PROJECTIVE GROUPS OF DEGREE LESS THAN $4p/3$
WHERE CENTRALIZERS HAVE NORMAL SYLOW $p$-SUBGROUPS

BY

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ABSTRACT. This paper proves the following theorem:

Theorem 1. Let $G$ be a finite primitive complex projective group of degree $n$
with a Sylow $p$-subgroup $P$ of order greater than $p$ for $p$ prime greater than five.
Let $n < 4p/3$, and if $p \equiv 1 \pmod{4}$, then $G$ is a trivial intersection set,
and for some nonidentity element $x$ in $G$, $C(x)$ does not have a normal Sylow $p$-subgroup.

1. Introduction. The main object of study in this paper is the case where a
projective group $G$ has a Sylow $p$-subgroup $P$ which is a trivial intersection set
and nonidentity elements of $G$ have centralizers which have normal Sylow $p$-
subgroups. The other possible cases were studied in earlier papers. These results
are combined in Theorem 3.

2. Notation. If $H$ is a subgroup of a group $G$, we let $H^G$ be the normal sub-
group of $G$ generated by $H$. The set of nonidentity elements of $H$ is called $H^*$.  
For characters $\mu$ and $\nu$ of $H$ we let $(\mu, \nu) = (1/|H|) \sum_{x \in H} \mu(x)\overline{\nu(x)}$, and we let
the squared norm $\|\mu\|^2 = (\mu, \mu)$.

A representation $X$ of a group $G$ is called quasiprimitive if it is irreducible
and its restriction to any normal subgroup of $G$ is the sum of equivalent irreducible
constituents.

3. Projective groups where centralizers of nonidentity elements have normal
Sylow $p$-subgroups. We shall prove the following theorem:

Theorem 2. Let $p$ be a prime greater than five. Let $G$ be a finite group with
a faithful, quasiprimitive, complex representation $X$ with character $X$ of dimen-
sion $n < 4p/3$, $n \neq p$. Let $P$ be an abelian Sylow $p$-subgroup of $G$. Throughout,
let $Z = Z(G)$, $C = C(P)$, $N = N(P)$, and $N_0 = \sum_{y \in P^*} C(y) - Z$. Let $P$ satisfy [4,
Hypothesis 4.1] (that is, $P$ and $N_0$ are trivial intersection sets and $N(N_0) = N$).
Let $(|Z|, p) = 1$. Then $|P| \leq p$.

By running through the classifications of groups of small degree, it can be
seen that the hypothesis \( p > 5 \) of Theorem 2 is unnecessary. By [2, (4A)], the assumption \( n \neq p \) is unnecessary. Also, \( 4p/3 \) can be replaced by a number asymptotic to \( \sqrt{2}p \) for large \( p \). The following proof of Theorem 2 does not require the use of [7].

**Proof.** Let \( X(G) \) be a counterexample with \( n \) minimal. Given that \( n \) is minimal, let \( |G| \) be minimal. Then \( |P| \geq p^2 \). Suppose \( X(G) \) is imprimitive on \( c > 1 \) subspaces and let \( K \) be the subgroup of \( G \) fixing these spaces setwise. By primitivity, \( K \) is represented faithfully by the \( c \) equivalent constituents of \( X | K \) of degree no larger than \( 2p/3 \). By [4], \( p^2 | [K : O_p(K)] \). As \( O_p(K) \subseteq G \), by primitivity, \( O_p(K) \subseteq Z \) and \( O_p(K) = \langle 1 \rangle \). Then \( p | [G : K] | c! \) and \( c = n \). Then \( K \) is an abelian normal subgroup of \( G \) and \( K = Z \). Then at most \( p \) divides \( c! \) and \( |G| \), which is a contradiction. If \( X(G) \) is a subgroup of a tensor product of two smaller dimensional groups, then, unless \( p = 7 \) and \( n = 9 \), using \( (p - 1)/2 > 4p/3 \geq n \) with [5] and using \( n/2 < 2p/3 < p - 1 \) with [4] we have by [6, Lemma 1] that \( p^2 | [G : O_p(G)] \), contrary to \( O_p(G) = \langle 1 \rangle \) and \( p^2 ||G| \). Even if \( p = 7 \) and \( n = 9 \), \( X(G) \) contains \( M \otimes I_3 \) and \( I_3 \otimes M \) where \( M \simeq SL(2,7) \) and a 7-element \( x \in I_3 \otimes M \) has \( M \otimes I_3 \subseteq C(x) \), so \( C(x) \) does not have a normal Sylow 7-subgroup, contrary to \( P \) being a T.I. set. Therefore, \( G \) is strongly primitive.

Let \( P^G \) be the normal subgroup of \( G \) generated by \( P \). By strong primitivity and [3, Theorem 51.7], \( X | P^G \) is irreducible. If \( X | P^G \) is imprimitive, then \( X | P^G \) is monomial; otherwise, the \( p \)-elements generating \( P^G \) would fix all spaces of primitivity. If \( X | P^G \) is monomial, then \( \langle 1 \rangle \neq O_p(P^G) \subseteq G \), contrary to \( O_p(G) = \langle 1 \rangle \). As \( P^G \) also satisfies [4, Hypothesis 4.1], \( G = P^G \) by minimality of \( |G| \). For any nonprincipal irreducible character \( \gamma \) of \( G \) we write

\[
\gamma | N = \pi + \rho
\]

where \( \rho \) is the sum of irreducible constituents of \( \gamma | N \) having \( P \) in the kernel and \( \pi \) is the sum of the others. As \( G = P^G \), \( ||\pi|| \geq 1 \).

