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SOME REMARKS CONCERNING THE VARIETIES GENERATED BY THE DIAMOND AND THE PENTAGON[1]

BY

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ABSTRACT. In 1945 M. P. Schützenberger exhibited two identities. He asserted that one provided an equational base for the diamond M_3 and the other a base for the pentagon N_5. Recently Ralph McKenzie produced another equational base for N_5. In the present paper the authors modify McKenzie's idea to verify Schützenberger's assertion for M_3. They also show Schützenberger's claim about N_5 is false.

Introduction. In this note we make some observations based on the preceding paper [2] by R. McKenzie. In §1 we modify the ideas in §2 of McKenzie's paper to obtain analogous results for $\mathcal{V}M_3$, the variety of lattices generated by the diamond. In particular, we provide a proof of the result announced by Schützenberger [3] that $\mathcal{V}M_3$ is characterized by the single identity:

$$\alpha. \quad \begin{aligned} x \cdot (y + z \cdot (u + v)) = {} & x \cdot (y + zu) + x \cdot (y + zv) + xz \cdot (u + v) \\ & + xu \cdot (z + yv) + xv \cdot (z + yu). \end{aligned}$$

This fact also follows from the much stronger results of Jónsson [1]; however, our proof of this result, like McKenzie's proof that certain identities characterize $\mathcal{V}N_5$, is model-theoretic in nature while Jónsson's results involve deeper lattice theoretic techniques.

In the article cited above, Schützenberger also asserted without proof that the variety $\mathcal{V}N_5$ generated by the pentagon is characterized by the identity:

$$\beta. \quad \begin{aligned} x \cdot [y + z \cdot (u + v)] = {} & x \cdot (xy + zu) + x \cdot (y + xzu) + x \cdot (xy + zv) \\ & + x \cdot (y + xzv) + xz \cdot (xzu + v) + xz \cdot (u + xzv). \end{aligned}$$

In §2 of our note we observe that β holds in some lattice not contained in $\mathcal{V}N_5$; thus β does not characterize $\mathcal{V}N_5$. Equational bases for $\mathcal{V}N_5$ have been found by McKenzie (see [2]).

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stimulating conversations concerning the questions considered here and in particular Professor McKenzie for making available an early version of his results.

1. Following McKenzie, a special term of type one (an ST1) is any lattice term of the form $\rho \cdot (\sigma + \tau)$ where ρ, σ, τ are each products of variables. A term in the dual form is called an ST2. McKenzie proved that for any ST1 ν and ST2 ϕ the inclusion $\nu \leq \phi$ (called a special inclusion) either holds in every lattice or else implies (modulo lattice theory Λ) the modular law. We will modify McKenzie's proof to show the following.

Lemma 1.1. *Every special inclusion ϵ either holds in all modular lattices or else implies (modulo Λ) the distributive law.*

If Θ is any equational theory we write $\sigma \leq_\Theta \tau$ instead of $\sigma \leq \tau \in \Theta$ and $\sigma \sim_\Theta \tau$ in place of $\sigma = \tau \in \Theta$. The equational theory of modular lattices is denoted by M; distributive lattices by Δ. The following is an analogue of McKenzie's Lemma 2.2.

Lemma 1.2. *For each term σ, there are finite, nonempty sets of terms F_1 and F_2 such that*

(i) *F_1 consist of ST1's ν satisfying $\nu \leq_M \sigma$; moreover $\sigma = \Sigma F_1 \in \mathbf{\Theta}_l[\alpha]$;*
(ii) *F_2 consist of ST2's ϕ satisfying $\sigma \leq_M \phi$; moreover $\sigma = \Pi F_2 \in \mathbf{\Theta}_l[\alpha]$.*

As an application of Lemmas 1.1 and 1.2 we prove α characterizes $\mathbf{\Theta}M_3$.

Theorem 1.3. $\mathbf{\Theta}M_3 = \mathbf{\Theta}_l[\alpha]$.

As an elementary application of the model-theoretic ideas to be used in the proof of 1.3 we first give a simple proof of the following well-known theorem.

Theorem 1.4. *The variety of lattices generated by the two element chain is the class of all lattices satisfying the distributive law $x \cdot (y + z) = xy + xz$.*

Proof. Clearly the two element chain satisfies the distributive law. Since the distributive law is self-dual, it is easily seen that from this law and Λ every term is equivalent to a term $\Sigma\sigma_i$ where σ_i is a product of variables and also to a term $\Pi\tau_j$ where τ_j is a sum of variables. Thus, every lattice inclusion $\nu \leq \phi$ is equivalent to a conjunction of inclusions of the form $\sigma \leq \tau$ where σ is a product and τ a sum of variables. Now, an inclusion $\sigma \leq \tau$ of this form will either hold in every lattice or fail in the two element chain depending upon whether or not some variable occuring in σ also occurs in τ. Thus, every identity holding in the two element chain follows from the distributive law and Λ.

