 3$ or 17. When $a = 11$, $\chi_{49}(x_2) = -15$, and so G has no elementary Abelian subgroup of order 8. Then, [10, Theorem 4.2.7] leads to a contradiction. When $b = 8$, $\chi_{49}(x_3) = -32$ so that $(\chi_{49} | \langle x_3 \rangle, 1\langle x_3 \rangle) < 0$, a contradiction.

If $p = 17$, a count of S_{17} subgroups yields $|G| = 2^3 3^4 5^2 7^2 17$ or $2^9 5^2 7^2 17$. In the former case, the coefficient $a(x_2, x_2, x_{17})$ implies that

$$\sum \chi^2(x_2) > |C(x_2)|, \tag{3.6}$$

where the summation is taken over all irreducible characters, χ, of G. This is a contradiction. In the latter case, the coefficient $a(x_2, x_2, x_{17})$ yields $|C(x_2)| = 2^9$, and then the work [15] of Suzuki leads to a contradiction.

If $p = 3$, a count of S_3 subgroups yields

$$|G| = 2^9 3 \cdot 5^2 7^2,\ 2^7 3 \cdot 5^2 7^2,\ 2^5 3 \cdot 5^2 7^2, \text{ or } 2^3 3 \cdot 5^2 7^2.$$

In the first two cases, the coefficient $a(x_2, x_2, x_3)$ and [10, Theorem 4.2.7] imply that (3.6) is satisfied, a contradiction. When $|G| = 2^5 3 \cdot 5^2 7^2$, the coefficient $a(x_2, x_2, x_3)$ implies that (3.6) is satisfied unless $\chi_{49}(x_2) = -7$. Then equation (2.1), Lemma 3.4 and the fact that χ_{49}, χ_{50} are of defect zero for the primes 7 and 5 respectively yields $a(x_2, x_3, x_5) = 129$. This is absurd since this coefficient must be a multiple of 5. Finally when $|G| = 2^3 3 5^2 7^2$, it follows from the coefficient $a(x_2, x_2, x_3)$ that $C(x_2)$ is solvable and then the works [8], [12], and [13] lead to a contradiction. This proves Lemma 3.10.

Lemma 3.11. *If G is a simple group satisfying hypotheses* I *and* II *with degree equation* $1 + 63 = 64$, *then G is isomorphic to* PSL(3, 4).

Proof. Here if x_3, x_5, and x_7 are as usual, Lemma 3.2, the relations above (2.1) and the coefficients $a(x_i, x_i, x_p)$, $i = 3, 5, 7$, yield $|G| = 2^6 3^b 5^c 7^d p$, where $b \leq 6$ and even, $c \leq 6$ and even, $d \leq 3$ and odd, and $p = 5, 31$, or 13. If $p = 5$, $c = 0$ and then Lemma 3.5 implies that G is isomorphic to PSL(3, 4). When $c = 6$, $\chi_{63}(x_5) = -62$, and then $(\chi_{63}|\langle x_5\rangle, 1\langle x_5\rangle) < 0$, a contradiction.

When $p = 31$, a count of S_{31} subgroups yields

$$|G| = 2^6 3^6 7^3 31,\ 2^6 3^2 7\, 31,\ 2^6 3^2 5^4 7\, 31, \text{ or } 2^6 3^6 5^4 7^3\, 31.$$

In the first two cases, Lemma 3.5 leads to a contradiction. In the third case, Lemma 3.1 applied to $B_0(7) \cap B_0(31)$ implies that the 15 characters of degree 64 are in $B_0(7)$ which is impossible. Finally, if $|G| = 2^6 3^6 5^4 7^3 31$, the coefficient $a(x_7, x_7, x_{31})$ yields $|C(x_7)|/3^2 5^2 7^3$, and $\chi_{63}(x_7) = 14$. Furthermore,

$$|C(x_7)| = \sum \chi^2(x_7) \geq 3572.$$

Thus $|C(x_7)| = 5^2 7^3,\ 3 5 7^3,\ 3^2 5 7^3,\ 3 5^2 7^3$, or $3^2 5^2 7^3$. In each of these cases there is an element, y_5, of order 5 in $C(x_7)$ which normalizes and, hence, centralizes an S_7 subgroup of $C(x_7)$. Then the coefficient $a(y_5, y_5, x_{31})$ is not integral, a contradiction.