Throughout, let \( t = [N : PZ] \). Let \( m = \gamma (1) \) and \( b = \rho (1) \). By [4, Lemma 2.11], \( \pi \) vanishes on \( N - (N_0 \cup Z) \). As in [4, Lemma 4.1],

\[
1 = ||\gamma ||^2 \geq (1/|G|)(|G|/|N|) \sum_{N_0} |\gamma(x)|^2 = -m^2/t|P| + (1/|N|) \sum_{N_0 \cup Z} (\pi (x) + \rho (x)) (\overline{\pi (x)} + \overline{\rho (x)})
\]

\[
= -m^2/t|P| + (1/|N|) \left[ \sum_{N} (\pi (x)\overline{\pi (x)} + \pi (x)\overline{\rho (x)} + \rho (x)\overline{\pi (x)}) + \sum_{N_0 \cup Z} |\rho (x)|^2 \right] = -m^2/t|P| + \|\pi \|^2 + (1/|N|) \sum_{N_0 \cup Z} |\rho (x)|^2 \geq -m^2/t|P| + \|\pi \|^2 + b^2/t.
\]
When $y = \chi$, throughout we let $\alpha$ and $\beta$ correspond to $\pi$ and $\rho$ respectively, $q = \beta(1)$, and

\[(3) \quad \chi|_N = \alpha + \beta,\]

where we show that $n \geq p + 1$, $\beta(1) \leq 1$, $\|\alpha\| = 1$, $\alpha|_P$ is a sum of $\alpha(1) = n - q$ distinct nonprincipal linear characters of $P$ permuted transitively by $N$, $|P| = p^2$, $P$ is elementary abelian, $O_{p'}(G) = Z$, and $C = PZ$.

Since $|P| > p^2$, $O_{p'}(G) = (1)$, and $O^2(G) \supseteq P^G = G$, by [6], we have $n > p - 1$ and $n \geq p + 1$ as $n \neq p$ by hypothesis. If $t = 1$, then $N = PZ$ is abelian and by (2), $p < n \leq \|\alpha\|^2 + q < 1 + n^2/p|P| < 1 + (4p/3)^2/p^2 = 25/9$, which is a contradiction. Therefore, $t \geq 2$ and by (2),

\[\|\alpha\|^2 \leq 1 + n^2/t|P| < 1 + (4p/3)^2/p^2 < 2,
\]

and $\|\alpha\| = 1$. Also, by (2), $q = b \leq n/\sqrt{|P|} < (4p/3)/p < 2$, and $q \leq 1$. More generally, in (2),

\[(4) \quad (\gamma|_P, 1_P) < \gamma(1)/(|P|)^{1/2}.\]

If $\alpha|_P$ has homogeneous constituents (Wedderburn components) of dimension $e$, then by summing (4) over the irreducible constituents of $\chi|_G - 1_G$, we have

\[(5) \quad (n - 1)e \leq (n - q/e)e^2 + q - 1 = ((\chi|_G - 1_G)|_P, 1_P) < (n^2 - 1)/(|P|)^{1/2}.\]

Then $e \leq (n + 1)/p < (4p/3 + 1)/p < 2$ and $e = 1$. As $\|\alpha\| = 1$, the distinct linear homogeneous spaces of $\alpha|_P$ are permuted transitively by $N$. Also, by (5), $|P| < (n + 1)^2 < (4p/3 + 1)^2 < p^2$ and $|P| = p^2$. If $P$ is cyclic, $[N:C]|\text{Aut}(P)| = p(p - 1)$ and $(n - q)|N:C| = p - 1$, contrary to $n - q > p - 1$. As $\beta(1) \leq 1$, $\chi|_P$ has all distinct linear constituents.

Suppose $O_{p'}(G) \supset Z$. Then by strong primitivity, $X|O_{p'}(G)$ is irreducible. Replacing elements $x$ of $X(P)$ by all unimodular scalar multiples of $x$, we get a group $P^*$ of exponent $p$, since $(n, p) = 1$, with $X(O_{p'}(G)) \subset P^*X(O_{p'}(G))$. By [6, Lemma 8] for $x \in P^*$ some scalar multiple $y$ of $x$ has all primitive $p$th roots of unity occurring equally often. As $(n, p) = 1$ and $\det x = \det y$, $x = y$ and trace $x$ is rational. Then by [9],

\[p^2 = |P/P \cap Z| \leq |P^*| \leq p^{[n/(p - 1)] + [n/p(p - 1)] + \cdots} = p\]

which is a contradiction. If $C \supset PZ$, we may find a $q$-element $v \in C - PZ$ for $q$ a prime unequal to $p$. By [2, proof of (3F)], since $\chi|_P$ is a sum of distinct linear constituents, $v \in O_q(G) \subset O_{p'}(G) - Z$, which is a contradiction.

Throughout, we write

\[(6) \quad \chi^2 = k1_G + \sum y_i\]
where \( k \) equals 0 or 1 and the \( y_i \) are irreducible nonprincipal characters of \( G \).

We now divide the proof into parts.

(A) \( G/Z \) is simple. If \( W \leq G \) and \( W \not\subseteq Z \), then \( W = G \). Also, \( G = G' \). Irreducible nonprincipal characters \( \gamma \) of \( G \) have degree greater than \( p \). If \( U \) is a subgroup of \( Z \) and \( S \) is a subset of \( G \), let \( \overline{S} \) be the image of \( S \) in \( G/U \). Then \( \overline{N}_0 \) is the similarly defined \( N_0 \) of \( G \), and \( \overline{N}_0 \) and \( \overline{P} \) are T.I. sets whose normalizer is \( \overline{N} \).

**Proof.** Let \( Z \subseteq K \triangleleft G \). As \( P^G = G \) and \( O_{p+}(G) = Z \), we have \( p^2 \mid |K| \) and \( p \mid |K| \), so \( K \) has a Sylow \( p \)-subgroup \( Q = P \cap K \) of order \( p \). As \( P \) is a T.I. set, \( P \) is a normal Sylow \( p \)-subgroup of \( C(Q) \). Then \( P \) has degree \( N(Q) \leq N(Q) \). By a Sylow theorem \( G = KN(Q) \). Then \( N(Q) \) and \( N(Q)/K \cap N(Q) \approx G/K \) have normal Sylow \( p \)-subgroups. Then \( O_{p}(G/K) = O_{p'}(G/K) = G/K \) since \( G = P^G \). Then \( G/K \) is a \( p \)-group and \( [G : K] = p \). As \( P \) is elementary abelian and \( ([N : P], p) = 1 \), by complete reducibility, \( N \) has an \( N/C \) complement \( R \) in \( P \). Then \( (R, N) = (R, P N \cap K) = (R, N \cap K) \subseteq R \cap K \subseteq R \cap (P \cap K) = R \cap Q = (1) \). By (3) with \( \alpha \) irreducible and \( \alpha(1) \geq n - 1 \), \( R \) consists of homologies contrary to \( \alpha \) (alternatively, some element in \( R \) violates \( \alpha \) (Theorem 8, page 96) and quasiprimity). Therefore, \( G/Z \) is simple.

