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$0 \leq X^2 \leq X$

BY

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ABSTRACT. This paper studies the structure of elements X satisfying $0 \leq X^2 \leq X$ in a Dedekind σ-complete partially ordered real linear algebra. The lollipop-shaped possible spectrum of X had been described previously. Three basic example types are described, each with possible spectrum a characteristic part of the lollipop and the possibility of splitting X into a sum of these types is considered. The matrix case is scrutinized. There are applications to operator theory. Contributions to the theory of convergence in partially ordered algebras are developed for technical purposes.

1. **Introduction and examples.** Let $\mathfrak{A}$ be a Dedekind σ-complete partially ordered linear algebra (dsc-pola), that is, a partially ordered real linear algebra with directed order, with positive unit 1 and with the property that monotone increasing bounded sequences have suprema ([1], [2]). This paper investigates the structure of elements X in $\mathfrak{A}$ satisfying the inequality in the title. This setting is very general and along with the natural applications to the theory of positive matrices there are also applications of the results to the theory of operators in Hilbert space. All contractions T satisfying $\|T - \mu T^2\| \leq 1 - \mu$ are constructed.

Part of the importance of elements satisfying the title inequality lies in their relation to order units. U is an *order unit* in $\mathfrak{A}$ iff for each $A \in \mathfrak{A}$ there is a real $m > 0$ such that $-mU \leq A \leq mU$. For example, the order units in the algebra of $n \times n$ real matrices are the positive matrices. If U is an order unit, then $-mU \leq U^2 \leq mU$ for some $m > 0$ and $X = m^{-1}U$ is an order unit satisfying $0 \leq X^2 \leq X$. Conversely, if X satisfies our inequality, it becomes an order unit in the *sub-dsc-pola of X-bounded elements* $\mathfrak{A}_X = \{A \in \mathfrak{A}: -mX \leq A \leq mX$ for some real $m > 0\}$.

1.1. It was shown in [3] that if $X \in \mathfrak{A}$ satisfies $0 \leq X^2 \leq X$, then spectrum X lies within the lollipop-shaped region $\mathcal{C} \cup \mathcal{S}$ where

$$\mathcal{C} = \{|\lambda| \leq \tfrac{1}{2}: |\lambda - \lambda^2| \leq \tfrac{1}{4}\}$$

and

$$\mathcal{S} = \{\lambda \text{ real}: 0 \leq \lambda \leq 1\}.$$

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Three basic example types.

Type S.

Proposition 1.2. *Let P be a positive idempotent in $\mathfrak{A}$. Then the set $\mathfrak{A}'_P = \{A \in \mathfrak{A}: PA = AP = A$ and $-mP \leq A \leq mP$ for some real $m > 0\}$, under its inherited order and algebraic structure, forms a dsc-pola in which P is both identity and an order unit.*

Proof. Only order completeness and closure under multiplication need be checked. Let $\{A_n\}_1^\infty$ be a monotone increasing sequence in $\mathfrak{A}'_P$ bounded by B in $\mathfrak{A}'_P$ and with supremum A in $\mathfrak{A}$. Then $A_n \leq A \leq B \leq mP$ for some positive real m and all n. Multiplying on both left and right by P, we get $A_n \leq PAP \leq B \leq mP$ which shows that PAP is the supremum of $\{A_n\}_1^\infty$ in $\mathfrak{A}'_P$. Check closure under multiplication by using the following exercise.

Exercise 1.3. If $-C \leq D \leq C$ and $-E \leq F \leq E$, then $-CE \leq DF \leq CE$.

Theorem 1.4 (Nakano, DeMarr, see Dai [1, Theorem 4.2]). *Suppose the identity 1 of a dsc-pola $\mathfrak{A}$ is an order unit. Then $\mathfrak{A}$ is commutative. If m is the largest and M is the smallest real number satisfying $m1 \leq A \leq M1$, then* spectrum *A is contained in the closed real interval $[m, M]$ and contains the end points of that interval.*

(We note that Nakano and DeMarr both prove a completely satisfactory spectral theorem for elements in $\mathfrak{A}$ under the above hypotheses. Actually Nakano [5] assumes $\mathfrak{A}$ is a lattice. DeMarr's hypotheses are even weaker than "1 is an order unit".)

X satisfying $0 \leq X^2 \leq X$ is said to be of type S if it is in $\mathfrak{A}'_P$ for some P. Since the identity in $\mathfrak{A}'_P$ is an order unit, it follows using 1.1 and the above "spectral theorem" that, for X in $\mathfrak{A}'_P$, $0 \leq X^2 \leq X$ iff $0 \leq X \leq P$. Thus a type S element has spectrum within $\mathfrak{S}$. (The complex number λ is not in the spectrum of X if and only if $\operatorname{Re}\lambda + i \operatorname{Im}\lambda - X$ has an inverse $B + iC$ in the formal complexification of $\mathfrak{A}$; see 2.5 of [3] for details.)

Typical example of a type S element: $\mathfrak{A}$ = all real functions on a set, with usual order, P is the constant function 1, and X is any function taking values in the closed interval $[0, 1]$.