In view of McKenzie's Lemma 2.1, our Lemma 1.1 and the above proof every special inclusion is either a lattice identity or equivalent to the modular law, distributive law or $x = y$. This fact was also observed independently by McKenzie.

Proof of Theorem 1.3. From Lemma 1.2 it follows that modulo $\boldsymbol{\Theta}_l[\alpha]$ every equation satisfied by the diamond M_3 is equivalent to a conjunction of special inclusions. Such a special inclusion does not imply the distributive law; thus, by Lemma 1.1 it belongs to $\mathbf{M}$. It is easily seen (and we will prove later) that α implies the modular law; thus, $\boldsymbol{\Theta}M_3 \subseteq \boldsymbol{\Theta}_l[\alpha]$.

It remains to show that α holds in M_3 which we do directly. Suppose the variables x, y, z, u, and v are, respectively, assigned to elements x', y', z', u' and v' in M_3. The right (resp. left) side of α is then assigned to an element RS (resp. LS) in M_3. By modularity each summand on the right side of α is contained in LS; hence it suffices to show LS is always a sum of elements on the right. Whenever u' and v' are comparable or one of $x', y', z' \in \{0, 1\}$, LS $\leq$ RS is easily checked. Now assume u' and v' are incomparable and $x', y', z' \notin \{0, 1\}$. Since $u' + v' = 1$, it suffices to show

$$x' \cdot (y' + z') = x' \cdot (y' + z'u') + x' \cdot (y' + z'v') + x'z' + x'u'(z' + y'v') + x'v' \cdot (z' + y'u').$$

Moreover, we may assume that x', y', z' are mutually incomparable; for if y', z' are comparable then $x' \cdot (y' + z') = x' \cdot (y' + z'u') + x'z'$ and, if x' is comparable with y' or z', then

$$x' \cdot (y' + z') = x'y' + x'z' = x' \cdot (y' + z'u') + x'z'$$

by modularity. Thus, there are only three remaining cases: $z' = u'$, $z' = v'$, $\{x', y'\} = \{u', v'\}$. In the first two cases $x' \cdot (y' + z') = x' \cdot (y' + z'u') + x' \cdot (y' + z'v')$ while in the last case $x' \cdot (y' + z') = x'u' \cdot (z' + y'v') + x'v' \cdot (z' + y'u')$. Hence, in every case LS $\leq$ RS completing the proof that $\alpha \in \boldsymbol{\Theta}M_3$.

Proof of Lemma 1.1. Let ϵ be any special inclusion

$$\rho \cdot (\sigma + \tau) \leq \phi + \chi \cdot \psi$$

which fails in some modular lattice. For any term π, let π^0 denote the set of all variables occuring in π. We wish to show that $\Delta \leq \boldsymbol{\Theta}_l[\epsilon]$. The proof is exactly the same as the proof of Lemma 2.1 in McKenzie's paper except for the last case where the sets $\rho^0 \cap \phi^0$, $\rho^0 \cap \psi^0$, $\tau^0 \cap \phi^0$, $\tau^0 \cap \chi^0$, and $\sigma^0 \cap \psi^0$ are empty while the sets $\rho^0 \cap \chi^0$, $\tau^0 \cap \psi^0$, and $\sigma^0 \cap \phi^0$ are nonempty. Suppose that, in addition, $\sigma^0 \cap \chi^0 \neq 0$. The assumption that various sets of variables are nonempty implies (modulo Λ) the inclusions $\sigma \leq \chi$, $\rho \leq \chi$, $\tau \leq \psi$, and $\sigma \leq \phi$. Hence, $\mathbf{M}$ implies $\rho \circ (\sigma + \tau) \leq \chi \cdot (\chi \cdot \phi + \psi) = \chi \cdot \phi + \chi \cdot \psi \leq \phi + \chi \cdot \psi$ contrary to our assumption that $\epsilon \notin M$. Hence, $\sigma^0 \cap \chi^0 = 0$ and in this last case the following five sets are pairwise disjoint:

$$\begin{aligned} &\rho^0 \cap \chi^0, \quad \sigma^0 \cap \phi^0, \quad \tau^0 \cap \psi^0, \\ &(\rho^0 - \chi^0) \cup (\tau^0 - \psi^0) \cup (\sigma^0 - \phi^0), \\ &(\phi^0 - \sigma^0) \cup (\psi^0 - \tau^0) \cup (\chi^0 - \rho^0). \end{aligned} \tag{1}$$

Choose three distinct variables v_0, v_1, v_2 not occuring in ϵ. Replacing the variables in the five sets listed in (1), respectively, by $v_0, v_1, v_2, v_0 + v_1 + v_2$, $v_0 \cdot v_1 \cdot v_2$ and all remaining variables by v_0, we see that $v_0 \cdot (v_1 + v_2) \leq v_1 + v_0 \cdot v_2$ belongs to $\Theta_l[\epsilon]$. Since this inclusion fails in both M_3 and N_5, it implies the distributive law; thus, $\Lambda \subseteq \Theta_l[\epsilon]$ as desired.