(B) \( N/C \cong L \) where \( L \) is a subgroup \( (L \) is fixed throughout this paper) of \( GL(2, p) \) of order \( t \). We view \( L \) as a 2-dimensional matrix group over \( GF(p) \). Also, column vectors and row vectors correspond to elements of \( P \) and linear characters of \( P \), respectively, on which \( L \) acts by matrix multiplication as \( N/C \) acts on \( P \) and characters of \( P \). If \( r \) is the matrix product of a row vector \( r \) with a column vector \( c \), then \( e^{2\pi i /p \cdot rc} \) equals the value of the corresponding character of \( P \) at the corresponding element of \( P \).

**Proof.** As \( C \) is the kernel of the action of \( N \) on \( P \), \( N/C \) is isomorphic to a subgroup of \( \text{Aut}(P) \cong GL(2, p) \). As \( C = PZ \), \( |L| = t \).

(C) Let \( \theta \) be an irreducible character of \( N \) not having \( P \) in its kernel. Let
\( \phi \) be a linear constituent of \( \theta \big|_P \) and \( H \) be the subgroup of \( N \) fixing \( \phi \). Then 
\( C \subseteq H \subseteq N_0 \cup Z \) and there exists a linear constituent \( \xi \) of \( \theta \big|_H \) with \( \xi \big|_P = \phi \) and \( \theta = \xi^N \), the induced character. Also, \( \theta \) vanishes on \( N - (N_0 \cup Z) \).

**Proof.** Let \( \theta \) be the character of the representation \( R \) on the space \( W \). Let 
\( U \) be the homogeneous space (Wedderburn component) for \( R \) restricted to \( P \) such that \( (\dim U)\phi \) is the character of the representation of \( P \) on \( U \). Let \( H \) be the subgroup of \( N \) fixing \( \phi \). Then \( H \) is also the subgroup of \( N \) fixing \( U \). Let \( \xi \) be the character of the representation of \( H \) on \( U \). Then by Frobenius reciprocity, 
\[
(\xi^N, \theta) = (\xi, \theta \big|_H) \geq 1.
\]
Since \( \theta \) is irreducible and \( \theta(1) = \dim W = [N : H]dim U = [N : H]\xi(1) = \xi^N(1), \theta = \xi^N \). Also, \( \xi \) is irreducible since \( \xi^N = \theta \) is irreducible.

Now, \( H/C \) corresponds to a \( p' \)-subgroup of the subgroup \( M \) of \( GL(2, p) \cong \text{Aut}(P) \) fixing a nonprincipal character of \( P \). As \( M \) is isomorphic to a normal extension of a group of order \( p \) by a cyclic group of order \( p - 1 \), \( H/C \) is cyclic. As elements of \( C = PZ \) are represented by scalars on \( U \) and \( \xi \) is irreducible, \( \xi \) is linear. As any element of \( H \) fixes the nonprincipal linear character \( \xi^P \) of \( P \), by the permutation lemma, it also fixes a nonidentity element of \( P \) and lies in \( N_0 \cup Z \). This proves (C) since \( \xi^N \) vanishes on \( N - (N_0 \cup Z) \) since \( N = MN_1 \) and \( WC N_0 U Z \).

(D) Let \( N/C \cong L \subseteq GL(2, p) \) as in (B). Then either

1. \( L/Z(L) \cong A_5 \) and \( |L| \mid 60(p - 1) \),
2. \( L/Z(L) \cong A_4 \) and \( |L| \mid 24(p - 1) \),
3. \( L/Z(L) \cong S_4 \) and \( |L| \mid 24(p - 1) \),
4. \( L \) is monomial, \( |L| \mid 2(p - 1)^2 \), and \( L \) contains a diagonal subgroup \( A \) with \( [L:A] \leq 2 \), or
5. \( L \) can be written as a monomial group in \( GL(2, p^2) \) where \( L \) contains a subgroup

\[
A = \begin{pmatrix}
\zeta & 0 \\
0 & \zeta^p
\end{pmatrix}
\]

for some \( \zeta \in GF(p^2) \) and \( [L:A] \leq 2 \).

Here, \( |L| \mid 2(p^2 - 1) \).

In cases I, II, and III, \( L \) is irreducible, \( |Z(L)| \mid (p - 1) \), and \(-I_2 \in Z(L)\).

**Proof.** We may consider \( L \) to be a faithful 2-dimensional representation of a \( p' \)-group over a field of characteristic \( p \). In the case of \( p' \)-groups, complex irreducible representations have a one to one correspondence with \( p \)-modular representations. Then \( L \) may be obtained from a finite complex \( p \)-integral 2-dimensional group by taking coefficients modulo an ideal dividing \( (p) \). By the classification in [1] of 2-dimensional complex linear groups, we have I, II, or III or \( L \) may be taken as monomial when written over a larger field. In the monomial cases there exists \( A \), an abelian subgroup of index 1 or 2 in \( L \). The character of the representation of \( A \) by our linear group \( L \) is a sum of two linear
characters \( \sigma \) and \( r \). If \( \sigma \) and \( r \) lie in \( \text{GF}(p) \), we have case IV. If they do not, since \( \sigma + r \) lies in \( \text{GF}(p) \), \( \sigma \) and \( r \) lie in \( \text{GF}(p^2) \) and are algebraic conjugates over \( \text{GF}(p) \). Then \( \sigma = r^p \).

(E) Let \( \theta, \phi, \) and \( H \) be as in (C). Then except in case IV, \( [H:C] \leq 5 \). If \( \theta = \alpha \), throughout this paper let \( b = [H:C] \). Then \( t = (n - q)b \). Suppose we are in case IV or V. Let \( \theta = \alpha \), let \( A \) be as in (D), and let \( M \subseteq L \) correspond to \( H/C \subseteq N/C \). Then \( M \cap A = \langle 1 \rangle \) and \( b \leq 2 \).

**Proof.** As in the proof of (C), \( H/C \) corresponds to the cyclic subgroup \( M \) of \( L \subseteq \text{GL}(2, p) \) of order dividing \( p - 1 \) and fixing the character \( \phi \). As elements of \( M \) have an eigenvalue 1, \( M \cap Z(\text{GL}(2, p)) = \langle 1 \rangle \). Therefore, \( [H:C] \leq 5 \). In case V, \( I_2 \) is the only element of \( A \) with an eigenvalue 1, so \( M \cap A = \langle 1 \rangle \) and \( [H:C] \leq 2 \).