Type C. Let

$$Y = X - X^2 \geq 0. \tag{1.5}$$

Then $X^2 - X + Y = 0$ and "solving" this equation for X, choosing the negative sign in the quadratic formula we obtain

$$X = g(Y) = (1 - \sqrt{1 - 4Y})/2 \quad \text{(false)}. \tag{1.6}$$

Simple examples ($X = 1$ for instance) show that this formula for X is invalid. Dis-

regarding this, note that the Maclaurin series

$$g(z) = \sum_{n=1}^{\infty} b_n z^n = \sum_{n=1}^{\infty} \frac{(2n-2)!}{n!(n-1)!} z^n \tag{1.7}$$

for the complex function $w = (1 - \sqrt{1 - 4z})/2$ has positive coefficients, converges absolutely in $|z| \leq \frac{1}{4}$, and for all such z

$$|w| \leq \tfrac{1}{2} \text{ and } w - w^2 = z \tag{1.8}$$

[4, Theorems 246–247].

Let $\mathfrak{A}_1$ be the ring of formal real power series absolutely convergent in $|z| \leq \frac{1}{4}$, partially ordered coefficientwise. If $X_2 = \sum_{n=1}^{\infty} b_n z^n \geq 0$, then $X_2 - X_2^2 =$ the series $z \geq 0$. Thus X_2 satisfies our inequality. Let $\mathfrak{A}_2$ be the sub-dsc-pola of $\mathfrak{A}_1$ formed of sums of X_2-bounded elements and multiples of the identity. X in $\mathfrak{A}$ is of type **C** if it is the image of X_2 under an order-preserving, suprema-preserving algebra homomorphism of $\mathfrak{A}_2$ into $\mathfrak{A}$.

The question of invertibility in the ring of power series with absolutely convergent coefficients was answered by N. Wiener (see [6, Ex. 12, p. 97] or [10, Lemma 6_{16}]). For $\mathfrak{A}_1$ we merely have a change of scale. An element $\sum_0^{\infty} c_n z^n \in \mathfrak{A}_1$ has spectrum the set of complex numbers $\{w\colon w = \sum_0^{\infty} c_n z^n,\ |z| \leq \frac{1}{4}\}$. It then follows from (1.8) that the spectrum of X_2 in $\mathfrak{A}_1$ is $\mathcal{C}$. The spectrum of X_2 in $\mathfrak{A}_2$ is $\mathcal{C}$ also, but we postpone a proof to §7. A type **C** element, as a homomorphic image of X_2, must have its spectrum within $\mathcal{C}$.

Type J. X is said to be of type **J** if $X = \frac{1}{2}W + Z$ where $W^2 = W$, $W \geq 0$, $Z^2 = 0$, $WZ = ZW = Z$, and $Z \geq -\frac{1}{2}W$. $X^2 = \frac{1}{4}W + Z$ so X satisfies our inequality. It is easy to check that spectrum X is $\{\frac{1}{2}\}$ if $W = 1$ and is $\{\frac{1}{2}\} \cup \{0\}$ otherwise. Specific example of a type **J** element: The matrix $X = \left[\begin{smallmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{smallmatrix}\right]$. Here $W = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $Z = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$.

Other examples could be constructed by taking direct sums of types **S**, **C**, and **J**. We ask to what extent a "decomposition of the spectrum" into the three types might be possible for all X satisfying $0 \leq X^2 \leq X$. Our main theorem will give positive results in this direction when $\frac{1}{2}$ is not in the spectrum. A complete answer is given for the dsc-pola of $n \times n$ matrices.

2. **Order limits.** If $B_n \in \mathfrak{A}$ we say $o\text{-}\lim B_n = B$ iff there exists a positive decreasing sequence A_n with greatest lower bound 0 such that $-A_n \leq B - B_n \leq A_n$ for all n. The o-limit is unique if it exists. "The o-limit of a sum is the sum of the o-limits." $o\text{-}\lim \gamma B_n = \gamma(o\text{-}\lim B_n)$ for γ real. o-limits preserve order, i.e., if $B_n \leq C_n$, then $o\text{-}\lim B_n \leq o\text{-}\lim C_n$. If B_n is monotone increasing, $o\text{-}\lim B_n = \sup B_n$. See [1] and [2] for more information on dsc-polas and order limits.

If $o\text{-}\lim_n \sum_{m=0}^{n} B_m$ exists, we denote it by $\sum_{m=0}^{\infty} B_m$. For $B_m \geq 0$, we write $\sum_{m=0}^{\infty} B_m < \infty$ to indicate convergence of the series.

We can now make sense of the right side of (1.6) for any element satisfying our inequality.

Proposition 2.1. *Suppose* $0 \leq X^2 \leq X$ *and let* $Y = X - X^2$. *Then* $g(Y) = \Sigma_{n=1}^{\infty} b_n Y^n \leq X$.

Denote the sum of the series by V.

Proof. Let $V_m = \Sigma_{n=1}^{m} b_n Y^n$. By induction we shall show that $X \geq V_m$ for all $m \geq 1$. In fact, this inequality implies that $X = X^2 + Y \geq V_m^2 + Y \geq V_{m+1}$, the right inequality following from the fact that as formal series

$$\left(\sum_{n=1}^{\infty} b_n z^n\right)^2 + z = \left(\sum_{n=1}^{\infty} b_n z^n\right). \tag{2.2}$$

For $m = 1$, $V_m = Y \leq X$. It follows that the series converges and $X \geq V$.