Before proceeding with the proof of Lemma 1.2 we need to derive several consequences of α, namely, α_1–α_4.

α_1. $x \cdot (y + u + v) = x \cdot (y + u) + x \cdot (y + v) + x \cdot (u + v)$,

α_2. $xy + zu = (x + zu) \cdot (y + zu) \cdot (z + xy) \cdot (u + xy)$,

α_3 $(= \alpha^d)$. $x + y \cdot (z + uv) = (x + y \cdot (z + u)) \cdot (x + y \cdot (z + v)) \cdot (x + z + uv) \cdot (x + u + z \cdot (y + v)) \cdot (x + v + z \cdot (y + u))$,

α_4 $(= \alpha_2^d)$. $(x + y) \cdot (z + u) = x \cdot (z + u) + y \cdot (z + u) + z \cdot (x + y) + u \cdot (x + y)$.

For the remainder of this section we let $\Theta = \Theta_l[\alpha]$. We first observe that the modular law belongs to Θ since

$$x \cdot (z + yx) \sim_\Lambda xx \cdot (z + (yx) \cdot x) \leq_\Theta x \cdot (yx + z \cdot (x + x)) \leq_\Lambda yx + zx.$$

Actually, the modular law is equivalent (modulo Λ) to the inclusion $\mathrm{RS}_\alpha \leq \mathrm{LS}_\alpha$.

Substituting $u + v$ for z in α gives $\alpha_1 \in \Theta$. Obviously $\mathrm{LS}_{\alpha_2} \leq_\Lambda \mathrm{RS}_{\alpha_2}$. Now,

$$\begin{aligned} \mathrm{RS}_{\alpha_2} &\sim_M [(y + zu) \cdot z + xy] \cdot [(x + zu) \cdot u + xy], \\ &\sim_M (xy + zu + yz) \cdot (xy + zu + ux), \\ &\sim_M \mathrm{LS}_{\alpha_2} + ux \cdot (xy + zu + yz), \end{aligned}$$

and

$$ux \cdot (xy + zu + yz) \sim_{\alpha_1} ux \cdot (xy + zu) + ux \cdot (xy + yz) + ux \cdot (zu + yz)$$

where each of these three terms are obviously $\leq_\Lambda \mathrm{LS}_{\alpha_2}$. Thus $\alpha_2 \in \Theta$. By duality $\alpha_4 \in \Theta$ once we have shown $\alpha_3 \in \Theta$.

It is easily seen that $\mathrm{LS}_{\alpha_3} \leq_M \mathrm{RS}_{\alpha_3}$; to illustrate we check the fourth factor on the right side: $\mathrm{LS}_{\alpha_3} \leq_\Lambda x + (y + uv) \cdot (z + uv) \sim_M x + uv + z \cdot (y + uv) \leq_\Lambda x + u + z(y + v)$. It now remains to show $\mathrm{RS}_{\alpha_3} \leq_\Theta \mathrm{LS}_{\alpha_3}$. Let γ be the product of the last four terms on the right side of α_3. Then

$$\mathrm{RS}_{\alpha_3} \sim_\Theta \gamma \cdot (x + yz) + \gamma \cdot (x + yu) + \gamma y \cdot (z + u) + \gamma z \cdot (y + xu) + \gamma u \cdot (y + xz).$$

Clearly $\gamma \cdot (x + yz) \leq \mathrm{LS}_{\alpha_3}$; we consider each of the remaining terms separately.