When $p = 13$, a count of S_{13} subgroups yields $|G| = 2^6 3^2 7\, 13$, $2^6 3^2 5^4 7\, 13$, or $2^6 3^4 5^2 7^3 13$. In the first case Lemma 3.5 leads to a contradiction.

When $|G| = 2^6 3^2 5^4 7\,13$, the coefficients $a(x_i, x_i, x_{13})$, $i = 3, 5$, yield $|C(x_7)|/3^2 5^2 7$ and $\chi_{63}(x_5) \equiv 13 \pmod{25}$. Then $\Sigma\chi^2(x_5) > |C(x_5)|$, unless

$$\chi_{63}(x_5) = 13 \quad \text{and} \quad |C(x_5)| = 25^4,$$

or

$$\chi_{63}(x_5) = -12 \quad \text{and} \quad |C(x_5)| = 35^4.$$

Let y_5 be an element of order 5 in $C(x_7)$. Then $|C(y_5)| = 5^2 7k$, $k \geq 1$. Consideration of the coefficient $a(y_5, y_5, x_{13})$ yields $\chi_{63}(y_5) \equiv 28 \pmod{35}$. Then it is clear that $y_5 \notin Z(S_5)$, and $\Sigma\chi^2(y_5) > |C(y_5)|$ unless

$$\chi_{63}(y_5) = -7 \quad \text{and} \quad |C(y_5)| = 25^2 7. \tag{3.7}$$

Now let A be an S_5 subgroup of $C(y_5)$. Clearly A is a normal subgroup of $C(y_5)$, and thus A centralizes the elements of order 7 in $C(y_5)$. But this implies that all nonidentity elements of A satisfy (3.7). This is a contradiction since there must be nonidentity elements of A conjugate to x_5. Thus $5 \nmid |C(x_7)|$. Now a count of S_7 subgroups yields $|N(S_7)| = 21$. Then Burnside's Theorem and the relations above (2.1) with $p = 7$ imply that $B_0(7)$ contains 3 characters whose degrees are congruent to ± 1 modulo 7 and 2 characters whose common degree is congruent to ± 3 modulo 7. Lemma 3.1 applied to $B_0(7) \cap B_0(13)$ implies that the identity character and χ_{64} are in $B_0(7)$. Then consideration of the tree for $B_0(7)$ (cf. [9]) yields that the other characters in $B_0(7)$ must be of degrees 13, 52, 52. Then consideration of the coefficient $a(x_2, x_2, x_7)$ yields $\chi_{13}(x_2) = -3$. Thus G has no elementary Abelian subgroup of order 8. Now [10, Theorem 4.2.7] leads to a contradiction.

Finally if $|G| = 2^6 3^4 5^2 7^3 13$, the coefficient $a(x_7, x_7, x_{13})$ yields $|C(x_7)|/3 5 7^3$ and $\chi_{63}(x_7) = 14$. Furthermore $5/|C(x_7)|$ else $\Sigma\chi^2(x_7) > |C(x_7)|$. But then if y_5 is an element of order 5 in $C(x_7)$, $7^3/|C(y_5)|$, and hence the coefficient $a(y_5, y_5, x_{13})$ leads to a contradiction.

It is shown in [1] that PSL(3, 4) satisfies hypotheses I and II. This proves Lemma 3.11.

Lemma 3.12. *If G is a simple group satisfying hypotheses* I *and* II, *then*

(1) *If G has degree equation* $1 + 80 = 81$, *then G is isomorphic to* PSL(2, 81), *and*

(2) *G cannot have degree equation* $1 + 125 = 126$.