2.3. Since we can also verify that $V_m^2 + Y \leq V_{2m}$, it appears most likely that $V - V^2 = Y$, which is a formula we must have for our main theorem. Unfortunately, this remains unproven in general, the difficulty being caused by the fact that multiplication is not necessarily o-continuous [2, p. 639]. (We say *multiplication is o-continuous* iff o-lim $A_n = A$ implies o-lim $BA_n = BA$ and o-lim $A_n B = AB$ for all B.)

We say *power series* $\Sigma_0^{\infty} c_n A^n$ and $\Sigma_0^{\infty} d_n A^n$ *multiply correctly* if their product is $\Sigma_{n=0}^{\infty} (\Sigma_{i=0}^{n} c_i d_{n-i}) A^n$. We conjecture that power series always multiply correctly.

Theorem 2.4. *Let* $A \geq 0$ *in* $\mathfrak{A}$ *and suppose* $\Sigma_0^{\infty} A^n < \infty$. *Then power series* $\Sigma_0^{\infty} c_n A^n$ *satisfying* $|c_n| \leq Mn^p$ (*some* M, p, *all* n) *o-converge and multiply correctly. Moreover,*

$$B \sum_0^{\infty} c_n A^n = \sum_0^{\infty} c_n B A^n \quad \textit{for all } B \in \mathfrak{A}. \tag{2.5}$$

This theorem, the proof of which we postpone to §6, is related to the following theorem of R. DeMarr.

Theorem 2.6 [2, Propositions 2 and 3]. *For* $A \geq 0$ *and* $\lambda \geq 0$, *the following are equivalent*:

(i) $(\lambda - A)^{-1} = \Sigma_0^{\infty} \lambda^{-1}(\lambda^{-1}A)^n$.

(ii) $(\lambda - A)^{-1} \geq 0$.

(iii) $\Sigma_0^{\infty} \lambda^{-1}(\lambda^{-1}A)^n < \infty$.

Proof. Without loss of generality, let $\lambda = 1$. For (ii) implies (i), let $F = (1 - A)^{-1}$ in DeMarr's Proposition 3. All else is spelled out there.

3. Main Theorem.

Theorem 3.1. *Suppose*

(1) $0 \leq X^2 \leq X$.

(2) $(\frac{1}{2} - X)^{-1}$ *exists.*

(3) $[(\frac{1}{2} - X)^{-1}]^2 \geq 0$.

Then there exists $P \geq 0$ *satisfying* $P^2 = P$, P *commuting with any element that commutes with* X. $\frac{1}{2}P \leq PX \leq P$. spectrum $PX \subseteq \mathcal{S}$. *If we let* $Q = 1 - P$ *and* $Y = X - X^2$, *then* $\Sigma_{n=1}^{\infty} b_n(QY)^n = QX = QV$ *and* spectrum $QX \subseteq \mathcal{C}$.

Proof. $(\frac{1}{2} - X)^2 = \frac{1}{4} - Y$. Hypothesis (3) says $(\frac{1}{4} - Y)^{-1} \geq 0$, or equivalently by Theorem 2.6, $(\frac{1}{4} - Y)^{-1} = \Sigma_{n=0}^{\infty} 4(4Y)^n \leq \infty$. It now follows from Theorem 2.4 and the fact that $\Sigma_{n=1}^{\infty} b_n(\frac{1}{4})^n < \infty$ that $V = \Sigma_{n=1}^{\infty} b_n Y^n$ is among the series multiplying correctly and we have

$$V - V^2 = Y = X - X^2. \tag{3.2}$$

Now define $D = X - V \geq 0$. Define $P = D(2X - 1)^{-1}$. That all elements mentioned so far commute with each other and with any element which commutes with X follows from (2.5). The following formulas are all straightforward algebraic consequences of (3.2):

$$\begin{aligned} &\text{(i) } D(X + V) = D, \\ &\text{(ii) } D(2X - 1) = D(1 - 2V) = D^2, \\ &\text{(iii) } 0 \leq D^2 4[(\tfrac{1}{2} - X)^{-1}]^2 = P^2 = D(2X - 1)(2X - 1)^{-2} = P. \end{aligned} \tag{3.3}$$

We also check $P - PX = (2X - 1)^{-1}D(1 - X) = (2X - 1)^{-1}D(X + V - X) \geq 0$ so $PX \leq P$. Also $2PX - P = P(2X - 1) = D \geq 0$ so $\frac{1}{2}P \leq PX$.

By (2.5), $\Sigma_{n=1}^{\infty} b_n(QY)^n = QV$ where $Q = 1 - P$. Moreover, $QX - QV = QD = 0$ so $QV = QX$. QX is not necessarily a type C element (V is), but its spectrum lies within $\mathcal{C}$ because it is the image of $X_2 = \Sigma_{n=1}^{\infty} b_n z^n$ under the algebraic homomorphism of $\mathfrak{A}_1$ to $\mathfrak{A}$ which sends the series $\Sigma_{n=1}^{\infty} c_n z^n$ to $\Sigma_{n=1}^{\infty} c_n(QY)^n$.

PX is a type S element. Because $\frac{1}{2}P \leq PX \leq P$, its spectrum as an element of $\mathfrak{A}_P'$ lies within $[\frac{1}{2}, 1]$. If $P \neq 1$, its spectrum as an element of $\mathfrak{A}$ will be its $\mathfrak{A}_P'$ spectrum with the point 0 added.