Case 1.

$$\begin{aligned} \gamma u \cdot (y + xz) &\sim_\Lambda u \cdot (y + xz) \cdot (x + y(z + v)) \cdot (x + z + uv) \cdot (x + v + z \cdot (y + u)) \\ &\leq_\Theta \mathrm{LS}_{\alpha_3} + \delta_1 + \delta_2 + \delta_3 + \delta_4 \end{aligned}$$

where

$$\delta_1 = u \cdot (y + xz) \cdot (x + yv) \cdot (x + z + uv),$$

$$\delta_2 = uy \cdot (z + v) \cdot (x + z + uv) \cdot (x + v + z \cdot (y + u)),$$

$$\delta_3 = u \cdot (y + xz) \cdot z \cdot (y + xv) \cdot (x + v + z(y + u)) \leq_{\mathrm{M}} \mathrm{LS}_{\alpha_3},$$

$$\delta_4 = u \cdot (y + xz) \cdot v \leq_{\theta} x + y \cdot (z + uv) = \mathrm{LS}_{\alpha_3}.$$

Now, $\delta_1 \sim_{\mathrm{M}} u \cdot (yv + x \cdot (xy + xz)) \cdot (x + z + uv) \leq_{\theta} \mathrm{LS}_{\alpha_3} + \delta_{11} + \delta_{12}$ where $\delta_{11} = u \cdot (yv + xy) \cdot (x + z + uv)$ and $\delta_{12} = u \cdot (yv + xz) \cdot (x + z + uv)$. It is easily seen that each $\delta_{1i} \leq_{\theta} \mathrm{LS}_{\alpha_3}$ by applying α_1 to δ_{1i} and then M and α to each summand not obviously $\leq \mathrm{LS}_{\alpha_3}$. Hence, $\delta_1 \leq_{\theta} \mathrm{LS}_{\alpha_3}$. Also,

$$\begin{aligned}
\delta_2 &\sim_{\mathrm{M}} uy \cdot ((z + uv) + x \cdot (z + v)) \cdot (x + v + z \cdot (y + u)) \\
&\leq_{\theta} \mathrm{LS}_{\alpha_3} + yu \cdot (xv + z + uv)\,(x + v + z \cdot (y + u)) \\
&\sim_{\mathrm{M}} \mathrm{LS}_{\alpha_3} + yu \cdot (z + v(x + uv))\,(x + v + z \cdot (y + u)) \\
&\leq_{\theta} \mathrm{LS}_{\alpha_3} + yu \cdot (z + vx) \cdot (x + v + z \cdot (y + u)) \\
&\sim_{\mathrm{M}} \mathrm{LS}_{\alpha_3} + yu \cdot (z \cdot (y + u) + (v + x)(z + vx)) \\
&\leq_{\theta} \mathrm{LS}_{\alpha_3} + yu(z \cdot (y + u) + vx) + yu \cdot (v + x)\,(z + vx).
\end{aligned}$$

Each of these two summands are easily seen to be $\leq_{\theta} \mathrm{LS}_{\alpha_3}$ by applying α and M where appropriate. Thus, $\delta_2 \leq_{\theta} \mathrm{LS}_{\alpha_3}$ and hence $yu(y + xz) \leq_{\theta} \mathrm{LS}_{\alpha_3}$.

Case 2.

$$\begin{aligned}
yy \cdot (z + u) &\sim_{\mathrm{M}} y \cdot (z + uv + x \cdot (z + u)) \cdot (x + y \cdot (z + v)) \\
&\quad \cdot (x + u + z \cdot (y + v))\,(x + v + z(y + u)) \\
&\leq_{\theta} \mathrm{LS}_{\alpha_3} + \delta_1 + \delta_2
\end{aligned}$$

where

$$\delta_1 = y \cdot (z + uv + xu) \cdot (x + y(z + v)) \cdot (x + u + z \cdot (y + v))\,(x + v + z \cdot (y + u))$$

and

$$\delta_2 = yu(x + z) \cdot (x + y(z + v))\,(x + v + z \cdot (y + u)) \leq_{\Lambda} yu \cdot (y + xz) \leq_{\theta} \mathrm{LS}_{\alpha_3}.$$

Now,

$$\begin{aligned}
\delta_1 &\sim_{\mathrm{M}} y \cdot (z + u \cdot (x + uv))\,(x + y \cdot (z + v)) \\
&\quad \cdot (x + u + z \cdot (y + v))\,(x + v + z \cdot (y + u)) \\
&\leq_{\theta} \mathrm{LS}_{\alpha_3} + \delta_{11} + \delta_{12}
\end{aligned}$$

where

$$\delta_{11} = y \cdot (z + ux) \cdot (x + y \cdot (z + v)) \cdot (x + u + z \cdot (y + v)) \cdot (x + v + z \cdot (y + u))$$

and

$$\delta_{12} = yu \cdot (x + uv) \cdot (x + y \cdot (z + v)) \leq_{\Lambda} yu \cdot (y + xz) \leq_{\theta} \mathrm{LS}_{\alpha_3}.$$