Suppose \( \theta = \alpha \). By (C), \( n - q = \alpha(1) = [N:H] = t/b \). Suppose further that we have case IV. Let \( A \) be the diagonal subgroup of \( L \) of index at most 2. Then if \( M \cap A = \langle 1 \rangle \), \( [H:C] \leq 2 \). Therefore, suppose that \( \text{diag}(\sigma, r) \) lies in \( (M \cap A)^\# \).

Then, as elements of \( M \) have an eigenvalue 1, \( \sigma \) or \( r = 1 \), say \( \sigma = 1 \). Then \( \phi \) corresponds to a multiple of \( (1, 0) \). Let \( \zeta \) be the homomorphism from \( A \) to \( \text{GF}(p)^\# \) with \( \zeta(\text{diag}(\pi, \rho)) = \pi \). Then \( M \cap A \) is the kernel of \( \zeta \) and \( [A:M \cap A] = |\zeta(A)| = |(p - 1)| \). Also, \( [L:M] = [L:A][A:M \cap A]/[M:M \cap A] \). Therefore, \( \alpha(1) = [N:H] = [L:M]/2(p - 1) \). Then, since \( \alpha(1) \leq n < 4p/3 < 2(p - 1) \), \( \alpha(1) \leq p - 1 \). Furthermore, \( n = \alpha(1) + q \leq \alpha(1) + 1 \leq p \), contrary to (A).

(F) Let \( \zeta \) and \( \xi \) be the characters of the symmetric and skew-symmetric tensors of \( \chi^2 \), respectively. Then, if \( -I_2 \in L \), \( (n + q)/2 = (1_p, \zeta|_p) \) and \( (n - q)/2 = (1_p, \xi|_p) \).

**Proof.** As multiplying a row vector by \( -I_2 \) corresponds to taking the complex conjugate of the corresponding character, if \( -I_2 \in L \) then all complex conjugates of constituents of \( \alpha|_p \) also occur in \( \alpha|_p \). By linearity of the homogeneous constituents of \( \chi|_p \), this concludes the proof of (F).

(G) \( (\chi|_p^2, 1_p) \) equals \( q \) or \( n \). If \( (\chi|_p^2, 1_p) = n \) then \( n - q \) is even.

**Proof.** If \( (\chi|_p^2, 1_p) \neq q \), then \( \alpha|_p \) contains a pair of complex conjugate linear characters. Then \( \alpha|_p \) consists entirely of pairs of complex linear characters since the action of \( N \) is transitive on the linear constituents of \( \alpha|_p \) and commutes with scalar multiplication (by \( -1 \) in particular and by elements of \( \text{GF}(p) \) in general) of the linear characters of \( P \). This proves (G) since \( \chi|_p \) has only linear homogeneous components.

(H) Let \( \gamma = \gamma_i \) be a nonprincipal constituent of \( \chi^2 \). Let \( \gamma|_N = \pi + \rho \) as in (I) with \( P \) in the kernel of \( \rho \) but not in the kernel of any constituent of \( \pi \). Then \( \|\pi\| = 1 \).
Proof. Let \( m = \gamma(1), \ b = \rho(1), \) and \( h = [H : C] \) with \( H \) being the \( H \) in (C) when \( \theta = \alpha \). Let \( \zeta \) and \( \xi \) be the characters of the symmetric and skew-symmetric tensors of \( \chi^2 \), respectively. We may assume that \( \|\pi\|^2 \geq 2 \), otherwise (H) holds. By (C) irreducible characters of \( N \) without \( P \) in the kernel have degree no larger than \( [N : C] = t \). Therefore, \( \|\pi\|^2 \geq \|\pi(1)/t\| = (m - b)/t \) where \( \{x\} \) is the smallest integer greater than or equal to \( x \). Then fixing \( p \) and using (2), we use the following to define various functions:

(7) \(-m^2/tp^2 + \max(2, (m - b)/t) + b^2/t < 1,\)

(8) \(b(t, m, b) = -m^2/tp^2 + \max(2, m/t + b^2/t - b/t) < 1, \) and

(9) \(g(t, m) = -m^2/tp^2 + \max(2, m/t) < 1.\)

For at least one of \( \mu = \zeta \) or \( \xi \) we have \( \mu = s1_G + y + \nu \) where \( \nu \) is a sum of nonprincipal irreducible characters \( \psi_i \) of \( G \) and \( s \leq k \leq 1 \). By (4), \( \psi_i(1) > (\nu_i|p, 1_p)p \). Summing this over \( i, \) we have

\[
\mu(1) - s - m = \nu(1) + (\nu|p, 1_p)p = [(\mu|p, 1_p) - s - b]p.
\]

Suppose that \(-1^2 \notin L\). Then by (F) and the above inequality, if \( \mu = \zeta \) we have

(10) \(m(n + 1)/2 - s - m \geq [(n + q)/2 - s - b]p\)

and if \( \mu = \xi \) we have

(11) \(m(n - 1)/2 - s - m \geq [(n - q)/2 - s - b]p.\)

In either case,

(12) \(m(n + 1)/2 - m \geq [(n - 1)/2 - 1 - b]p\)

and

(13) \(b \geq (n - 1)/2 - 1 - m(n + 1)/2p + m/p = f(n, m)\)

define \( f(n, m) \). For fixed \( m, \) \( f(n, m) \) decreases as a function of \( n \geq (p - 1)/2 \). Since \( p < n < 4p/3 \) by (A),

(14) \(b \geq f(n, m) \geq f(4p/3, m) = m/p - (2/9)p - 13/6.\)

We no longer assume that \(-1^2 \notin L\). The cases \( p = 7, 11, 13, 17, 19, \) and \( 23 \) were examined by computer for \( n = p + 1, \ldots, [4p/3]; \ q = 0 \) or \( 1; \ s = 0 \) or \( 1; \ m = p + 1, \ldots, -s + m(n + 1)/2; \ b = 1, \ldots, 5; \) and \( b = 0, \ldots, -s + (n + q)/2. \) The computer ruled out cases for any of the following reasons:

1. Inequality (7) failed.

2. \( n - q \) and \( p \) failed to satisfy any of the following two propositions:
\( P_1: (n - q) \text{ divides } 60(p - 1) \text{ or } 24(p - 1), \)
\( P_2: (n - q) \text{ divides } 2(p - 1)^2 \text{ or } 2(p^2 - 1). \) (This uses (D).)

