TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 173, November 1972

AUTOMORPHISMS OF ω_1-TREES

BY

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ABSTRACT. The number of automorphisms of a normal ω_1-tree T, denoted by $\sigma(T)$, is either finite or $2^{\aleph_0} \leq \sigma(T) \leq 2^{\aleph_1}$. Moreover, if $\sigma(T)$ is infinite then $\sigma(T)^{\aleph_0} = \sigma(T)$. Moreover, if T has no Suslin subtree then $\sigma(T)$ is finite or $\sigma(T) = 2^{\aleph_0}$ or $\sigma(T) = 2^{\aleph_1}$. It is consistent that there is a Suslin tree with arbitrary precribed $\sigma(T)$ between $2^{\aleph_0}$ and $2^{\aleph_1}$, subject to the restriction above; e.g. $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_{324}$ and $\sigma(T) = \aleph_{17}$. We prove related results for Kurepa trees and isomorphism types of trees. We use Cohen's method of forcing and Jensen's techniques in L.

0. **Introduction.** A normal ω_1-tree, being a partially ordered set with $\aleph_1$ elements, admits at most $2^{\aleph_1}$ automorphisms. Also, the number of isomorphism types of ω_1-trees is not larger that $2^{\aleph_1}$. In §1, we shall show that the number of automorphisms a normal ω_1-tree admits is either finite, or at least $2^{\aleph_0}$; as a matter of fact, it is $2^{\aleph_0}$ or $2^{\aleph_1}$, with the exception of Suslin trees.

In §2, we investigate Suslin trees. Inspired by Jensen's result [6], which says that it is consistent to have rigid Suslin trees, and Suslin trees which admit $\aleph_1$, or $\aleph_2$ automorphisms, we prove a general consistency result saying that the number of automorphisms of a Suslin tree can be anything between $2^{\aleph_0}$ and $2^{\aleph_1}$ it is allowed to be. We shall also show that the number of isomorphism types of Suslin trees can be $2^{\aleph_1}$.

In §3, we follow Jensen's line by showing that results analogous to those of §2 can be obtained for Suslin continua.

In §4, Kurepa trees will be considered. We shall establish the consistency of rigid Kurepa trees, of Kurepa trees with $2^{\aleph_0}$ of $2^{\aleph_1}$ automorphisms, and of as many as $2^{\aleph_1}$ isomorphism types of Kurepa trees.

Finally, we shall observe that there are Aronszajn trees which admit $2^{\aleph_1}$ automorphisms, and that we do not know whether the existence of rigid normal ω_1-trees can be proved.

Throughout the paper, we utilize the axiom $\Diamond$ of Jensen, cf. [6]. Also, we use the method of forcing and assume the reader to be familiar with it.

Received by the editors November 25, 1970.

AMS (MOS) subject classifications (1969). Primary 0260, 0265; Secondary 0415.

Key words and phrases. Normal ω_1-tree, Suslin tree, Suslin continuum, Kurepa tree, rigid tree.

(1) The work on this paper was partially supported by NSF Grant GP-22937.

1. **Normal** ω_1**-trees.** A partially ordered set $(T, \leq)$ is called a *tree* if for each $x \in T$, the set $\hat{x} = \{y \in T: y < x\}$ is well ordered by $\leq$. The *order* $o(x)$ of $x \in T$ is the order type of $\hat{x}$. If α is an ordinal then U_α, the α *the level* of the tree, is the set of all $x \in T$ whose order is α. The *length* of the tree T is $l(T) = \sup\{o(x) + 1: x \in T\}$; an α*-tree* is a tree whose length is α. Maximal well-ordered subsets of T are called *branches* of the tree; the *length* of a branch b is the order type of b. An α-branch is a branch of length α.

If T is a tree (we shall often write T instead of $(T, \leq)$) and if α is an ordinal, then $T|\alpha$ is the tree consisting of all the elements of T whose order is less than α. If T_1 and T_2 are trees then T_2 is an *extension* of T_1 (and T_1 is a *restriction* of T_2) if $T_1 = T_2|\alpha$ for some α. For each $x \in T$, the set $\{y \in T: y \geq x\}$ is a tree; we denote it T_x.

Definition. A *normal* ω_1*-tree* is a tree T with the following properties:

(1) the length of T is ω_1,

(2) for each $\alpha < \omega_1$, U_α is at most countable; U_0 consists of exactly one element,

(3) for each $x \in T$, the length of T_x is ω_1,

(4) each $x \in T$ has exactly $\aleph_0$ immediate successors,

(5) if $o(x) = o(y) = \alpha$ is a limit ordinal, and if $\hat{x} = \hat{y}$, then $x = y$.

A *normal binary* ω_1*-tree* is defined similarly, except that in (4), each x has exactly two immediate successors. A *normal countable* (*binary*) *tree* is a countable tree which is a restriction of a normal (binary) ω_1-tree.

Since a normal ω_1-tree has $\aleph_1$ elements, we may always assume that the elements of the tree are countable ordinals. On the other hand, we have another representation of normal ω_1-trees. Let ${}^{\omega_1}\omega = \bigcup_{\alpha<\omega_1} {}^{\alpha}\omega$ be the set of all functions s whose domain is a countable ordinal and whose values are the natural numbers; similarly, ${}^{\omega_1}2 = \bigcup_{\alpha<\omega_1} {}^{\alpha}2$ is the set of all such functions s with values 0 and 1. The *length* of s, $l(s)$, is the domain of s. Any normal ω_1-tree can be represented by a set $T \subseteq {}^{\omega_1}\omega$ (ordered by the natural ordering of ${}^{\omega_1}\omega$), which satisfies the following conditions:

(1) for each $s \in T$ and each $n \in \omega$, $\widehat{sn} \in T$,

(2) for each $s \in T$ and each $\alpha < \omega_1$, there is $t \in T$ such that $l(t) = \alpha$ and $t \subseteq s$ or $t \supseteq s$,

(3) for each $\alpha < \omega_1$, $T \cap {}^{\alpha}\omega$ is at most countable.