Comments. spectrum X = spectrum $PX \cup$ spectrum QX. Thus we have also proved the main result of [3] assuming our additional hypotheses (2) and (3). In any event, the heart of [3] is a comparatively unmotivated proof that

$$\text{for } \lambda > \tfrac{1}{4}, \quad \sum_{n=1}^{\infty} \lambda^{-1}(\lambda^{-1}Y)^n < \infty. \tag{3.4}$$

This can now be obtained easily from Proposition 2.1 by comparison of series.

Hypothesis (2) is equivalent to the existence of $(\frac{1}{4} - Y)^{-1}$. (3.4) says that $0 \leq (\lambda - Y)^{-1} = \Sigma_{n=0}^{\infty} \lambda^{-1}(\lambda^{-1}Y)^n$. Hypothesis (3) merely requires right continuity of these relations at $\lambda = \frac{1}{4}$. It is not clear that (3) is necessary. However this may turn out, DeMarr's generalization of the Perron-Frobenius Theorem provides weak conditions on $\mathfrak{A}$ guaranteeing the required continuity.

Theorem 3.5 (DeMarr [12]). *Suppose* (a) *for each* $B \in \mathfrak{A}$ *there exists real* $\lambda > 0$ *and* $C \geq 0$ *in* $\mathfrak{A}$ *with* $-C \leq B \leq C$ *and* $\Sigma_{n=1}^{\infty} (\lambda^{-1}C)^n < \infty$; *or suppose (stronger)* (b) $\mathfrak{A}$ *has an order unit. Then for* $A \geq 0$, $(\mu - A)^{-1}$ *does not exist where* $\mu = \inf\{\lambda > 0 : \Sigma_{n=1}^{\infty} (\lambda^{-1}A)^n < \infty\}$.

4. **The matrix case.** Suppose $\mathfrak{A}$ is the dsc-pola of $n \times n$ real matrices and $0 \leq X^2 \leq X$ in $\mathfrak{A}$. Since order units abound the conclusion of the Main Theorem holds unless $\frac{1}{2} \in$ spectrum X. In that case we must work hard. Choose $0 < \alpha < 1$ with α so close to 1 that $(\frac{1}{2}\alpha, \frac{1}{2}] \cap$ spectrum $\alpha X = \emptyset$. Since $0 \leq (\alpha X)^2 \leq \alpha X$, the Main Theorem holds for αX. The spectral projection P for the part of the spectrum of αX lying in $(\frac{1}{2}, 1]$ is a nonnegative matrix. This P is the same spectral projection for X. $\frac{1}{2}P \leq PX \leq P$. It follows from the deeper structure of $\mathfrak{A}'_P$ (see [1, Chapter I]) that $PX = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_n P_n$ where $\frac{1}{2} < \lambda_1 < \lambda_2 < \cdots < \lambda_n \leq 1$ are the eigenvalues of X in $(\frac{1}{2}, 1]$, P_i are positive projections and $P_i P_j = 0$ if $i \neq j$.

Now let λ be an eigenvalue of Y in $\mathcal{C}$ and let P_λ be the associated spectral projection. Then $P_\lambda X = XP_\lambda = \lambda P_\lambda + Z_\lambda$ where $P_\lambda Z_\lambda = Z_\lambda P_\lambda = Z_\lambda$ and Z_λ is nilpotent (look at the Jordan form).

$$P_\lambda Y = P_\lambda(X - X^2) = (\lambda - \lambda^2)P_\lambda + (1 - 2\lambda)Z_\lambda + Z_\lambda^2 \tag{4.1}$$

and this is the spectral part of Y corresponding to the eigenvalue $\lambda - \lambda^2$. The function $z = w - w^2$ is a 1-1 holomorphic function from the interior of $\mathcal{C}$ onto $\{|z| < \frac{1}{4}\}$ with inverse $w = \Sigma\, b_n z^n$. It follows (see [7, Vol. I, pp. 113–114]) that, for λ in the interior of $\mathcal{C}$, $\Sigma\, b_n (P_\lambda Y)^n = P_\lambda X$.

Now suppose λ is on the boundary of $\mathcal{C}$, i.e. $|\lambda - \lambda^2| = \frac{1}{4}$. $\Sigma\, b_n Y^n$ converges and hence $\Sigma\, b_n (P_\lambda Y)^n$ converges. This latter series can converge [7, Vol. I, p. 111, Theorem 7] iff the series $\Sigma\, b_n z^n$ and its formal derivatives up to degree $m - 1$ converge at $\lambda - \lambda^2$ where m is the index of $\lambda - \lambda^2$ in Y = the degree of $\lambda - \lambda^2$ in the minimal polynomial of Y = the order of nilpotence of $(1 - 2\lambda)Z_\lambda + Z_\lambda^2$. The relevant convergence facts are in [4, Theorems 246–247]. $\Sigma\, b_n z^n$ converges to $(1 - \sqrt{1 - 4z})/z$ for all $|z| = \frac{1}{4}$. $\Sigma\, nb_n z^{n-1}$ converges to $(1 - 4z)^{-\frac{1}{2}}$ for all $|z| = \frac{1}{4}$ except at $z = \frac{1}{4}$ where it diverges. $\Sigma\, n(n-1)b_n z^{n-2}$ diverges when $|z| = \frac{1}{4}$. Hence for $\lambda - \lambda^2 \neq \frac{1}{4}$, that is $\lambda \neq \frac{1}{2}$, we have $0 = ((1 - 2\lambda)Z_\lambda + Z_\lambda^2)^2$, and since Z_λ is nilpotent, this means $Z_\lambda^2 = 0$. For $\lambda = \frac{1}{2}$, $0 = (0 \cdot Z_\lambda + Z_\lambda^2)^1 = Z_\lambda^2$ also.