3. \( P_1 \) fails and \( b > 2. \) (This uses (D) and (E).)

4. If \( b > 2 \) or \( P_2 \) fails) and \( P_3 \) holds where \( P_3 \) is the proposition:

\( P_3: \) Inequality (10) fails and \( \lfloor n(n - 1)/2 - s - m < 0 \) or (11) fails].

(Here, if \(-I_2 \in L, \) then if \( \mu = \zeta, \) we have (10), and if \( \mu = \xi, \) we have \( s + m \leq n(n - 1)/2 \) and (11). Therefore, \(-I_2 \in L \) implies \( P_3 \) fails. However, if \( b > 2 \) or \( P_2 \) fails, then by (D) and (E), \(-I_2 \in L. \))

The only cases left by the computer satisfied one of the following:

1. \( p = 7, \ n - q = 8, \) and \( P_3 \) holds.
2. \( p = 7, \ n - q = 9, \ b \leq 3, \ m \geq 30, \) and the left side of (7) is greater than .4.
3. \( p = 13, \ n - q = 16, \ b = 2, \) and \( P_3 \) holds.
4. \( p = 19, \ n - q = 24, \ b = 2, \) and \( P_3 \) holds.

In cases 1, 3, and 4, \( P_3 \) holds, so \(-I_2 \in L. \) Then we have case IV or V in (D). Let \( A \) be the diagonal subgroup of \( L \) of index at most 2 (in case V, \( A \) may have to be written in \( \text{GF}(2, p^2). \)) As \( 2 \mid b(n - q)/2 = t/2 \mid |A|, \) in case V we have \(-I_2 = \text{diag}(-1, (-1)^p) \in A, \) which is a contradiction. Therefore, we have case IV. Let \( D \) be the group of all nonsingular diagonal matrices in \( \text{GL}(2, p). \) Then \( A \subseteq D \) and \( |D| = (p - 1)^2. \) In all cases 1, 3, and 4, two divides \( b(n - q)/2 \) and \( |A| \) to at least as high a power as it divides \( (p - 1)^2 \) and \( |D|. \) Then a Sylow 2-subgroup of \( A \) is a Sylow 2-subgroup of \( D \) and contains \(-I_2, \) which is a contradiction.

Therefore, we have case 2. As \( n - q = 9 \nmid 2(p^2 - 1) \) and neither 60, 12, nor 24 divide \( 9b, \) in (D) we have case IV. Then by (E), \( M \cap A = \{1\} \) and \( b = 1 \) or 2 where \( M \subseteq L \) corresponds to \( H/C \subseteq N/C. \) Changing coordinates, we take
\[
A = \{\text{diag}(x, y) \mid x, y \in \text{GF}(7), \ x, y = 1, 2, 4\}.
\]

If \( b = 2, \) we have \( \binom{0}{\psi - 1} \psi \in M \) for some \( \psi. \) Changing coordinates by \( v \rightarrow (\text{diag}(\psi, 1)v \) for \( v \in \{\text{column vectors}\} \cong P, \) we take \( \psi = 1. \) Further changing coordinates by a scalar matrix, or by a diagonal matrix if \( b = 1, \) we take \( (1, 1) \) to correspond to a linear constituent of \( \alpha | p. \) Then \( \{x, y \mid x, y = 1, 2, 4\} \) is the set of linear constituents of \( \alpha | p \) and \( \{x, y \mid x, y = 1, 2, 3, 4, 5, 6\} \) is the set of 81 constituents of \( \chi | p \) with multiplicities. If \( b = 1, \) then under the action of \( N, \) the constituents of \( \mu | p \) lie in four orbits of length 9: \( O_2, O_2, O_3, \) and \( O_4, \) or five orbits of length 9: \( O_1, O_2, O_2, O_3, \) and \( O_4, \) if \( \mu \) corresponds to the skew-symmetric or symmetric tensors, respectively, of \( \chi^2. \) Here \( O_1 \) is represented by \( (2, 2), \) \( O_2 \) by \( (3, 3), \) \( O_3 \) by \( (2, 2), \) and \( O_4 \) by \( (3, 2). \) If \( b = 2, \) then \( O_3 \) and \( O_4 \) are joined into an orbit \( O_{34} \) of length 18. As \( 9 \mid |O_i| \) for all \( i, \) \( 9 \mid m \) and \( m = 36 \) or 45. If \( \|n\|^2 \max(2, \lfloor (m - b)/2 \rfloor) = \lfloor m/9b \rfloor, \) then the right side of (2) exceeds 1.4, which is a contradiction. By (C), orbits of linear constituents of \( \chi | p \) correspond to irreduc-
ible constituents of $X|N$. If $b = 2$, then $O_{34}$ corresponds to the only possible irreducible constituent of $\pi$ of degree 18. As $m \geq 36$, $\pi$ has at least two irreducible constituents of degree 9 and $\|\pi\|^2 < |m/18|$, which is a contradiction. Therefore, $b = 1$. Then in application of (C) to any irreducible constituents of $X|N$, $H = PZ = C$. If $\nu$ and $\mu$ both correspond to $O_2$, then by (C), $\nu$ and $\mu$ are induced from linear constituents $\phi|P\pi$ and $\theta|P\pi$, respectively, with $\phi|P = \theta|P$.

As $X|Z$ is a multiple of $\phi|Z$ and $\theta|Z$, $\phi|Z = \theta|Z$, $\phi = \theta$, and $\nu = \mu$. As $\|\pi\|^2 = 1$ or $\|\pi\|^2 = |m/9|$, $\pi$ cannot contain both constituents corresponding to $O_2$ as they are identical. Then $m = 36$, a case not reported by the computer as (7) fails.