If we replace $n \in \omega$ by $n = 0, 1$ in (1) then we get a representation of normal binary ω_1-trees. The order of each s is $l(s)$. If $l(s)$ is a limit ordinal then $s = \bigcup_{t \in b} t = \lim b$ where $b = \{t: t \subsetneqq s\}$ is a branch in $T|l(s)$; thus we shall often make no distinction between b and $\lim b$. Also, we shall frequently use expressions like $T = \lim_{\alpha\to\omega_1} T|\alpha$. All results we shall prove for normal trees hold for normal binary trees as well.

An *isomorphism* of two trees is a one-to-one mapping of T_1 onto T_2 which preserves the partial ordering. An *automorphism* of T is an isomorphism of T onto itself. A tree T is called *rigid* if the only automorphism of T is the identity. By $\sigma(T)$ we denote the cardinality of the set of all automorphisms of T.

A set A of elements of T is an *antichain* of T if any two elements of A are incomparable. A normal ω_1-tree is called a *Suslin tree* if it has no uncountable antichain.

Before proceeding to our first theorem we make two more remarks on the terminology. A *subtree* naturally means a subset of a tree with the same partial ordering. An element $x \in T$ is said to *split* in T if there are at least two immediate successors of x in T.

Theorem 1. *If T is a normal ω_1-tree then either $\sigma(T)$ is finite or $2^{\aleph_0} \leq \sigma(T) \leq 2^{\aleph_1}$. Moreover, if no subtree of T is a Suslin tree then either $\sigma(T)$ is finite or $\sigma(T) = 2^{\aleph_0}$ or $\sigma(T) = 2^{\aleph_1}$.*

Proof. Let U be the set of all $x \in T$ such that T_x is not rigid. U is a subtree of T and if $x \leq y$ and $y \in U$ then $x \in U$. Note that every automorphism of T is an automorphism of U. We shall consider two cases.

Case I. U has an infinite antichain A. For each $x \in A$, T_x has at least one nontrivial automorphism π_x. If $Y \subseteq A$, let π_Y be the automorphism of T defined as follows: for $z \in \bigcup\{T_x : x \in Y\}$, let $\pi z = \pi_x z$ where $x \in Y$ is the unique x such that $z \in T_x$; otherwise, let $\pi z = z$. For different $Y \subseteq A$ we get different π_Y; it follows that $\sigma(T) \geq 2^{\aleph_0}$.

Case II. U does not have an infinite antichain. Then only finitely many elements of U split in U and U has only finitely many branches. Let U_0 be the set of all $x \in T$ with the property that there exist two different immediate successors x_1 and x_2 of x, such that both T_{x_1} and T_{x_2} are rigid, and T_{x_1} is isomorphic to T_{x_2}. Clearly, $U_0 \subseteq U$, and since U has only finitely many branches, U_0 is cofinal in U. Now we consider three subcases:

Subcase (a). U_0 is infinite. Since U_0 does not have infinite antichains U_0 has an infinite chain C (a linearly ordered subset). For each $x \in C$, let x_1, x_2 be successors of x, as in the definition of U_0, and let π_x be an isomorphism of T_{x_1} and T_{x_2}. If $Y \subseteq C$, let π_Y be the automorphism of T defined as follows: for $z \in \bigcup\{T_{x_1} : x \in Y\}$ let $\pi z = \pi_x z$, if $z \in \bigcup\{T_{x_2} : x \in Y\}$ let $\pi z = \pi_x^{-1} z$ (in either case, x is unique); otherwise, let $\pi z = z$. For different $Y \subseteq A$ we get different π_Y; it follows that $\sigma(T) \geq 2^{\aleph_0}$.

Subcase (b). For each $x \in U$, let $s(x)$ be the set of all immediate successors y of x such that T_y is rigid and is isomorphic to some $T_{Y'}$, where $y' \neq y$ is another immediate successor of x. Assume that for some $x \in U$, $s(x)$ is infinite. It is obvious that $\sigma(T_x) \geq 2^{\aleph_0}$.

Subcase (c). U_0 is finite and for each $x \in U_0$, $\sigma(x)$ is finite. In this case, $\sigma(T)$ is finite.

Now suppose that the additional assumption of the theorem holds. Let us consider the following two cases:

Case I. At most countably many elements of U split in U. Let $\alpha < \omega_1$ be such that no element of U of order α or bigger splits in U. Then U has only countably many branches longer than α.

Subcase (a). U has an ω_1-branch B. We let $U_0 \subseteq U$ be as in Case II above. U_0 is cofinal in B and hence U_0 contains an uncountable chain. By the same argument as in Subcase II(a) above, we get $2^{\aleph_1}$ automorphisms of T; thus $\sigma(T) = 2^{\aleph_1}$.

Subcase (b). All branches of U are countable. Hence all branches of U are shorter than some $\beta < \omega_1$. Consequently, $\sigma(T)$ is at most the number of all permutations of the βth level, i.e. $\sigma(T) \leq 2^{\aleph_0}$.

Case II. Uncountably many elements of U split in U. Using the assumption about Suslin trees we shall show that U has an uncountable antichain. By the same argument as earlier in this proof, we conclude that $\sigma(T) = 2^{\aleph_1}$.