We can now calculate explicity $\Sigma\, b_n P_\lambda Y = (\Sigma\, b_n(\lambda - \lambda^2)^n)P_\lambda + (\Sigma\, nb_n(\lambda - \lambda^2)^{n-1}(1 - 2\lambda))Z_\lambda$ which is $\lambda P_\lambda + Z_\lambda = P_\lambda X$ when $\lambda \neq \frac{1}{2}$ but is λP_λ for $\lambda = \frac{1}{2}$. Let Q be the spectral projection associated with the spectral set $\mathcal{C} - \{\frac{1}{2}\}$. Summing over the spectral values in the set, we obtain $\Sigma\, b_n (QY)^n = QX$.

$P_{\frac{1}{2}}X$ is a type **J** element if $Z_{\frac{1}{2}} \neq 0$, whereas if $Z_{\frac{1}{2}} = 0$, it is at once types **C**, **S**, and **J**. To verify this, only $P_{\frac{1}{2}} \geq 0$ still must be proven. As we have seen, $P_{\frac{1}{2}}$ is the spectral projection associated to the maximum eigenvalue $\frac{1}{4}$ of the nonnegative matrix Y and $\frac{1}{4}$ is a simple root of the minimal polynomial of Y.

Lemma 4.2. *Let the maximum eigenvalue of a nonnegative matrix A be a simple root of its minimal polynomial. Then the associated spectral projection is nonnegative.*

Proof. Without loss of generality assume the maximum eigenvalue is 1 and denote the associated projection by P_1. The other eigenvalues of modulus 1 must be roots of unity (this is proven for irreducible nonnegative matrices in [7, Vol. II, p. 53], and can be extended without difficulty to reducible matrices using [11, p. 46, line 11]). $B = n^{-1}(A + A^2 + \cdots + A^n)$ is nonnegative and also has spectral projection P_1 at eigenvalue 1, but if n is chosen correctly, B has all its other eigenvalues of modulus strictly less than 1. The "simple root" hypothesis implies $P_1B = P_1$. $B = P_1 + C$, where $P_1C = CP_1 = 0$, and C has spectral radius less than 1 so $C^n \to 0$. Then $B^n \to P_1$ and P_1 is nonnegative.

5. **Banach algebras and operator theory.** Let $\mathbf{B}$ be a (perhaps complex) normed algebra, $\mathbf{R}$ be the real numbers and $\mathfrak{A} = \mathbf{B} \oplus \mathbf{R}$, with coordinatewise addition and multiplication and $A = (B, \mu) \geq 0$ iff $\|B\| \leq \mu$. $\mathfrak{A}$ is order-complete (a dsc-pola) iff $\mathbf{B}$ is norm-complete (a Banach algebra) [8, Theorem 2] and we shall assume this completeness.

Again let us suppose that $X = (S, \mu) \in \mathfrak{A}$ satisfies $0 \leq X^2 \leq X$. This is equivalent to

$$\|S\| \leq \mu \quad \text{and} \quad \|S - S^2\| \leq \mu - \mu^2. \tag{5.1}$$

Only $0 \leq \mu \leq 1$ need be considered.

5.2 First consider $0 \leq \mu \leq \frac{1}{2}$ and apply Proposition 2.1. $Y = (S - S^2, \mu - \mu^2)$.

$$V = g(Y) = \left(\sum_1^\infty b_n(S - S^2)^n, g(\mu - \mu^2)\right) = \left(\sum_1^\infty b_n(S - S^2)^n, \mu\right).$$

$V \leq X$ or, in other words, $\|S - \Sigma_1^\infty b_n(S - S^2)^n\| \leq 0$. We have here a short proof of the following unpublished result of DeMarr.

Application 5.3. *Let $\|R\| \leq \frac{1}{4}$ in a Banach algebra. Then $S = \Sigma_1^\infty b_n R^n$ is the unique solution to the equation $S - S^2 = R$ in $\|S\| \leq \frac{1}{2}$.* The uniqueness is proved above. That $\|S\| \leq \Sigma\, b_n \|R\|^n \leq \frac{1}{2}$ and that S solves the equation are well known (norm-convergent power series in Banach algebras multiply correctly).

Corollary 5.4. *Let A satisfy $\|I - A\| \leq 1$ in a Banach algebra. Then there is a unique B in the algebra with $\|I - B\| \leq 1$ and $B^2 = A$.*

Proof. Let $S = \frac{1}{2}(I - B)$ and $R = \frac{1}{4}(I - A)$.

We have also shown in 5.2 that when $0 \leq \mu \leq \frac{1}{2}$, then $X = V$, X is a type $\mathbf{C}$ element and the Main Theorem's conclusion holds. We now show that when $\mu > \frac{1}{2}$ then $\frac{1}{2}$ is not a spectral value of X, which together with the existence of an order

unit $(0, 1)$ in $\mathfrak{A}$ implies that the Main Theorem holds for this Banach algebra case using only the hypothesis $0 \leq X^2 \leq X$.