We may now assume that $p \geq 29$. As $n < 4p/3$ and $t = (n - q)b \leq 5n$, $t \leq 20p/3$ and $2t \leq 40p/3 < p^2/2$. For fixed $t$, $g(t, m)$ is a decreasing function of $m$ for $0 \leq m \leq 2t$, increasing for $2t \leq m \leq p^2/2$, and decreasing for $p^2/2 \leq m$. For $m \leq p^2/2$,

$$\begin{align*}
g(t, m) &\geq g(t, 2t) = -(2t)^2/tp^2 + 2 = 2 - 80/3p > 1.
\end{align*}$$

Then by (9), $m \geq p^2/2$. Also,

$$\begin{align*}
f(4p/3, m) &= m/p - (2/9)p - 13/6 \geq p/2 - (2/9)p - 13/6 \\
&= (5/18)p - 13/6 > (5/18)29 - 13/6 > 1.
\end{align*}$$

Suppose that $-I_2 \notin L$. Then (14) holds. For $b \geq 1$ and $t$ and $m$ fixed, $b(t, m, b)$ is an increasing function of $b$. By (16) we may combine (8) and (14) and define $k(t, m)$:

$$\begin{align*}
k(t, m) &= b(t, m, f(4p/3, m)) \leq b(t, m, b) < 1.
\end{align*}$$

As $m \geq p^2/2 > 2t$,

$$\begin{align*}
b(t, m, b) &= -m^2/tp^2 + m/t + b^2/t - b/t.
\end{align*}$$

The $m^2$ terms cancel in $k(t, m)$. The coefficient of $m$ in $k(t, m)$ is

$$\begin{align*}
\end{align*}$$

Therefore, for $m \geq p^2/2$, by (16) and (15)

$$\begin{align*}
k(t, m) &\geq k(t, p^2/2) \geq b(t, p^2/2, 1) = g(t, p^2/2) > 1.
\end{align*}$$

This contradicts (17).

Therefore, we have $p \geq 29$, $m \geq p^2/2$, and $-I_2 \notin L$. By (D) we then have case IV or V. Let $x = (4p - 1)/3$. By (E), $b \leq 2$ and $t = (n - q)b \leq 2n \leq 2x$. Also, $m \leq n(n + 1)/2 \leq x(x + 1)/2 < 2p(4p + 3)/9 < p^2$. As $g(t, m)$ is decreasing in $m$ for $m \geq p^2/2$, we have by (9),
\[ g(t, m) = \frac{(m/t)(1 - m/p^2)}{(x(x + 1)/2)/2x}
\]

\[ = \frac{(x + 1)(2p^2 - x^2 - x)/8p^2}{(4p + 2)(18p^2 - 16p^2 + 8p - 1 - 12p + 3)/216p^2}
\]

\[ \geq 4(29) + 2)(1/108)(1 - 2/29) = (3186)/3132 > 1,
\]

which is a contradiction.

(I) Let \( \gamma \) be a nonprincipal irreducible character of \( G \) with \( \gamma|_N = \pi + \rho \) as
in (1). Let \( \|\pi\| = 1 \) and \( \pi(1) = \alpha(1) \). Then \( \rho(1) \leq 1 \). For the remainder of
the proof, let \( \omega = \gamma_1 \) be a fixed nonprincipal irreducible constituent of \( \chi^2 \) with \( \omega|_P \)
containing \( \phi^2 \) where \( \phi \) is a fixed linear constituent of \( \alpha|_P \). Write

\[ \omega|_N = \sigma + \tau \]

as in (1). Then \( \|\sigma\| = 1, \sigma(1) = \alpha(1), \) and \( \tau \leq 1 \).

Proof. By (4), \( pp(1) \leq \pi(1) + \rho(1) \) and \( \rho(1) \leq \pi(1)/(p - 1) < 4p/3(p - 1) < 2. \)
As \( \phi^2 \) is a constituent of \( \chi^2|_P \), \( \omega = \gamma_1 \) exists. By (H), \( \|\pi\| = 1 \). As \( \phi^2 \) and \( \phi \)
are fixed by the same subgroup \( H \) of \( N \), by (C) we have \( \sigma(1) = [N : H] = \alpha(1) \).
Letting \( \gamma = \omega \), we have \( \tau(1) \leq 1 \). If \( q = 1 \) and \( \tau(1) = 0 \), then by (A), \( p < \omega(1) = \sigma(1) = \alpha(1) < \chi(1) \), contrary to (A) and minimality
of \( \chi(1) \) for a counterexample since by (A) the kernel of \( \omega \) lies in \( Z \).

(J) Let \( \gamma \) be a nonprincipal irreducible character of \( G \) with \( \gamma|_N = \pi + \rho \) as
(1). Let \( \|\pi\| = 1 \) and \( \gamma|_Z \) be multiples of the same linear character of
\( Z \). Let \( s = \sigma(1) \) and \( r = \pi(1) \). Then \( s = \rho = (s\pi - r\sigma)^G \), the induced character,
and vanishes outside of \( N^G_0 = \bigcup_{g \in G} g^{-1}N_0g \). Also, \( sp(1) = r\tau(1) \). Furthermore,
if \( \pi = \sigma \), then \( \gamma = \omega \).

Proof. Suppose that \( \pi \neq \sigma \). By hypothesis, \( s\pi - r\sigma \) vanishes on \( Z \). By (C),
\( \pi \) and \( \sigma \) vanish on \( N - (N_0 \cup Z) \) and \( s\pi - r\sigma \) vanishes on \( N - N_0 \). As \( N_0 \) is a
T.I. set, by [3, Lemma 38.15], \( \|s\pi - r\sigma \|^2 = \|s\pi - r\sigma \|^2 = s^2 + r^2 \). By
Frobenius reciprocity,

\[ ((s\pi - r\sigma)^G, s\pi - r\omega) = (s\pi - r\sigma, s\pi - r\omega|_N) = (s\pi - r\sigma, s\pi + sp - r\sigma - rr) = s^2 + r^2. \]

Then \( ((s\pi - r\sigma)^G, s\pi - r\omega) = \|s\pi - r\sigma\|^2 \), and \( \|s\pi - r\omega\|^2 \) all equal \( s^2 + r^2 \),
so by the Cauchy-Schwarz inequality, \( s\pi - r\omega = (s\pi - r\sigma)^G \) which has support on
\( N^G_0 \) since \( s\pi - r\sigma \) has support on \( N_0 \). As \( 1 \neq N^G_0 \),

\[ 0 = s\pi - r\omega(1) = s\pi(1) + sp(1) - r\sigma(1) - r\tau(1) = sp(1) - r\tau(1). \]