Assume that U does not have an uncountable antichain; we shall construct a subtree which is a Suslin tree. Let V be the set of all $x \in U$ such that there are uncountably many $y \in U$ such that $y \geq x$ and y splits in U. V has elements of arbitrary high order, hence V is uncountable; also, if $x \in V$ and $y \leq x$ then $y \in V$. We shall show that there is some $z \in V$ such that $T_z \cap U \subseteq V$. If not, then $U - V$ is cofinal in U. However, let A be a maximal antichain in $U - V$, i.e. $A \subseteq U - V$ is an antichain and every $y \in U - V$ is comparable to some $a \in A$. Since A is at most countable, let $z \in V$ be of order bigger than the orders of all $a \in A$. Let $y \geq z$, $y \in U - V$. Now y is comparable to some $a \in A$, i.e. $y \geq a$, and hence $z \geq a$; we reached the contradiction, since $z \in V$ and $a \notin V$. Having $z \in V$ such that $T_z \cap U = T_z \cap V$, let $W = T_z \cap U$. For each $x \in W$, there are uncountably many $y \in W$, $y \geq x$, such that y splits in W. Let X be the set of all $y \in W$ which split in W. X is an uncountable tree, every $y \in W$ splits, every $T_x \cap X$ $(x \in X)$ is uncountable and X has no uncountable antichain. As a matter of fact, X is almost a Suslin tree, except that its elements may not have $\aleph_0$ immediate successors; all we know is that they split. Thus we take $Y \subseteq X$ consisting of all elements of X whose order (in the tree X) is a limit ordinal. Now, Y is a Suslin tree.

2. **Suslin trees.** If T is a Suslin tree then by Theorem 1, either $\sigma(T)$ is finite or $2^{\aleph_0} \leq \sigma(T) \leq 2^{\aleph_1}$. If we are interested in what values $\sigma(T)$ takes, the best we can hope for is a relative consistency result. This is because the existence of Suslin trees is independent of the axioms of Zermelo-Fraenkel set theory ZFC (see [15], [2], [13], [5], [7]).

The question of automorphism properties of Suslin trees was first investigated by Jensen in [6]. He proved that his axiom ($\Diamond$) implies the existence of rigid Suslin trees and Suslin trees with $\aleph_1$ automorphisms; also that $V = L$ implies the existence of Suslin trees with $\aleph_2$ automorphisms. Here we generalize this result to show, by the way of consistency, that $\sigma(T)$ can be anything it is allowed to be.

First we prove a refinement of Theorem 1 to show what $\sigma(T)$ is allowed to be.

Theorem 2. *If T is a normal ω_1-tree and if $\sigma(T)$ is infinite, then $\sigma(T)^{\aleph_0} = \sigma(T)$.*

Proof. Let $\sigma(T) = \kappa$ and let $U = \{x \in T: \sigma(T_x) = \kappa\}$. We consider two cases.

Case I. U has an infinite antichain. Then we can construct $\kappa^{\aleph_0}$ automorphisms of T.

Case II. U does not have an infinite antichain.

Subcase (a). U has an ω_1-branch B. We let U_0 be the set of all $x \in U$ with the property that there exist different immediate successors x_1, x_2 of x such that neither x_1 nor x_2 are in U and T_{x_1} is isomorphic to T_{x_2}. U_0 is cofinal in U, and hence we may use the argument from Subcase II(a) Theorem 1, to get $\sigma(T) = 2^{\aleph_1}$.

Subcase (b). Each branch of U is at most countable. Thus there exists $\alpha < \omega_1$ such that $U \subseteq T|\alpha$. Let x_n, $n \in \omega$, be all the elements of the αth level. Obviously,

$$\prod_{n=0}^{\infty} \sigma(T_{x_n}) \leq \sigma(T).$$

On the other hand,

$$\sigma(T) \leq 2^{\aleph_0} \prod_{n=0}^{\infty} \sigma(T_{x_n}).$$

This is proved as follows: $2^{\aleph_0}$ is the number of all permutations of x_n, $n \in \omega$; for any $n, m \in \omega$, the number of isomorphisms between T_{x_n} and T_{x_m} is at most $\sigma(T_{x_n})$. Thus

$$\prod_{n=0}^{\infty} \sigma(T_{x_n}) \leq \kappa \leq 2^{\aleph_0} \cdot \prod_{n=0}^{\infty} \sigma(T_{x_n}).$$

Now, using the fact that $\kappa \geq 2^{\aleph_0}$, we see that either $\kappa = 2^{\aleph_0}$ or $\kappa = \prod_{n=0}^{\infty} \lambda_n$ where λ_n are cardinals less than κ. A simple cardinal arithmetic shows that $\kappa^{\aleph_0} = \kappa$.

Before we present the consistency proof, we shall discuss briefly the method we use. We assume that the reader is familiar with the method of forcing. Given a transitive model $\mathfrak{M}$ of ZFC, the *ground model*, we consider a partially ordered *set* $(P, \leq)$ *of forcing conditions* in $\mathfrak{M}$ and assume that there exists a set $G \subseteq P$ which is *generic* (over $\mathfrak{M}$). Then $\mathfrak{M}[G]$ is a model of ZFC, the *generic extension* of $\mathfrak{M}$. By saying that there is a generic extension of $\mathfrak{M}$ which satisfies a certain theorem we mean that there is a $(P, \leq)$ such that $\mathfrak{M}[G]$ satisfies that theorem whenever G is generic.