Lemma 5.5. spectrum $X \subseteq \{\mu\} \cup \{|\lambda| < \frac{1}{2}: |\lambda - \lambda^2| \leq \mu - \mu^2\}$. *The same inclusion holds for* spectrum S.

Proof. spectrum $X = \{\mu\} \cup$ spectrum S. For $\lambda \in$ spectrum S, (5.1) implies $|\lambda| \leq \mu$ and $|\lambda - \lambda^2| \leq \mu - \mu^2$. For $|\lambda| \geq \frac{1}{2}$, λ must be real (1.1), but if $\frac{1}{2} \leq \lambda < \mu$ then $\lambda - \lambda^2 > \mu - \mu^2$.

We now use our Main Theorem to describe all linear operators S on Hilbert space satisfying (5.1) for fixed μ. For $\mu \leq \frac{1}{2}$, such S are described in Application 5.3. Now suppose $\mu > \frac{1}{2}$. We have $P \neq 0$. $P = (E, \gamma)$ with $E^2 = E$, $\gamma^2 = \gamma$ and $\|E\| \leq \gamma$. Thus $\gamma = 1$ and $\|E\| = \|E^2\| \leq \|E\|^2$ so $\|E\| = 0$ or $\|E\| = 1$.

In case $E = 0$, $Q = (1, 0)$ and the equality $QX = QV$ yields $S = \Sigma_1^\infty b_n(S - S^2)^n$. Moreover, $\|S - S^2\| \leq \mu - \mu^2 \leq \frac{1}{4}$ so S is again of the sort described in Application 5.3.

If $\|E\| = 1$, $PX = (ES, \mu)$. spectrum $PX = \{\mu\}$ in $\mathfrak{A}_P'$ so by Theorem 1.4. $\mu P \leq PX \leq \mu P$. $PX = \mu P$. $ES = \mu E$. $S = \mu E + S_1$ where $S_1 = (1 - E)S$. Thus $QX = (S_1, 0)$ and from $QX = QV$ we obtain $S_1 = \Sigma_1^\infty b_n R^n$ where $R = (1 - E)(S - S^2)$. We next show that $\|R\| \leq \mu - \mu^2$ and thus that S_1 is among the operators described in 5.3.

An idempotent operator of norm 1 on Hilbert space, such as E, must be selfadjoint. (This can be checked, as suggested by another proof by J. Ledbetter, by expanding $\langle (1 - E)E^*x, (1 - E)E^*x\rangle$ to show that $(1 - E)E^* = 0$.) $1 - E$ must also be a selfadjoint idempotent and hence also of norm 1. $\|(1 - E)(S - S^2)\| \leq \|(S - S^2)\| \leq \mu - \mu^2$.

We summarize our results as a theorem about $T = \mu^{-1}S$.

Application 5.6. T is a linear contraction operator on Hilbert space satisfying $\|T - \mu T^2\| \leq 1 - \mu$ for given $\mu > 0$ iff it is of the form $T = E + \Sigma_1^\infty \mu^{-1} b_n R^n$ where E is a selfadjoint projection, $\|R\| \leq \mu - \mu^2$ and $ER = RE = 0$.

5.7. If $0 \leq X^2 \leq X$, then $\mathfrak{A}_X$, the sub-dsc-pola of X-bounded elements, is a Banach algebra under the norm $\|A\|_X = \inf\{m: -mX \leq A \leq mX\}$. Exercise 1.3 shows this norm is submultiplicative. Norm completeness is most enjoyably proven by checking order completeness in the dsc-pola $\mathfrak{A}_X \oplus \mathbf{R}$ described earlier in this section. (A_n, λ_n) is monotone increasing and bounded by (A, λ) in $\mathfrak{A}_X \oplus \mathbf{R}$ iff $\lambda_n X - A_n$ and $\lambda_n X + A_n$ are monotone increasing and bounded respectively by $\lambda X - A$ and $\lambda X + A$. Then λ_n is monotonically increasing, and $\sup \lambda_n$ and o-$\lim A_n$ exist. In fact, $2(\sup \lambda_n)X = \sup(\lambda_n X + A_n) + \sup(\lambda_n X - A_n)$ and $2\ o\text{-}\lim A_n = \sup(\lambda_n X + A_n) - \sup(\lambda_n X - A_n)$ and because o-limits preserve order $\sup(A_n, \lambda_n) = (o\text{-}\lim A_n, \sup \lambda_n)$.

We now apply Application 5.3 to the Banach algebra $\mathfrak{A}_X$.

Theorem 5.8. *Suppose* $0 \leq X^2 \leq X$ *in* $\mathfrak{A}$. *For each* R *satisfying* $-\frac{1}{4}X \leq R \leq \frac{1}{4}X$, $S = \Sigma_1^\infty b_n R^n$ *is the unique element of* $\mathfrak{A}$ *satisfying* $S - S^2 = R$ *and* $-\frac{1}{2}X \leq S \leq \frac{1}{2}X$.

This is a short proof for another unpublished result of DeMarr who used the statement of 5.8 to prove 5.3.

6. **Normal order convergence.** Our goal here is the proof of Theorem 2.4, but it does not take very much longer to give an outline of a new theory of convergence in dsc-polas for which multiplication is continuous.

Lemma 6.1. *Let* $A_n \geq 0$, $\Sigma A_n < \infty$, *and* $B \geq 0$. *Then* $\Sigma BA_n \leq B \Sigma A_n$.