Suppose that \( \pi = \sigma \) and \( \omega \neq \gamma \). Then \( \pi(1) = \sigma(1) = \alpha(1) \) and by (I), \( \rho(1) \leq 1 \).
Let \( D\sigma \) be the determinant of the corresponding representation. As \( G = G' \), \( \gamma \) and
\( \omega \) are unimodular. Then if \( \tau(1) = \rho(1) = 1 \), we have for \( x \in N \),
As $\tau(1)$ and $\rho(1) \leq 1$, in any event $\rho(x)\tau(x)$ is real and nonnegative. Let $e = (1/|G|)\sum_{x \in N_0^G} |\gamma(x)|^2$ and $f = (1/|G|)\sum_{x \in N_0^G} |\omega(x)|^2$. Then since $\pi$ and $\sigma$ vanish off $N_0 \cup Z$ by (C), we have

$$(1/|G|) \sum_{x \in N_0^G} \gamma(x)\bar{\omega}(x) = (1/|G|)(|G|/|N|) \sum_{x \in N_0^G} (\pi(x) + \rho(x))(\sigma(x) + \tau(x))$$

$$= (1/|N|) \sum_{x \in N_0^G \cup Z} (\pi(x)\sigma(x) + \pi(x)\tau(x) + \rho(x)\sigma(x) + \rho(x)\tau(x)) - \gamma(1)\omega(1)/|G|^2 \tau$$

$$= (\pi, \sigma) + (\pi, \tau) + (\rho, \sigma) + \sum_{x \in N_0^G \cup Z} \rho(x)\tau(x) - \gamma(1)\omega(1)/|G|^2 \tau$$

$$\geq 1 + 0 + 0 - (t + 1)(4p/3 + 1)/p^2 \tau > 1/2.$$
\[
\gamma_i(u) = d_i \omega(u)/(n - q).
\]

Summing the last two equations over \(i\), we have

\[
(19) \quad (\chi^2_p, 1_p) - k = \sum \rho_i(1) = \sum d_i r(1)/(n - q) = D r(1)/(n - q)
\]

and, for \(u \notin N^G_0\),

\[
(20) \quad \chi^2(u) - k = \sum \gamma_i(u) = \sum d_i \omega(u)/(n - q) = D \omega(u)/(n - q).
\]

Suppose that \(r(1) = 1\) and \(N_0 \notin PZ\). Then for some \(v \in P^\#\), \(|N_0 \cup Z| \geq |C(v)| \geq 2|PZ|\). As \(r(1) = 1\), \((1/|N|) \sum_{N_0 \cup Z} |r(x)|^2 = |N_0 \cup Z|/|N|\). By the second to last line of (2), for \(\gamma = \omega\),

\[
(21) \quad [(4p - 1)/3 + 1]^2/tp^2 \geq (n - q + 1)^2/tp^2 \geq \omega(1)^2/tp^2 > |N_0 \cup Z|/|N| \geq 2/p.
\]

Then \(p < 11\), \(p = 7\), \(q = 0\), and \(n = 9\). Even then the 2 in \(2/t\) in (21) cannot be replaced by a larger integer or (21) would fail. As \(N_0 \cup Z\) contains only entire cosets of \(PZ\), \(|N_0 \cup Z| = 2|PZ| = |C(v)|\). As \(N(N_0) = N\), \(C(v) = N_0 \cup Z\) corresponds to a normal subgroup \(E\) of order 2 of \(L\) where the element \(x\) of \(E^\#\) has an eigenvalue 1. As \(E \subseteq Z(L)\), by diagonalizing \(x\), we have \(L \subseteq D\) where \(D\) is the group of all nonsingular diagonal matrices in \(GL(2, p)\). As \(9 = (n - q) \| L\|\), \(L\) contains \(|\text{diag}(y, w)| \ y, w = 1, 2, 4\), a Sylow 3-subgroup of \(D\). Then \(L\) contains at least five elements with an eigenvalue 1, and \(|N_0 \cup Z| \geq 5|PZ|\), contrary to (21).

Still suppose that \(r(1) = 1\). Then \(N_0 \subseteq PZ\) and only \(p\)-singular elements lie in \(N_0\) and \(N^G_0\). There exists some \(y \in N - PZ\). If \(y \in N^G_0\), then \([y]_p \in P^\#\) and \(y \in C([y]_p) \subseteq N_0 \cup Z \subseteq PZ\), which is a contradiction. Therefore, \(y \in N - (N^G_0 \cup U)\).

By (C), \(\omega(y) = r(y)\) and \(\chi(y) = \beta(y)\). Then by (20) and (G),

\[
2 \geq |\beta^2(y) - k| = |\chi^2(y) - k| = D |\omega(y)|/(n - q) = D |\beta(y)|/(n - q) \geq (n^2 - n)/n \geq n - 1,
\]

which is a contradiction.

Therefore, \(r(1) \neq 1\). By (I), \(q \leq r(1) \leq 1\), so \(q = r(1) = 0\). Then by (19),

\[
((\chi^2_p, 1_p) = k \leq 1.
\]

By (G), \(0 = q = (\chi^2_p, 1_p) = k\). Then for \(u \notin N^G_0\), by (20)

\[
\chi^2(u) = D \omega(u)/(n - q) = n^2 \omega(u)/n = \chi(1) \omega(u).
\]

(L) \(\chi(x) = 0\) for \(x \notin Z \cup N^G_0\).

Proof. The group \(\overline{G} = G/\Omega^1_4(\Omega_2(Z))\) has the faithful representation \(Y\) corresponding to \(\omega\) since by (A) the kernel of \(\omega\) lies in \(Z\). By (A), \(Y(\overline{G})\) is a counterexample to the theorem. As \(\omega(1) = \chi(1)\) by (K), by minimality of \(|G|\), \(|\Omega^1_4([Z], 2)| = 1\) and \((|Z|, 2) = 1\). Let \(\zeta\) be a primitive \(|G|\)th root of unity, \(Q\) be the rationals, and \(K = Q(\zeta)\). Let \(\mu\) be the automorphism of \(K\) taking \([\zeta]\) to \([\zeta]_2\), and \([\zeta]_2\) to \(( [\zeta]_2)^2\) and \([\zeta]_2\), respectively. Then \(\omega|_Z = \chi^\mu|_Z\) and by (J), for \(u \notin N^G_0\), \(\omega(u) = \chi^\mu(u)\). Then by (K), for \(u \notin N^G_0\),
Let \( u \in N^G_0 \) and \( \chi(u) \neq 0 \), and let \( \nu \) be any automorphism of \( K \). There exists an integer \( e \) such that \( \zeta^e = \zeta^e \) and \( (e, |G|) = 1 \). Then \( u^e \notin N^G_0 \), otherwise, \( u^e \notin Z \), \( (u^e) = (u^e) \) centralizes a \( p \)-singular element and \( u \in N^G_0 \). Then by (22),

\[
(\chi^e(u)) = (\chi(u^e))^2 = \chi(1)\chi^e(u) = \chi(1)\chi^e(u).
\]