Proof. As usual, let $x_2 \in C(S_2)$, $x_3 \in C(S_3)$, $x_5 \in C(S_5)$, and $x_7 \in C(S_7)$. When the degree equation is $1 + 80 = 81$, the coefficients $a(x_i, x_i, x_p)$, $i = 2, 5, 7$, the relations above (2.1) and Lemma 3.2 yield $|G| = 2^a 3^4 5^c 7^d p$, where $a \leq 10$ and even, $c \leq 5$ and odd, $d \leq 4$ and even and $p = 79$ or 41. If $c = 5$, the coefficient $a(x_5, x_5, x_p)$ implies that $\chi_{80}(x_5) = -45$, and then $(\chi_{80}|\langle x_5\rangle, 1\langle x_5\rangle)$ < 0, a contradiction. Now a count of S_p subgroups yields

$$|G| = 2^4 3^4 5\,79,\ 2^{10}3^4 5^3 41,\ 2^4 3^4 5\,41, \quad\text{or}\quad 2^4 3^4 5^3 7^4 41.$$

In the first two cases, Lemma 3.6 leads to a contradiction, and in the third case Lemma 3.6 implies that G is isomorphic to PSL(2, 81). In the last case, the coefficient $a(x_7, x_7, x_{41})$ yields $\chi_{80}(x_7) = 31, -18$ or -67. Now $\Sigma\chi^2(x_7) >$ $|C(x_7)|$, a contradiction, when $\chi_{80}(x_7) = 31$. When $\chi_{80}(x_7) = -18$ or -67, $(\chi_{80}|\langle x_7\rangle, 1\langle x_7\rangle) < 0$, also a contradiction.

It is an easy matter to verify that PSL(2, 81) satisfies hypotheses I and II. This proves (1).

When the degree equation is $1 + 125 = 126$, the coefficients $a(x_i, x_i, x_p)$, $i = 2, 3, 7$, the relations above (2.1), and Lemma 3.2 yield $|G| = 2^a3^b5^37^dp$, where $a \leq 13$ and odd, $b \leq 8$ and even, $d \leq 3$ and odd, and $p = 127$ or 31. If $p = 127$, the work [11] of Feit leads to a contradiction. If $p = 31$, a count of S_{31} subgroups yields $|G| = 2^{11}3^2 5^3 7\,31,\ 2^7 3^8 5^3 7\,31$, or $2^{11}3^6 5^3 7^3 31$. In the first case, consideration of $a(x_2, x_2, x_{31})$ yields $\Sigma\chi^2(x_2) > |C(x_2)|$, a contradiction. In the second case, the coefficient $a(x_3, x_3, x_{31})$ yields $\chi_{125}(x_3) = -118$. But then $(\chi_{125}|\langle x_3\rangle, 1\langle x_3\rangle) < 0$, a contradiction. When $|G| = 2^{11}3^6 5^3 7^3 31$, Lemma 3.4, the coefficients $a(x_2, x_2, x_{31})$ and $a(x_5, x_5, x_{31})$, and the fact that χ_{125} is of defect 0 for the prime 5 yield

$$|C(x_5)|/3^2 7\,5^3. \tag{3.8}$$

Furthermore, (3.8) holds for any 5-element, x_5. Let y_7 be an element of order 7 in $C(x_5)$. Consider $\langle y_7x_5\rangle = H$ of order 35. Since $(\chi_{125}|_H, 1_H)$ is a nonnegative integer, $\chi_{125}(y_7) \equiv 20 \pmod{35}$. Now the coefficient $a(y_7, y_7, x_{31})$ yields $\Sigma\chi^2(y_7) > |C(y_7)|$, a contradiction. Thus $7 \nmid |C(x_5)|$, for any 5-element, x_5. Thus an S_5 of $C(x_5)$ is normal in $C(x_5)$. Thus the set of S_5 subgroups of G is a T.I. set. Thus the number of S_5's in G is congruent to 1 modulo 125. Thus $|N(S_5)| =$ $2\,7^3 5^3$. But then there is an element, z_7, of order 7 in $N(S_5)$ so that $5^3/|C(z_7)|$. The coefficient $a(z_7, z_7, x_{31})$ now leads to a contradiction. This proves (2).