Proof. $B \Sigma^\infty A_n \geq B \Sigma^m A_n = \Sigma^m BA_n$.

6.2. We shall say that *nao*-lim $B_n = B$ iff there exists $C_m \geq 0$ with $\Sigma_1^\infty C_j < \infty$ such that for each $m > 0$ there exists N_m so that $-C_m \leq B - B_n \leq C_m$ whenever $n \geq N_m$. (For convenience, when we say a sequence is *nao*-convergent, we assume the sequence of C's is given and, without loss of generality, we shall assume $N_{m+1} > N_m$.)

6.3. If *nao*-lim $B_n = B$, then *o*-lim $B_n = B$.

Proof. Let $A_n = \Sigma_{j=m}^\infty C_j$, where $N_m \leq n < N_{m+1}$. Then $A_n \geq 0$, $A_n \downarrow 0$, and $-A_n \leq B - B_n \leq A_n$.

6.4. "The sum of the *nao*-limits is the *nao*-limit of the sum."

6.5. Multiplication is *nao*-continuous. If *nao*-lim $B_n = B$, then *nao*-lim $AB_n = AB$ for all A.

Proof. That the order in $\mathfrak{A}$ is directed implies that $A = A_1 - A_2$ where $A_1 \geq 0$ and $A_2 \geq 0$. Thus we may assume $A \geq 0$. Then $AC_m \geq 0$, $\Sigma AC_j < \infty$ and $-AC_m \leq AB - AB_n \leq AC_m$.

6.6. If *nao*-lim $B_n = 0$ and *o*-lim $A_n = 0$, then *nao*-lim $A_nB_n = 0$.

Proof. We may even weaken the hypothesis on the A_n by assuming only $-A \leq A_n \leq A$, all A_n, fixed A. Then $-AC_m \leq A_nB_n \leq AC_m$ (use Exercise 1.3).

6.7. If *nao*-lim $B_n = B$ and *nao*-lim $A_n = A$, then *nao*-lim $B_nA_n = BA$.

Proof. Using 6.5 and 6.6, the standard proof for limits of products works here.

6.8. We say B_n is a *nao*-Cauchy sequence iff there exists C_m as in 6.2 so that for each $m > 0$ there exists N_m such that whenever $n_1 \geq N_m$ and $n_2 \geq N_m$, then $-C_m \leq B_{n_1} - B_{n_2} \leq C_m$. A sequence is *nao*-convergent iff it is *nao*-Cauchy.

Proof. Let B_n be *nao*-Cauchy. Let $D_m = B_{N_{m+1}} - B_{N_m}$. $0 \leq D_m + C_m \leq 2C_m$. Thus $\Sigma (D_m + C_m)$ converges, and hence ΣD_m converges. Note that $B_{N_1} + \Sigma_1^j D_m = B_{N_j}$. Let *o*-lim$_j$ $B_{N_j} = B$. For $N_j > n \geq N_m$, $-C_m \leq B_n - B_{N_j} \leq C_m$. Thus $-C_m \leq B_n - B \leq C_m$.

6.9. Suppose $A_n \geq 0$, ΣA_n is *nao*-convergent and $-A_n \leq B_n \leq A_n$. Then ΣB_n is *nao*-convergent.

Proof. $-\Sigma_{n1}^{n2} A_n \le \Sigma_{n1}^{n2} B_n \le \Sigma_{n1}^{n2} A_n$ so ΣB_n is *nao*-Cauchy.

6.10. If $A_n \ge 0$, $A_n' \ge 0$, and ΣA_n and $\Sigma A_n'$ are *nao*-convergent then $\Sigma_{n=1}^m (\Sigma_{j=1}^m A_j A_{n-j}')$ *nao*-converges to $(\Sigma_1^\infty A_n)(\Sigma_1^\infty A_n')$.

Proof.

$$\left(\sum_1^m A_n\right)\left(\sum_1^m A_n'\right) \le \sum_{n=1}^{2m}\left(\sum_{j=1}^n A_j A_{n-j}'\right) \le \left(\sum_1^{2m} A_n\right)\left(\sum_1^{2m} A_n'\right)$$

and the conclusion follows from 6.7 and the definition of *nao*-convergence.

6.11. Suppose in addition in 6.10 that $-A_n \le B_n \le A_n$ and $-A_n' \le B_n' \le A_n'$. Then $\Sigma_{n=1}^m (\Sigma_{j=1}^n B_j B_{n-j}')$ *nao*-converges to $(\Sigma_1^\infty B_n)(\Sigma_1^\infty B_n')$.

Proof. Write ΣB_n as $\Sigma (A_n + B_n) - \Sigma A_n$ and similarly for B_n'.

6.12. Suppose $A \ge 0$ and $\Sigma_0^\infty A^n$ *o*-converges. Then the series *nao*-converges.

Proof. Let $C_n = \Sigma_0^\infty A^n - \Sigma_0^{m-1} A^n = \Sigma_{n=m}^\infty A^n \le A^m \Sigma_0^\infty A^n$ and note that $\Sigma_0^\infty [A^m \Sigma_0^\infty A^n] < \infty$.