A generic extension has the same ordinals as the ground model. Under some circumstances, $\mathfrak{M}[G]$ also has the same cardinals and the cofinality function $cf(\alpha)$ as $\mathfrak{M}$. We recall the following well-known fact: Call two conditions p, q *compatible* if there is some r such that $p \geq r$ and $q \geq r$. If κ is a cardinal then $(P, \leq)$ is said to satisfy the *κ-chain condition* (κ-c.c.) if every set of pairwise incompatible conditions has power less than κ. $(P, \leq)$ is *countably closed* if every countable set of pairwise compatible conditions has a lower bound.

Lemma 2.1. (a) *If, in $\mathfrak{M}$, κ is a regular cardinal and $(P, \leq)$ satisfies the κ-c.c., then κ is a regular cardinal in $\mathfrak{M}[G]$.*

(b) *If $(P, \leq)$ is countably closed in $\mathfrak{M}$, then every countable set of ordinals in $\mathfrak{M}[G]$ is an element of $\mathfrak{M}$. Consequently, $\omega_1^{\mathfrak{M}[G]} = \omega_1^{\mathfrak{M}}$.*

Now we present the theorem:

Theorem 3. *Let $\mathfrak{M}$ be a transitive model of ZFC which satisfies $2^{\aleph_0} = \aleph_1$; let κ be a cardinal in $\mathfrak{M}$ and assume $\kappa^{\aleph_0} = \kappa$ in $\mathfrak{M}$. Then there is a generic extension $\mathfrak{M}[G]$ which has the same cardinals and the same cofinality and satisfies the following:*

There exists a Suslin tree T which admits exactly κ automorphisms.

Thus, if we choose $\mathfrak{M}$ such that, e.g., $2^{\aleph_1} = \aleph_{17}$ and $\kappa = \aleph_5$, then, in $\mathfrak{M}[G]$, we have a Suslin tree T with $\sigma(T) = \aleph_5$, although $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_{17}$.

Proof. We use the following set $(P, \leq)$ of forcing conditions: P is the set of all pairs (T, f) where

(i) T is a countable normal binary tree, $T \subseteq {}^{\omega_1}2$,

(ii) $\mathrm{dom}(f) \subseteq \kappa$, $|\mathrm{dom}(f)| \leq \aleph_0$, f is one-to-one and $\mathrm{rng}(f)$ is a group of automorphisms of T,

(iii) $(T_2, f_2) \leq (T_1, f_1)$ iff T_2 is an extension of T_1 and, for all $i \in \kappa$, $f_2(i) \supseteq f_1(i)$.

Let G be a fixed generic set of conditions. To begin with, it is easy to see that $(P, \leq)$ is countably closed, since if $\{(T_n, f_n): n \in \omega\}$ is pairwise compatible, then $(T, f) = (\lim_{n\to\infty} T_n, \lim_{n\to\infty} f_n)$ is a condition stronger than each (T_n, f_n). Thus every countable set of ordinals in $\mathfrak{M}[G]$ is in $\mathfrak{M}$. We shall show that $(P, \leq)$ satisfies the $\aleph_2$-c.c., and consequently, by Lemma 2.1, $\mathfrak{M}[G]$ has the same cardinals and cofinality as $\mathfrak{M}$. First, a combinatorial lemma:

Lemma 2.2. *Assume $2^{\aleph_0} = \aleph_1$. Let W be a family of countable subsets of κ, $|W| > \aleph_1$. Then there exists $Z \subseteq W$, $|Z| > \aleph_1$, and a countable $S \subseteq \kappa$ such that $X \cap Y = S$ for any two distinct elements X, Y of Z.*

Proof. Obviously, it is sufficient to prove the lemma for $\kappa = \omega_2$. The argument is essentially due to Marczewski [10]. We can assume $\kappa = \aleph_2$. For each $X \in W$, let $X_\alpha (\alpha < \omega_1)$ be the αth element of X (in the natural order). There exists α such that $|\{X_\alpha: X \in W\}| > \aleph_1$; let α_0 be the least one. We can construct $W_1 \subseteq W$, $|W_1| > \aleph_1$ such that, for any distinct $X, Y \in W_1$, the α_0th element of X is greater than all elements of Y (or conversely). Since $|\{\{X_\alpha: \alpha < \alpha_0\}: X \in W_1\}| \leq 2^{\aleph_0} = \aleph_1$, there is $Z \subseteq W_1$, $|Z| > \aleph_1$ such that $\{X_\alpha: \alpha < \alpha_0\}$ is the same for each $X \in Z$.

With this lemma at hand, it is not difficult to see that if W is a set of conditions and $|W| > \aleph_1$, then there exist $Z \subseteq W_1$, $|Z| > \aleph_1$, and a countable $S \subseteq \kappa$ such that

$$\text{if } (T, f) \neq (T', g) \in Z, \text{ then } T = T', \operatorname{dom}(f) \cap \operatorname{dom}(g) = S \text{ and } f \mid S = g \mid S.$$

Thus (T, f) and (T', g) are compatible for any pair of conditions in Z; hence the $\aleph_2$-c.c. is satisfied.

Let us define $(\mathcal{T}, \mathfrak{f})$ by

$$(\mathcal{T}, \mathfrak{f}) = \lim_{(T, f) \in G} (T, f).$$

Obviously, $\mathcal{T}$ is a normal binary ω_1-tree and $\mathfrak{f}$ is a one-to-one map of κ onto a group of automorphisms of $\mathcal{T}$. To show that $\mathcal{T}$ is a Suslin tree, we prove the following lemma:

Lemma 2.3. *If (T, f) is a condition of limit length α and if A is a maximal antichain in T, then there is a condition (T', f') stronger than (T, f), of length $\alpha + 1$, such that A is maximal in T'.*

The argument which we used in [2] (see also [3]) and which we will not repeat here, shows that this lemma is enough to show that $\mathcal{T}$ is a Suslin tree.