Now,

$$\delta_{11} \leq_{\theta} \mathrm{LS}_{\alpha_3} + \delta_{111} + \delta_{112} + \delta_{113}$$

where

$$\delta_{111} = y \cdot (z + ux) \cdot (x + yv)\,(x + u + z \cdot (y + v)),$$
$$\delta_{112} = y(z + ux)\,(z + v)\,(x + u + z(y + v)) \cdot (x + v + z(y + u)),$$
$$\delta_{113} = yv \cdot (z + ux)\,(x + u + z \cdot (y + v)) \leq_{\Lambda} \delta_{111}.$$

Now,

$$\begin{aligned}\delta_{111} &\sim_{M} y \cdot (ux + z \cdot (x + yv)) \cdot (x + u + z \cdot (y + v)) \\ &\leq_{\theta} \mathrm{LS}_{\alpha_3} + yv \cdot (z + ux) \cdot (x + u + z \cdot (y + v))\end{aligned}$$

where the last term $\leq_{\Lambda} yv \cdot (y + xz) \leq_{\theta} \mathrm{LS}_{\alpha_3}$ by Case 1 with u and v permuted. A similar argument shows $\delta_{112} \leq_{\theta} \mathrm{LS}_{\alpha_3}$. Hence, $yy \cdot (z + u) \leq_{\theta} \mathrm{LS}_{\alpha_3}$.

Case 3.

$$\begin{aligned}yz \cdot (y + xu) &\sim_{\Lambda} z \cdot (y + xu) \cdot (x + y \cdot (z + v)) \cdot (x + u + z \cdot (y + v)) \\ &\leq_{\theta} \mathrm{LS}_{\alpha_3} + \delta_1 + \delta_2 + \delta_3\end{aligned}$$

where $\delta_1 = z \cdot (y + xu)\,(x + yv) \cdot (x + u + z \cdot (y + v))$, $\delta_2 = z \cdot (y + xu) \cdot (y + xv)$, and $\delta_3 = z \cdot (y + xu) \cdot v \cdot (y + zx) \cdot (x + u + z \cdot (y + v)) \leq_{M} \mathrm{LS}_{\alpha_3}$.
Now,

$$\begin{aligned}\delta_1 &\sim_{M} z \cdot (yv + x \cdot (y + xu)) \cdot (x + u + z \cdot (y + v)) \\ &\leq_{\theta} \mathrm{LS}_{\alpha_3} + z \cdot (yv + xu) \cdot (x + u + z \cdot (y + v)) \\ &\sim_{M} \mathrm{LS}_{\alpha_3} + z \cdot (xu + yv \cdot (x + u + z \cdot (y + v)) \\ &\leq_{\theta} \mathrm{LS}_{\alpha_3} + \delta_{11} + \delta_{12} + \delta_{13}\end{aligned}$$

where $\delta_{11} = z \cdot (xu + yv \cdot (x + u)) \leq_{\theta} \mathrm{LS}_{\alpha_3} + zu \cdot (yv + xu) \leq_{M} \mathrm{LS}_{\alpha_3}$, $\delta_{12} = z \cdot (x + u)\,(yv + xuz \cdot (y + v)) \leq_{M} \mathrm{LS}_{\alpha_3}$, and $\delta_{13} = z \cdot (y + v) \cdot (yv + xu) \sim_{M} z \cdot (yv + xu \cdot (y + v)) \leq_{\theta} \mathrm{LS}_{\alpha_3} + z \cdot (yv + xvu) \leq_{\theta} \mathrm{LS}_{\alpha_3}$ since $z \cdot (yv + xuv) \leq_{\theta} x + yv(z + uv)$. Hence, $\delta_1 \leq_{\theta} \mathrm{LS}_{\alpha_3}$. Now,

$$\delta_2 \sim_{M} z \cdot (y + xu \cdot (y + xv)) \leq_{\theta} \mathrm{LS}_{\alpha_3} + z \cdot (y + xuv) \leq_{\theta} \mathrm{LS}_{\alpha_3}$$

as with δ_{13}. Hence, $yz \cdot (y + xu) \leq_{\Theta} \mathrm{LS}_{\alpha_3}$.