Lemma 3.13. *If G is a simple group satisfying hypotheses* I *and* II, *then*

(1) *G cannot have degree equation* $1 + 224 = 225$;

(2) *G cannot have degree equation* $1 + 2400 = 2401$ *and*

(3) *G cannot have degree equation* $1 + 4374 = 4375$.

Proof. As usual, let $x_i \in C(S_i)$, $i = 2, 3, 5$ and 7. When the degree equation is $1 + 224 = 225$, the coefficients $a(x_j, x_j, x_p)$, $j = 2, 3, 5, 7$, and the relations above (2.1) yield $|G| = 2^a3^b5^c7^dp$, where $a \leq 11$ and odd, $b \leq 8$ and even, $c \leq 4$

and even, $d \leq 5$ and odd, and $p = 223$ or 113. A count of S_p subgroups yields $|G| = 2^5 3^2 5^2 7\, 223$ or $2^7 3^4 5^4 7^5 113$. In the former case, Lemma 3.1 applied to $B_0(7) \cap B_0(223)$ implies that $B_0(7)$ contains the 111 characters of degree 225, which is absurd. In the latter case, the coefficient $a(x_7, x_7, x_{113})$ yields $\chi_{225}(x_7) = -118$ so that $(\chi_{225} | \langle x_7 \rangle, 1\langle x_7 \rangle) < 0$, a contradiction. This proves (1).

When the degree equation is $1 + 2400 = 2401$, the coefficients $a(x_j, x_j, x_p)$, $j = 2, 3, 5$, the relations above (2.1) and Lemma 3.2 yield $|G| = 2^a 3^b 5^c 7^4 p$, where $a \leq 19$ and odd, $b \leq 13$ and odd, $c \leq 8$ and even, and $p = 2399$ or 1201. Next a count of S_p subgroups yields $|G| = 2^{15} 3^3 5^6 7^4 2399$, $2^{13} 3^3 5^6 7^4 1201$, $2^5 3\, 5^2 7^4 2399$, or $2^5 3\, 5^2 7^4 1201$. In the first two cases the coefficient $a(x_2, x_2, x_p)$ implies that $\Sigma \chi^2(x_2) > |C(x_2)|$, a contradiction. In the third case, Lemma 3.1 applied to $B_0(3) \cap B_0(2399)$ implies that the 1199 characters of degree 2401 are in $B_0(3)$, which is absurd. In the final case, application of Lemma 3.1 to $B_0(3) \cap B_0(1201)$ implies that $B_0(3)$ contains a character of degree 2402. But then $\Sigma \chi^2(1) > |G|$, a contradiction. This proves (2).

Finally if the degree equation is $1 + 4374 = 4375$, the coefficients $a(x_i, x_i, x_p)$, $i = 2, 3, 5, 7$, and the relations above (2.1) yield $|G| = 2^a 3^b 5^c 7^d p$, where $a \leq 25$ and odd, $b \leq 9$ and odd, $c \leq 6$ and even, $d \leq 3$ and odd and $p = 4373$ or 547. When $p = 4373$, if $b = 9$ then the coefficient $a(x_3, x_3, x_{4373})$ implies that $\Sigma \chi^2(x_3) > |C(x_3)|$, a contradiction. Then a count of S_{4373} subgroups of G yields no choices for $|G|$. When $p = 547$, if $d = 1$, then Lemma 3.1 applied to $B_0(7) \cap B_0(547)$ implies that the 273 characters of degree 4374 are in $B_0(7)$, a contradiction. Then a count of S_{547} subgroups of G yields no choices for $|G|$. This proves (3).

We have now considered all of the possible degree equations which result from the solutions listed in §2, and we have used them to show that PSL(2, 5), PSL(2, 7), PSL(2,9), PSL(2, 8), PSL(2, 16), PSL(2, 25), PSL(2, 27), PSL(2, 81) and PSL(3, 4) are the only simple groups which satisfy hypotheses I and II.

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