Proof. There is a certain pattern in the arguments used in this and subsequent proofs. In all cases we construct a branch of a countable normal tree of limit length, satisfying certain conditions. Rather than describing the construction, which is in each case based on diagonalization, we shall refer to the following (extramathematical)

Diagonal Principle. If T is a countable normal tree of limit length, then there exists a branch through T which satisfies a countable number of prescribed conditions.

(E.g., given a countable set B of branches of T, there exists a branch which is not in B.)

As for Lemma 2.3, we can use the Diagonal Principle to show that for each $x \in T$ there exists an α-branch b_x through x such that, for all $\pi \in \operatorname{rng}(f)$, $\pi(b_x)$ intersects A (this is because A is maximal). We let

$$B = \{\pi b_x \colon x \in T \text{ and } \pi \in \operatorname{rng}(f)\}.$$

Further we set $T' = T \cup B$ and f' the obvious extension of f onto T'. Clearly, T' is a normal tree of length $\alpha+1$, $\operatorname{rng}(f')$ is a group of automorphisms of T', and A is a maximal antichain in T'.

It remains to be proved that $\mathfrak{T}$ admits at most κ automorphisms.

Lemma 2.4. (*In* $\mathfrak{M}[G]$). *For each automorphism ρ of $\mathfrak{T}$ and each $x \in \mathfrak{T}$ there exist $y > x$ and $\pi \in \operatorname{rng}(\mathfrak{f})$ such that $\rho \mid \mathfrak{T}_y = \pi \mid \mathfrak{T}_y$.*

Proof. Assume that the lemma is false, that there exist ρ and $x \in \mathfrak{T}$ such that, for each $\pi \in \operatorname{rng}(\mathfrak{f})$ and each $y \in \mathfrak{T}$, $y > x$, there is some $z \in \mathfrak{T}$, $z > x$, such that $\rho(z) \neq \pi(z)$. In other words, there exists a name $\underline{\rho}$ for ρ, a condition $(T_0, f_0) \in G$ and $x_0 \in T_0$ such that

$$(T_0, f_0) \Vdash \underline{\rho} \text{ is an automorphism of } \mathfrak{T}$$

and, for each $(T, f) \leq (T_0, f_0)$, each $\pi \in \operatorname{rng}(f)$ and each $y \in T$, $y > x_0$, there exist a condition $(T', f') \leq (T, f)$ and some $z \in T'$, $z > y$, such that

$$(T', f') \Vdash \underline{\rho}(z) \neq \pi(z).$$

We shall construct a sequence $(T_0, f_0) \geq (T_1, f_1) \geq \cdots \geq (T_n, f_n) \geq \cdots$ of conditions and sequences $x_0 < x_1 < \cdots < x_n < \cdots$, $\tilde{x}_0 < \tilde{x}_1 < \cdots < \tilde{x}_n < \cdots$ of elements of $T_0, T_1, \cdots, T_n, \cdots$. Let $n \mapsto (u(n), v(n))$ be some reasonable one-to-one correspondence between ω and $\omega \times \omega$. Having constructed (T_n, f_n), x_n and $\tilde{x}_n$, we let (T_{n+1}, f_{n+1}), x_{n+1} and $\tilde{x}_{n+1}$ be such that

(a) $x_{n+1} > x_n$, $x_{n+1} \in T_{n+1} - T_n$,

(b) $(T_{n+1}, f_{n+1}) \Vdash \underline{\rho}(x_{n+1}) = \tilde{x}_{n+1} \neq \pi(x_{n+1})$,

where π is the $u(n)$th element of $\operatorname{rng}(f_{v(n)})$ (each $\operatorname{rng}(f_i)$ is supposed to have a fixed enumeration by integers). We let

$$(T_\omega, f_\omega) = \lim_{n\to\infty} (T_n, f_n), \qquad b = \lim_{n\to\infty} x_n, \; \tilde{b} = \lim_{n\to\infty} \tilde{x}_n.$$

Clearly, b and $\tilde{b}$ are α-branches of T_ω (where α is the length of T_ω), and $(T_\omega, f_\omega) \Vdash \underline{\rho}(b) = \tilde{b}$, whereas $\tilde{b} \neq \pi(b)$ for each $\pi \in \operatorname{rng}(f_\omega)$.

As easily seen, there is a tree $T \supseteq T_\omega$ of length $\alpha + 1$ such that $b \in T$, $\tilde{b} \notin T$ and T is closed under all $\pi \in \operatorname{rng}(f_\omega)$. Namely, consider

$$U_\alpha = \{\pi b_z \colon \pi \in \operatorname{rng}(f_\omega),\ z \in T_\omega\} \cup \{\pi b \colon \pi \in \operatorname{rng}(f_\omega)\},$$

where, for each $z \in T_\omega$, b_z is an α-branch such that $\pi b_z \neq \tilde{b}$ for each $\pi \in \operatorname{rng}(f_\omega)$. Also, f_ω naturally extends to an f for T. Since $b \in T$ and $\tilde{b} \notin T$, we have $(T, f) \Vdash b \in \mathfrak{T} \wedge \tilde{b} \notin T$; thus $(T, f) \Vdash \underline{\rho}$ is not an automorphism of $\mathfrak{T}$, a contradiction.