Case 4.

$$\begin{aligned} y \cdot (x + yu) \sim_{\Lambda} & (x + yu) \cdot (x + y \cdot (z + v)) \\ & \cdot (x + z + uv) \cdot (x + v + z \cdot (y + u)) \\ & \leq_{\Theta} \mathrm{LS}_{\alpha_3} + \delta_1 + \delta_2 + \delta_3 + \delta_4 \end{aligned}$$

where $\delta_1 = (x + yu) \cdot (x + yv) \cdot (x + z + uv)$, $\delta_2 = y \cdot (x + yu)(z + v) \cdot (x + z + uv) \cdot (x + v + z \cdot (y + u))$, $\delta_3 = z \cdot (x + yu)(y + xv) \cdot (x + v + z \cdot (y + u))$, $\delta_4 = v \cdot (x + yu)(y + xz) \cdot (x + z + uv)$. Now,

$$\delta_1 \sim_{\mathrm{M}} (x + yu) \cdot (x + yv \cdot (x + z + uv)) \sim_{\mathrm{M}} x + yv(x + yu) \cdot (x + z + uv) \leq_{\Lambda} x + \delta_4$$

and $\delta_i \leq_{\Theta} \mathrm{LS}_{\alpha_3}$ (for $i = 2, 3, 4$) follows from Cases 2, 3, and 1 respectively with variables u and v interchanged. Hence, $y \cdot (x + yu) \leq_{\Theta} \mathrm{LS}_{\alpha_3}$ completing the proof that $\alpha_3 \in \mathbf{\Theta}_l[\alpha]$.

Proof of Lemma 1.2. We only prove (i); (ii) follows by duality. The proof is only a slight modification of McKenzie's proof of Lemma 2.2. Let F (resp. G), with or without subscripts, always denote a finite, nonempty set of ST1's (resp. ST1's and terms of the type $\zeta + \sigma$, where ζ and σ are products of variables).

First, note that given F, we can find a G with $\Sigma\, G \leq_{\mathrm{M}} \Sigma\, F$ and $x \cdot \Sigma\, F \sim_{\Theta} \Sigma\{x \cdot \phi \colon \phi \in G\}$. Indeed, by α_1,

$$x \cdot \sum F \sim_{\Theta} \sum \{x \cdot (\phi_0 + \phi_1) \colon \phi_0, \phi_1 \in F\}$$

and each $\phi_0 + \phi_1 \leq_{\mathrm{M}} \Sigma\, F$. By several applications of α: if $\phi_\kappa = \zeta_\kappa \cdot (\sigma_{\kappa 0} + \kappa_{\kappa 1})$, then (where κ, λ, δ range over $\{0, 1\}$)

$$\begin{aligned} x \cdot (\phi_0 + \phi_1) \sim_{\Theta} & \sum_{\kappa,\lambda} x \cdot (\zeta_0 \cdot \sigma_{0\kappa} + \zeta_1 \cdot \sigma_{1\ \ }) + \sum_{\kappa} x \cdot \zeta_\kappa \cdot (\sigma_{\kappa 0} + \sigma_{\kappa 1}) \\ & + x \cdot \zeta_0 \cdot \sigma_{10} \cdot \sigma_{11} \cdot (\sigma_{00} + \sigma_{01}) \\ & + \sum_{\kappa,\lambda,\delta} x \cdot \sigma_{\kappa\lambda} \cdot (\zeta_\kappa + \zeta_{1-\kappa} \cdot \sigma_{\kappa\, 1-\lambda} \cdot \sigma_{1-\kappa\, \delta}) \\ & + \sum_{\kappa,\lambda} x \cdot \sigma_{1\kappa}\, \sigma_{0\lambda} \cdot (\sigma_{1\ 1-\kappa} \cdot \zeta_0 + \zeta_1 \cdot \sigma_{0\ 1-\lambda}). \end{aligned} \tag{2}$$

G is a subset of F together with the above five types of ϕ's associated with pairs ϕ_0, ϕ_1 in F. We need only check that for each of the types of elements ϕ described, $\phi \leq_{\mathrm{M}} \Sigma\, F$. For ϕ a member of F or one of the first three types above it is obvious that $\phi \leq_{\mathrm{M}} \Sigma\, F$. Suppose $\phi = \sigma_{1\kappa} \cdot \sigma_{0\lambda} \cdot (\sigma_{1\ 1-\kappa} \cdot \zeta_0 + \zeta_1 \cdot \sigma_{0\ 1-\lambda})$, i.e., is of the fifth type. Then

$$\phi \leq_{\Lambda} (\sigma_{00} + \sigma_{01}) \cdot (\sigma_{1\ 1-\kappa} \cdot \zeta_0 + \zeta_1 \cdot (\sigma_{00} + \sigma_{01})) \leq_{\mathrm{M}} \sigma_{1\ 1-\kappa} \cdot \zeta_0 \cdot (\sigma_{00} + \sigma_{01}) + \zeta_1.$$

Therefore,

$$\phi \leq_M \sigma_{1\kappa} \cdot (\zeta_1 + \phi_0 \cdot \sigma_{1\,1-\kappa}) \leq_\Lambda (\sigma_{10} + \sigma_{11}) \cdot (\zeta_1 + \phi_0 \cdot (\sigma_{10} + \sigma_{11}))$$
$$\leq_M \phi_1 + \phi_0 \leq \sum F.$$