As we let \( \nu \) run over all automorphisms of \( K \), \( \mu \) runs over all automorphisms of \( K \). Taking the product of (23) as \( \nu \) ranges over all automorphisms of \( K \),

\[
\left( \prod_{\gamma \in K} \chi^\nu(\gamma) \right)^2 = \chi(1)^{\text{aut } K} \prod_{\gamma \in K} \chi^\nu(\gamma) \quad \text{and} \quad \prod_{\gamma \in K} \chi^\nu(\gamma) = \chi(1)^{\text{aut } K},
\]

As \( |\chi^\nu(\gamma)| \leq \chi(1) \) for all \( \nu \in \text{aut } K \), by (24) this is always equality. Then \( |\chi(\gamma)| = \chi(1) \) and \( u \in Z \).

We are now able to complete the proof of the theorem. By (K), \( q = 0 \), and \( \chi_N = \alpha \), so by (L),

\[
1 = \|\chi\|^2 = \frac{1}{|G|} \sum_{\gamma \in G} |\chi(\gamma)|^2
\]

\[
= \frac{1}{|G|} \left[ \sum_{\gamma \in Z} |\chi(\gamma)|^2 + \sum_{\gamma \in N^G_0} |\chi(\gamma)|^2 \right] = \frac{|Z|}{|G|} n^2 + \frac{1}{|G||G|/|N|} \sum_{\gamma \in N^G_0} |\chi(\gamma)|^2
\]

\[
< \frac{|Z|}{|N|} n^2 + \frac{1}{|N|} \sum_{\gamma \in N^G_0} |\chi(\gamma)|^2 = \left( \frac{1}{|N|} \right) \sum_{\gamma \in Z \cup N^G_0} |\chi(\gamma)|^2 \leq \|\alpha\|^2 = 1,
\]

which is a contradiction.

4. The case \( p \equiv -1 \pmod{4} \). The following theorem combines Theorem 2 and [7, Theorem 4].

Theorem 3. Let \( p \) be prime greater than 5 with \( p \equiv -1 \pmod{4} \). Let \( G \) be a finite group with a faithful, quasiprimitive, complex representation \( \chi \) with character \( \chi \) of dimension \( n < 4p/3 \), \( n \neq p \), and if \( p = 7 \), \( n \leq 8 \). Then \( p^2 \nmid |G/Z(G)| \).

The hypothesis \( p > 5 \) of Theorem 3 is unnecessary by the classification of 2-dimensional groups in the case \( p = 3 \). Given Theorem 2 it is easier to prove the combination Theorem 3 and [7, Theorems 2, 3, and 4] than it is to prove [7, Theorems 2, 3, and 4] since the stronger induction hypothesis allowed in proving the former combination eliminates some of the cases that had to be studied in the proof of [7, Theorems 2, 3, and 4]. We now prove Theorem 3.

Proof. We may replace elements \( x \) of \( X(G) \) by all unimodular scalar multiples.
of $x$. This does not affect quasiprimitivity or change $G/Z(G)$ within isomorphism. Therefore, we may assume that $X(G)$ is unimodular. Let $P$ be a Sylow $p$-subgroup of $G$. Then $P$ is abelian, otherwise some element in $(P' \cap Z(P))^a$ contradicts [1, Theorem 8, p. 96] and quasiprimitivity. Then as $(n, p) = 1$, $(|Z(G)|, p) = 1$. We may assume that $X(G)$ is a counterexample to our theorem, so $|P| > p$. Then by [7, Theorem 4], $P$ is a T.I. set and there exists a subgroup $E \subseteq G$ with $F = C(E)$ (possibly equal to $G$) having the following properties: $P \subseteq F$ and $X | F$ has a strongly primitive constituent $Y$ of degree larger than $p$. Also, $Y$ is faithful on $P$ and $\gamma(P) \subseteq \gamma(F)$ satisfy [4, Hypothesis 4.1]. Furthermore, all homogeneous subspaces of $\gamma(P)$ are linear and $C_{\gamma(F)}(\gamma(P)) = \gamma(P)Z(\gamma(F))$. Also, $(|Z(\gamma(F))|, p) = 1$. Finally, $N_{\gamma(F)}(\gamma(P))$ acts transitively on the nonprincipal homogeneous subspaces of $\gamma(P)$.

By the above, $\gamma$ has degree unequal to $p$. $\gamma$ is quasiprimitive, $|\gamma(P)| = |P| > p$, $(|Z(\gamma(F))|, p) = 1$, and $\gamma(P) \subseteq \gamma(F)$ satisfy [4, Hypothesis 4.1]. Therefore, $\gamma(F)$ contradicts Theorem 2. This completes the proof.

We now prove Theorem 1 from the abstract. Let $G$ be a linear group corresponding to $\mathcal{G}$ satisfying the hypothesis of Theorem 1. By [7, Theorem 1, (b)], $P$ is a trivial intersection set. By Theorem 3, $p \equiv 1 \pmod{4}$. By replacing elements $x$ of $G$ by all unimodular scalar multiples, we may assume that $G$ is unimodular. As $p \nmid n$, $(|Z(G)|, p) = 1$. Suppose that for all $x \in \mathcal{G}$, $C(x)$ has a normal Sylow $p$-subgroup. As in the proof of Theorem 3, $P$ is abelian. Suppose that $N_0$ is defined as in Theorem 2, and for some $y, g \in G$, $y \in N_0 \cap N_0^g$. Let $H$ be the preimage of $C(y)$. Then $C(y) \subseteq H$ and $C(\mathcal{G})$ and $H$ have normal Sylow $p$-subgroups, so $C(y)$ has a normal Sylow $p$-subgroup $Q$ which we may assume is contained in a Sylow $p$-subgroup $P^b$ of $G$. Now, $y$ centralizes some nonidentity elements $u$ and $v$ of $P$ and $P^g$, respectively. As $u, v \in P \subseteq P^b$ and $P$ is a T.I. set, $P = P^b = P^g$ and $P$ satisfies [4, Hypothesis 4.1]. Then by Theorem 2, $|P| \leq p$, which is a contradiction.

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