Now, if ρ is any automorphism of $\mathfrak{T}$, then by Lemma 2.4 we can find a maximal antichain A in $\mathcal{T}$ such that, for each $y \in A$, there is $\pi \in \operatorname{rng}(\mathcal{P})$ such that $\rho \mid \mathcal{T}_y = \pi \mid \mathcal{T}_y$. Since $|A| \le \aleph_0$, we can conclude that $\mathcal{T}$ admits at most $\kappa^{\aleph_0}$ automorphisms. As mentioned above, every countable subset of κ in $\mathfrak{M}[G]$ is in $\mathfrak{M}$, so that $(\kappa^{\aleph_0})_{[G]} = (\kappa^{\aleph_0})_{\mathfrak{M}} = \kappa$. This completes the proof of Theorem 3.

Next we prove the consistency of $2^{\aleph_1}$ isomorphism types of Suslin trees. We use Jensen's axiom ($\diamondsuit$) which was invented to show that there are Suslin trees in L and to get the results on automorphism properties of Suslin trees in L, cf. [5], [6].

The formulation of ($\diamondsuit$) we use here is as it appears in [8].

A set C of ordinals less than a limit ordinal α is *closed unbounded* in α if it is closed and unbounded in the order-topology of the ordinals less than α. A set $A \subseteq \omega_1$ is *stationary* if it intersects every closed unbounded subset of ω_1.

Axiom ($\diamondsuit$). *There is a sequence $\{h_\alpha: \alpha \in \omega_1\}$ of functions with the property that, for every function $h \subseteq \omega_1 \times \omega_1$, the set $\{\alpha \in \omega_1: h \mid \alpha = h_\alpha\}$ is stationary.*

We recall that ($\diamondsuit$) holds in L.

Theorem 4. *If ($\diamondsuit$) holds, then there are $2^{\aleph_1}$ different isomorphism types of Suslin trees.*

Proof.(2) We shall construct binary Suslin trees T_F for every $F \in {}^{\omega_1}2$ such that, for any $F \neq G$, T_F and T_G are not isomorphic. To do that we construct countable normal trees T_f for each $f \in {}^{\underline{\omega_1}}2$ such that $l(T_f) = \alpha + 1$ where $\alpha = \operatorname{dom}(f)$, and that $T_f \subseteq T_g$ whenever $f \subseteq g$. Then we let $T_F = \lim_{f \subseteq F} T_f$ for each $F \in {}^{\omega_1}2$. The elements of T_F will be countable ordinals.

Having constructed T_f, $f \in {}^{\underline{\omega_1}}2$, we construct $T_{f\frown 0}$ and $T_{f\frown 1}$ simply by adjoining $\aleph_0$ immediate successors to each element of the last level of T_f; moreover, we can easily make $T_{f\frown 0} \neq T_{f\frown 1}$. If α is a limit ordinal and $f \in {}^{\alpha}2$, let $\hat{T}_f$ be the tree $\hat{T}_f = \lim_{g \to f} T_g$ of length α; clearly, if $f \neq g \in {}^{\alpha}2$ then $\hat{T}_f \neq \hat{T}_g$. To construct T_f we consider three cases.

Case I. The set $A = \operatorname{rng}(h_\alpha)$ is a maximal antichain in $\hat{T}_f$. Then we construct T_f in such a way that A is maximal in T_f (this is to make sure that the resulting ω_1-trees are Suslin trees, as in [5]); to get T_f we use the Diagonal Principle.

Case II. The function h_α is an isomorphism of $\hat{T}_f$ and some $\hat{T}_g$, $g \neq f$. Then we construct T_f and T_g in such a way that h_α cannot be extended to an isomorphism of T_f and T_g: For each $x \in \hat{T}_f$, let b_x be an α-branch of $\hat{T}_f$ through x and let b be an additional α-branch of $\hat{T}_f$. We construct T_f by adjoining one

(2) Originally (cf. [4]) we proved this theorem with $\aleph_2$ instead of $2^{\aleph_1}$ and got the consistency of $2^{\aleph_1}$ by a different method. The present proof was suggested by K. Hrbáček.

element on the top of each branch b_x, and also of b, and T_g by adjoining elements to the branches $h_\alpha(b_x)$ only.

Case III. If neither I or II hold, we let T_f be an arbitrary normal $(\alpha + 1)$-tree which extends $\hat{T}_f$.

Now, each T_F is a Suslin tree, as mentioned in Case I. To show that T_F and T_G are not isomorphic if $F \neq G$, assume that π is an isomorphism of T_F and T_G. Let α_0 be an ordinal such that $F|\alpha_0 \neq G|\alpha_0$. By $(\Diamond)$ there is a limit ordinal $\alpha > \alpha_0$ such that $\pi|\alpha$ is an isomorphism of $\hat{T}_{F|\alpha}$ and $\hat{T}_{G|\alpha}$ and that $\pi|\alpha = h_\alpha$. By Case II, however, h_α cannot be extended to an isomorphism of T_F and T_G; a contradiction.

3. **Remarks on Suslin continua.** A Suslin continuum is a complete dense linear ordering whose intervals satisfy the countable chain condition but it is not imbeddable in the real continuum. The notions of Suslin trees and Suslin continua are closely related; one can easily construct a Suslin continuum from a Suslin tree and vice versa (cf. [3]).

Jensen [6] investigated the automorphism properties of Suslin continua in L (or, assuming $\Diamond$). The results are the same as mentioned for Suslin trees in §2, because there is a relationship between automorphisms of Suslin trees and of Suslin continua.

Suppose that T is a normal binary Suslin tree, we may assume that $T \subseteq {}^{\omega_1}2$. Consider the set of all branches of T, ordered by the lexicographical ordering of ${}^{\omega_1}2$. This is a linearly ordered set; moreover, the ordering is complete. If we remove the endpoints and the left sides of all gaps, we get a Suslin continuum S.