Thus, $\phi \leq_M \Sigma F$. For $\phi = \sigma_{\kappa\lambda} \cdot (\zeta_\kappa + \zeta_{1-\kappa} \cdot \sigma_{\kappa\,1-\lambda} \cdot \sigma_{1-\kappa\,\delta})$ of the fourth type a similar argument shows

$$\phi \leq_M \zeta_\kappa \cdot (\sigma_{\kappa 0} + \sigma_{\kappa 1}) + \zeta_{1-\kappa} \cdot \sigma_{1-\kappa\,\delta} \cdot (\sigma_{\kappa 0} + \sigma_{\kappa 1}) \leq \phi_\kappa + \phi_{1-\kappa} \leq \sum F.$$

This completes our preliminary remarks. (i) is proved by induction on the formation of terms. The only nontrivial part of the argument is the passage over products: assume that $\sigma_\kappa \sim_\Theta \Sigma F_\kappa$ and $\Sigma F_\kappa \leq_M \sigma_\kappa$ (for $\kappa = 0, 1$) and consider $\sigma_0 \cdot \sigma_1$. By the above we have sets G_0, G_1 such that

$$\sigma_0 \cdot \sigma_1 \sim_\Theta \sum \{\sigma_0 \cdot \phi_1 : \phi_1 \in G_1\} \sim_\Theta \sum \{\phi_0 \cdot \phi_1 : \phi_1 \in G_1, \phi_0 \in G_0\}$$

where each term $\phi_0 \cdot \phi_1 \leq_M \sigma_0 \cdot \sigma_1$. Thus, it only remains to consider the simple terms $\phi_0 \cdot \phi_1$. Suppose $\phi_\kappa = \zeta_\kappa \cdot (\sigma_{\kappa 0} + \sigma_{\kappa 1})$ for $\kappa = 0, 1$. We apply α_4, α_1 and then α repeatedly as in the construction of G above to see that $\phi_0 \cdot \phi_1$ is equivalent modulo Θ to sums of ST1's of the type that occur in (2). Repeating our previous argument shows that, for each such ST1 ν, $\nu \leq_M \phi_0 \cdot \phi_1$ and hence $\nu \leq_M \sigma_0 \cdot \sigma_1$ as desired. This completes the proof of Lemma 1.2.

The following corollary which is an easy consequence of Lemmas 1.1 and 1.2 was suggested to us by Professor McKenzie.

Corollary 1.5. *An inclusion $\sigma \leq \tau$ is valid in M_3 iff every special inclusion $\nu \leq \phi$, for which $\nu \leq_M \sigma$ and $\tau \leq_M \phi$, is provable in* M.

2. We will now show that Schützenberger's identity β does not characterize $\mathfrak{O}N_5$. Since β is rather complicated it is useful to observe that it is equivalent to the following two identities:

β_1. $x \cdot (y + z) = x \cdot (xy + z) + x \cdot (y + xz)$,

β_2. $x \cdot (w + y \cdot (u + v)) = xy \cdot (u + v) + x \cdot (yu + w) + x \cdot (yv + w)$.

The identity β_1 is just McKenzie's η_3; β_2 is the dual of η_7.

Suppose Q_3 is the lattice given in Figure 1. Observe that Q_3 is subdirectly irreducible, self-dual and has a nontrivial automorphism ϕ. Our results are based on the following lemma.

Lemma 2.1. *β_2 holds in Q_3.*

As a consequence of this lemma, Q_3 gives a counterexample to Schützenberger's claim.

Theorem 2.2. *The identity β holds in Q_3 but Q_3 is not a member of $\mathfrak{O}N_5$.*

Proof. That $Q_3 \notin \mathfrak{O}N_5$ is mentioned in McKenzie's paper. For a short direct proof we need only observe that McKenzie's identity η_1, $x \cdot (y + u) \cdot (y + v) \leq x \cdot (y + uv) + xu + xv$, fails in Q_3 when f, e, g and h are assigned to x, y, u, and v

respectively. The fact that $\beta \in \Theta Q_3$ follows from Lemma 2.1, the self-duality of Q_3 and the remark that $\beta_1 \in \Theta_l[\beta_2^d]$. To see that β_1 is a consequence of Λ and β_2^d, $x + w(y + uv) = (x + y + uv) \cdot (x + w \cdot (y + u)) \cdot (x + w \cdot (y + v))$, we first observe

(1) $$w \cdot (x + y) \leq_{\beta_2^d} x + w \cdot (y + xw).$$

This follows from β_2^d by setting $u = x$ and $v = w$. Thus,

$$x \cdot (xy + z) + x \cdot (y + xz) \sim_{\beta_2^d} [x(xy + z) + y] \cdot x \cdot (y + z) \sim_{\beta_2^d} x \cdot (y + z)$$

where the last equality holds by (1).