6.13. We prove Theorem 2.4. If $A \ge 0$ and $\Sigma_0^\infty A^n < \infty$ then $(\Sigma A^n)(\Sigma A^n) = \Sigma (n+1)A^n$ *nao*-converges. By induction the power series for $(\Sigma A^n)^p$ *nao*-converges. But these series have coefficients of the order of n^{p-1}. Thus series $\Sigma c_n B^n$ multiply correctly if $-A \le B \le A$ and $|c_n| \le Mn^p$.

6.14. Suppose $X \ge 0$ and $\Sigma c_n X^n$ converges whenever $\Sigma |c_n| < \infty$ (for instance when $0 \le X^2 \le X$). Then these series all *nao*-converge and therefore multiply correctly.

Proof. Define $C_m = \Sigma_{n=N_m}^\infty |c_n| X^n$ where N_m is chosen so that $\Sigma_{m=1}^\infty (\Sigma_{n=N_m}^\infty |c_n|) < \infty$.

7. Type C elements.

7.1. *The spectrum of* X_2 *in* $\mathfrak{A}_2$. Recall $\mathfrak{A}_2$ is the algebra of formal real series absolutely convergent in $\{|z| \le 1/4\}$ with non-zero-degree coefficients bounded by a multiple of those of $X_2 = \Sigma_1^\infty b_n z^n$.

Under the norm $\|\Sigma_0^\infty c_n z^n\| = |c_0| + \inf\{m > 0: -mX_2 \le \Sigma_1^\infty c_n z^n \le mX_2\}$, $\mathfrak{A}_2$ is the Banach algebra $(\mathfrak{A}_1)_{X_2}$ with unit added.

The norm of the backward shift operator

$$z^L : \sum c_n z^n \to \sum c_{n+1} z^n \tag{7.2}$$

in $\mathfrak{A}_2$ may be computed as $\|z^L\| = \sup_n (\|z^L z^n\|/\|z^n\|) = \sup_n (b_{n-1}^{-1}/b_n^{-1}) = 4$ (recall 1.7). One can similarly check that the operator norm of $(z^L)^n = 4^n$.

We wish to show that the spectrum of X_2 in $\mathfrak{A}_2$ is $\mathcal{C}$. Since $\mathfrak{A}_2 \subseteq \mathfrak{A}_1$, this spectrum contains $\mathcal{C}$. By (1.1) we need only check that $(\lambda - X_2)^{-1}$ exists for $1/2 < \lambda \le 1$. As a function on the complex disc $\{|z| \le 1/4\}$,

$$\frac{1}{\lambda - X_2} = \frac{X_2 - (1-\lambda)}{z - (\lambda - \lambda^2)} \tag{7.3}$$

and we are done after verifying that, for $\frac{1}{2} < \lambda < 1$, $X_2 - (1 - \lambda)$ vanishes at $\mu = \lambda - \lambda^2$ (use 1.6) and that the operator

$$(z - \mu)^L : f(z) \to (f(z) - f(\mu))/(z - \mu) \tag{7.4}$$

in $\mathfrak{A}_2$ is the bounded operator $\Sigma_{n=1}^{\infty} \mu^{n-1}(z^L)^n$. (The interested reader will find it easier to verify this himself than to locate [9, 5.16 (c) (2)].)

Additional information about type C elements would be learned by further study of $\mathfrak{A}_2$. For instance, the above discussion shows that for X a type C element and $\lambda > \frac{1}{2}$, $(\lambda - X)^{-1} = \Sigma_0^{\infty} \lambda^{-n-1}X^n \leq \|\Sigma_{n=1}^{\infty} \mu^{n-1}(z^L)^n\| X \leq (\Sigma_1^{\infty} \mu^{n-1}4^n)X = (2/(2\lambda - 1))^2 X$.

8. **Example.** The following example shows that "splitting the lollipop" as described in this paper for special cases is not possible in general. Let $\mathfrak{A}$ consist of all sequences A_n $(n = 2, 3, 4, \cdots)$ of real 3×3 matrices, algebraic operations defined by termwise matrix operations and partially ordered entrywise. Let X be the sequence

$$X_n = \left(\frac{1}{2} - \frac{1}{n}\right)\underbrace{\begin{pmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{pmatrix}}_{R_n} + \left(\frac{1}{2} + \frac{1}{n}\right)\underbrace{\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}}_{P_n}. \tag{3.5}$$

One may check that $P_n^2 = P_n$, $R_n^2 = R_n$, $R_nP_n = P_nR_n = 0$, $X_n - X_n^2 = (\frac{1}{4} - 1/n^2)(R_n + P_n)$ and thus $0 \leq X^2 \leq X$. It is also easy to check that the conclusion of the Main Theorem holds. In fact, the uniqueness of spectral decomposition for matrices shows that P must be the constant sequence P_n. But now restrict consideration to the sub-dsc-pola of X-bounded elements $\mathfrak{A}_X$. $P \notin \mathfrak{A}_X$ because at best $P_n \leq (n/2)X_n$ and the conclusion of the Main Theorem does not hold for X in $\mathfrak{A}_X$. Note that $(\frac{1}{2} - X)^{-1}$ is not within $\mathfrak{A}_X$ either.

The above matrix-sequence counterexample object was discovered by K. Koh for use as a ring-theory counterexample.

With the above pathology as a guide we conjecture that if $0 \leq X^2 \leq X$ in $\mathfrak{A}$, then $\mathfrak{A}$ can be embedded as a sub-dsc-pola of a dsc-pola $\mathfrak{B}$ in which the spectrum of X "splits".

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