To get the results for Suslin continua from those for Suslin trees, Jensen uses the following relationship between automorphisms of Suslin trees and of Suslin continua:

If C is a closed unbounded subset of ω_1, let $T^C = \{x \in T\colon o(x) \in C\}$.

Lemma 3.1. (a) *If C is a closed unbounded subset of ω_1, if π is an automorphism of T^C and if π preserves the lexicographical ordering of ${}^{\omega_1}2$ then π determines a unique automorphism $\tilde{\pi}$ of S.*

(b) *If ϕ is an automorphism of S then there exist a closed unbounded set $C \subseteq \omega_1$ and an automorphism π of T^C such that $\phi = \tilde{\pi}$.*

The automorphism $\tilde{\pi}$ is defined in the natural way, since the branches of T^C are dense in S. It follows that if $C_1 \subseteq C_2$ and $\pi_1 = \pi_2 | T^{C_1}$ then $\tilde{\pi}_1 = \tilde{\pi}_2$.

Using Lemma 3.1 we can prove the analogs of Theorems 3 and 4.

Theorem 3a. *Under the assumptions of Theorem 3 there is a generic extension which preserves the cardinals and cofinality and satisfies the following:*

There exists a Suslin continuum S which admits exactly κ automorphisms.

The proof is analogous to that of Theorem 3. We modify the forcing conditions as follows: Let Λ be the set of all limit countable ordinals. We let $\mathrm{rng}(f)$ to a group of automorphisms of T^{Λ}, preserving the lexicographical ordering of ${}^{\omega_1}2$. Otherwise, (T, f) is as before. The proof proceeds similarly. The same argument shows that $\mathcal{T}$ is a Suslin tree; $\mathfrak{f}$ provides us with κ automorphisms of the corresponding Suslin continuum S. To show that S admits at most κ automorphisms, consider an automorphism ϕ of S. By Lemma 3.1 there is a closed unbounded subset C of ω_1 and an automorphism ρ of $\mathcal{T}^C$ such that $\phi = \tilde{\rho}$. Now, we can prove an analog of Lemma 2.4. For each $x \in \mathcal{T}$ there exist $y > x$ and $\pi \in \mathrm{rng}(\mathfrak{f})$ such that

$$\rho \mid \mathcal{T}_y^{C \cap \Lambda} = \pi \mid \mathcal{T}_y^{C \cap \Lambda}.$$

(The proof goes exactly as before, we only have to carry a name $\underset{\sim}{C}$ for C throughout.) By the remark following Lemma 3.1 it is obvious that $\tilde{\pi}$ and $\tilde{\rho}$ agree in the interval of S determined by y. Thus ϕ is obtained as a combination of countably many elements of $\mathrm{rng}(\mathfrak{f})$ and hence S admits at most κ automorphisms.

In a similar way, we can modify the construction of the trees T_F in the proof of Theorem 4 to obtain

Theorem 4a. *If* $(\Diamond)$ *holds then there are* $2^{\aleph_1}$ *different isomorphism types of Suslin continua.*

To construct T_f when $\mathrm{dom}(f)$ is a limit ordinal α, we slightly change Case II: There exist $g \neq f$ and a set C closed unbounded in α and h_α is an isomorphism of $\hat{T}_f^C$ and $\hat{T}_g^C$. Otherwise, we proceed as before. We establish that, for no closed unbounded subset C of ω_1, T_F^C and T_G^C are isomorphic if $F \neq G$. This implies that the Suslin continua corresponding to T_F and T_G are not isomorphic.

4. **Kurepa trees.** We may ask whether results similar to those in §2 can be proved for other types of ω_1-trees, particularly for Kurepa trees.

A *Kurepa tree* is a normal ω_1-tree which has at least $\aleph_2$ ω_1-branches. By Silver [11], the existence of Kurepa trees is not provable in ZFC. However, by Stewart [14] and Solovay [12], the existence of Kurepa trees is consistent relative to ZFC. We prove the following theorem.

Theorem 5. *The following is consistent relative to ZFC*:

(a) *there is a rigid Kurepa tree,*

(b) *there is a Kurepa tree* T *with* $\sigma(T) = 2^{\aleph_0}$,

(c) *there is a Kurepa tree* T *with* $\sigma(T) = 2^{\aleph_1}$,

(d) *there are* $2^{\aleph_1}$ *different isomorphism types of Kurepa trees.*

Using hybrids of Kurepa and Suslin trees we could possibly get Kurepa trees with $\sigma(T)$ between $2^{\aleph_0}$ and $2^{\aleph_1}$, but if we are interested purely in Kurepa trees then the expected infinite values of $\sigma(T)$ are $2^{\aleph_0}$ and $2^{\aleph_1}$.

Proof. We could prove (a) and (c) by refinement of Stewart's and Solovay's arguments. However, we use a different approach. First, the following lemma yields (c).

Lemma 4.1. *If T is a Kurepa tree then there is a Kurepa tree T' with $\sigma(T') = 2^{\aleph_1}$.*

Proof. Let T_0 to be the following binary tree:

$$T_0 = \{s \in {}^{\omega_1}2 \colon s(\alpha) = 1 \text{ for only finitely many } \alpha\}.$$

T_0 is a normal binary ω_1-tree. Moreover, $\sigma(T_0) = 2^{\omega_1}$: for $f \in {}^{\aleph_1}2$, let π_f be the following automorphism of T_0:

$$\text{if } s \in {}^{\alpha}2,\ s \in T_0, \text{ then } \pi(s) = \breve{s}$$

where

$$\breve{s}(\xi) = \begin{cases} s(\xi) & \text{if } f(\xi) = 0, \\ 1 - s(\xi) & \text{if } f(\xi) = 1. \end{cases}$$

Clearly, $\pi_f \neq \pi_g$ if $f \neq g$.