Proof of Lemma 2.1. Suppose the elements x', y', w', u', and v' in Q_3 are substituted for the variables x, y, w, u and v in β_2 respectively. The left (similarly, the right) side of β_2 is assigned the value LS (similarly RS). It is obvious that RS $\leq$ LS; thus, it suffices to show LS is always equal to a sum of the values on the right. This is obviously true if either $w' \geq y'$ or $w' \geq u' + v'$ or u' and v' are comparable. We assume

(2) $$w' \not\geq y', \ w' \not\geq u' + v', \text{ and } u' \text{ and } v' \text{ are incomparable.}$$

In view of the automorphism ϕ and the fact that LS $\leq$ RS whenever $w' \in \{0, 1\}$, it is enough to show LS $\leq$ RS whenever $w' \in \{a, f, g, e, c\}$. If w' is incomparable with $u' + v'$, then either $w' = a$ and $u' + v' = b$ or $w' = g$ and $u' + v' \in \{e, b\}$. In the first case either $u' = h$ or $v' = h$ so LS $= x' = x' \cdot (y'u' + w') + x' \cdot (y'v' + w')$; in the second, if $u' + v'$, $y' \in \{1, b, h\}$ then LS $= x' = x' \cdot (y'u' + w') + x' \cdot (y'v' + w')$ and if either $u' + v' = e$ or $y' \in \{\alpha, f, e, d\}$, LS $= x'a = x' \cdot (y'u' + w') + x' \cdot (y'v' + w')$. Hence, we may assume

(3) $$w' < u' + v'.$$

If $w' < y'$, then $y' \cdot (u' + v') \geq w'$ so LS $= x'y' \cdot (u' + v')$; hence, we may also assume

(4) $$w' \text{ and } y' \text{ are incomparable.}$$

From (2), (3), (4) it remains to consider four cases

(6.1) $w' = a$, $y' \in \{b, h\}$, $u' + v' = 1$,

(6.2) $w' \in \{f, e\}$, $y' = g$, $u' + v' \in \{1, a, b\}$,

(6.3) $w' = g$, $y' \in \{f, e, b, h, d\}$, $u' + v' \in \{1, a\}$,

(6.4) $w' = c$, $y' \in \{d, h\}$, $u' + v' \in \{1, a, b, e\}$.

To illustrate, we consider (6. 4). If $u' + v' \in \{a, e\}$, either u' or v' belong to $\{d, e, f\}$; thus, LS $= x' \cdot e = x' \cdot (y'u' + w') + x' \cdot (y'v' + w')$. On the other hand, if $u' + v' \in \{1, b\}$, either u' or v' belongs to $\{b, h\}$; so either $y' \cdot u' = y'$ or $y' \cdot v' = y'$. Thus LS $= x' \cdot (w' + y') = x' \cdot (y'u' + w') + x' \cdot (y'v' + w')$ as desired. The other cases are, likewise, easily checked. We conclude that $LS \leq RS$, and hence β_2, always holds in Q_3.

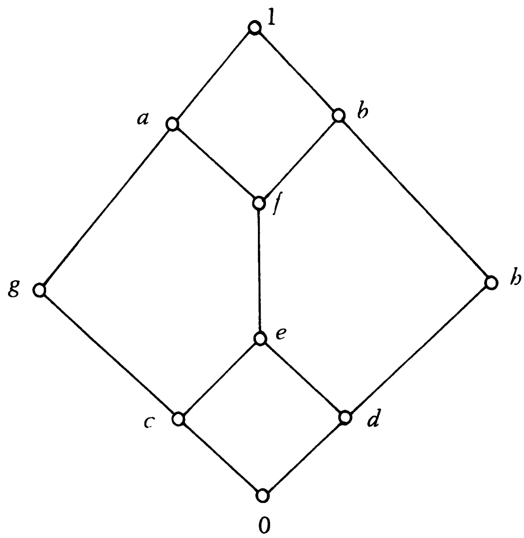

Figure 1

BIBLIOGRAPHY

1. B. Jónsson, *Equational classes of lattices*, Math. Scand. 22 (1968), 187–196. MR 40 #66.

2. R. McKenzie, *Equational bases and nonmodular lattices varieties*, Trans. Amer. Math. Soc. **174** (1972), 1–43.

3. M. Schützenberger, *Sur certains axiomes de la théorie des structures*, C. R. Acad. Sci. Paris 221 (1945), 218–220. MR 7, 235.

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