Now we define $T' = T \otimes T_0$ as follows:

$$T' = \{(u, v) \colon u \in T,\ v \in T_0 \text{ and } o(u) = o(v)\},$$
$$(u, v) \leq (u', v') \text{ iff } u \leq u' \text{ and } v \leq v'.$$

T' is a normal ω_1-tree. For each automorphism π of T_0 we have an automorphism $\tilde{\pi}$ of T' defined as follows:

$$\tilde{\pi}(u, v) = (u, \pi v).$$

Thus $\sigma(T') = 2^{\aleph_1}$. And finally, b is a branch in T' iff there are branches b_1 and b_2 in T and T_0 such that $b = \{(u, v) \in T' \colon u \in b_1 \text{ and } v \in b_2\}$. Hence T' is a Kurepa tree.

Next, call an ω_1-tree T *totally rigid* if for no $x \neq y \in T$, T_x and T_y are isomorphic. We show

Lemma 4.2. *If T is a Kurepa tree and if ($\Diamond$) holds, then there is a totally rigid Kurepa tree.*

The assumptions of Lemma 4.2 are compatible since both are true in L.

Proof. We may assume that the elements of T are countable ordinals and that there are uncountably many countable ordinals which are not elements of T. By recursion on the levels of T we shall construct a tree T' on countable ordinals which is "fatter" than T and is totally rigid.

Let $\{h_\alpha : \alpha < \omega_1\}$ be a sequence of functions given by the axiom ($\Diamond$). Suppose that we have already constructed $T' \mid \alpha$ $(\alpha < \omega_1)$; by the induction hypothesis, $T' \mid \alpha \supseteq T \mid \alpha$. We consider three cases.

Case I. For some β, $\alpha = \beta + 1$. Then we construct the αth level of T' in such a way that it includes the αth level of T, and that every element of the βth level of T' splits into $\aleph_0$ successors.

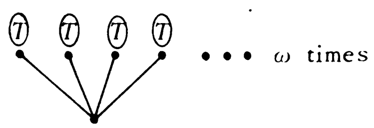
Case II. α is a limit number, and for no $x \neq y \in T' \mid \alpha$, h_α is an isomorphism of $(T' \mid \alpha)_x$ and $(T' \mid \alpha)_y$. Then we construct the αth level of T such that for each $x \in T' \mid \alpha$ there is some $y > x$ in the αth level of T'.

Case III. α is a limit number, and there exist $x \neq y \in T' \mid \alpha$ such that h_α is an isomorphism of $(T' \mid \alpha)_x$ and $(T' \mid \alpha)_y$. First we do the same as in Case II; now, when we have extended countably many branches of $T' \mid \alpha$, we may use the Diagonal Principle to find a branch b such that b goes through x, $h_\alpha(b) \neq b$ and $h_\alpha(b)$ is not among those branches extended so far. In addition to the branches extended so far, we extend b, to complete the construction of the αth level. Clearly, h_α does not extend to an isomorphism of $(T' \mid \alpha + 1)_x$ and $(T' \mid \alpha + 1)_y$.

We let $T' = \lim_{\alpha \to \omega_1} T' \mid \alpha$. T' is totally rigid; for, if h is an isomorphism of T_x and T_y $(x \neq y)$, then by ($\Diamond$), there is a limit ordinal α such that $h \mid \alpha = h_\alpha$ and h_α is an isomorphism of $(T' \mid \alpha)_x$ and $(T' \mid \alpha)_y$; this has been taken care of in Case III. Also, T' is a normal ω_1-tree, and obviously, T' has at least those ω_1-branches which T has; hence T' is a totally rigid Kurepa tree.

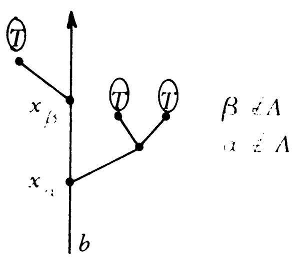
Now, we can easily prove (b) and (d) of Theorem 5.

(b) Consider the following tree T':



where T is some rigid Kurepa tree. Obviously, $\sigma(T') = 2^{\aleph_0}$.

(d) Let T be a totally rigid binary Kurepa tree. For every $A \subseteq \omega_1$, we construct a binary Kurepa tree T_A as follows. We start with a single ω_1-branch $b = \{x_\alpha : \alpha < \omega_1\}$ and, to each $x_\alpha \in b$, we graft a tree as follows, depending on whether $\alpha \in A$ or not.



Clearly, if $A \neq A'$ then T_A and $T_{A'}$ are not isomorphic.

5. **A remark on Aronszajn trees.** An *Aronszajn tree* is a normal ω_1-tree which has no ω_1-branch. The first construction of such a tree was given by Aronszajn (cf. [9]); Gaifman and Specker [1] proved that there are $2^{\aleph_1}$ different isomorphism types of Aronszajn trees.

Given an Aronszajn tree T, we can use the argument in Lemma 4.1 to construct an Aronszajn tree $T' = T \otimes T_0$ such that $\sigma(T') = 2^{\aleph_1}$.

If T is a normal ω_1-tree with $\sigma(T) < 2^{\aleph_1}$ and T is not a Suslin tree then, for some $x \in T$, T_x is rigid. By [13] the existence of Suslin trees is not provable in ZFC, so that we cannot prove the existence of a tree with $\sigma(T) < 2^{\aleph_1}$ without proving the existence of a rigid tree.

We conclude with the following problem; we are inclined rather to believe in the negative solution:

Problem. Is it provable in ZFC that there exists a rigid normal ω_1-tree?

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