PROJECTIVE GROUPS OF DEGREE LESS THAN $4p/3$
WHERE CENTRALIZERS HAVE NORMAL SYLOW $p$-SUBGROUPS

BY
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ABSTRACT. This paper proves the following theorem:

Theorem 1. Let $\overline{G}$ be a finite primitive complex projective group of degree $n$ with a Sylow $p$-subgroup $\overline{P}$ of order greater than $p$ for $p$ prime greater than five. Let $n \neq p$, $n < 4p/3$, and if $p \equiv 7 \pmod{4}$, $n < 8$. Then $p \equiv 1 \pmod{4}$, $\overline{P}$ is a trivial intersection set, and for some nonidentity element $\overline{x}$ in $\overline{G}$, $C(\overline{x})$ does not have a normal Sylow $p$-subgroup.

1. Introduction. The main object of study in this paper is the case where a projective group $\overline{G}$ has a Sylow $p$-subgroup $\overline{P}$ which is a trivial intersection set and nonidentity elements of $\overline{G}$ have centralizers which have normal Sylow $p$-subgroups. The other possible cases were studied in earlier papers. These results are combined in Theorem 3.

2. Notation. If $H$ is a subgroup of a group $G$, we let $H^G$ be the normal subgroup of $G$ generated by $H$. The set of nonidentity elements of $H$ is called $H^*$. For characters $\mu$ and $\nu$ of $H$ we let $(\mu, \nu) = (1/|H|) \sum_{x \in H} \mu(x)\overline{\nu(x)}$, and we let the squared norm $\|\mu\|^2 = (\mu, \mu)$.

A representation $X$ of a group $G$ is called quasiprimitive if it is irreducible and its restriction to any normal subgroup of $G$ is the sum of equivalent irreducible constituents.

3. Projective groups where centralizers of nonidentity elements have normal Sylow $p$-subgroups. We shall prove the following theorem:

Theorem 2. Let $p$ be a prime greater than five. Let $G$ be a finite group with a faithful, quasiprimitive, complex representation $X$ with character $\chi$ of dimension $n < 4p/3$, $n \neq p$. Let $P$ be an abelian Sylow $p$-subgroup of $G$. Throughout, let $Z = Z(G)$, $C = C(P)$, $N = N(P)$, and $N_0 = \sum_{y \in P^*} C(y) - Z$. Let $P$ satisfy [4, Hypothesis 4.1] (that is, $P$ and $N_0$ are trivial intersection sets and $N(N_0) = N$). Let $(|Z|, p) = 1$. Then $|P| \leq p$.

By running through the classifications of groups of small degree, it can be...
seen that the hypothesis $p > 5$ of Theorem 2 is unnecessary. By [2, (4A)], the assumption $n \neq p$ is unnecessary. Also, $4p/3$ can be replaced by a number asymptotic to $\sqrt{2}p$ for large $p$. The following proof of Theorem 2 does not require the use of [7].

Proof. Let $X(G)$ be a counterexample with $n$ minimal. Given that $n$ is minimal, let $|G|$ be minimal. Then $|P| \geq p^2$. Suppose $X(G)$ is imprimitive on $c > 1$ subspaces and let $K$ be the subgroup of $G$ fixing these spaces setwise. By primitivity, $K$ is represented faithfully by the $c$ equivalent constituents of $X|_K$ of degree no larger than $2p/3$. By [4], $p^2 \not\mid [K : O_p(K)]$. As $O_p(K) \not\leq Z$, by primitivity, $O_p(K) \subset Z$ and $O_p(K) = \langle 1 \rangle$. Then $p \mid [G : K] | c!$ and $c = n$. Then $K$ is an abelian normal subgroup of $G$ and $K = Z$. Then at most $p$ divides $c!$ and $|G|$, which is a contradiction. If $X(G)$ is a subgroup of a tensor product of two smaller dimensional groups, then, unless $p = 7$ and $n = 9$, using $((p - 1)/2) > 4p/3 > n$ with [5] and using $n/2 < 2p/3 < p - 1$ with [4] we have by [6, Lemma 1] that $p^2 \not\mid [G : O_p(G)]$, contrary to $O_p(G) = \langle 1 \rangle$ and $p^2 \not\mid |G|$. Even if $p = 7$ and $n = 9$, $X(G)$ contains $M \otimes I_3$ and $I_3 \otimes M$ where $M \simeq SL(2,7)$ and a 7-element $x \in I_3 \otimes M$ has $M \otimes I$, $C(x)$ having a normal Sylow 7-subgroup, contrary to $P$ being a T.I. set. Therefore, $G$ is strongly primitive.

Let $P^G$ be the normal subgroup of $G$ generated by $P$. By strong primitivity and [3, Theorem 51.7], $X|_{P^G}$ is irreducible. If $X|_{P^G}$ is imprimitive, then $X|_{P^G}$ is monomial; otherwise, the $p$-elements generating $P^G$ would fix all spaces of primitivity. If $X|_{P^G}$ is monomial, then $1 \neq O_p(P^G) \not\leq G$, contrary to $O_p(G) = \langle 1 \rangle$. As $P^G$ also satisfies [4, Hypothesis 4.1], $G = P^G$ by minimality of $|G|$. For any nonprincipal irreducible character $\gamma$ of $G$ we write

$$\gamma|_N = \pi + \rho$$

where $\rho$ is the sum of irreducible constituents of $\gamma|_N$ having $P$ in the kernel and $\pi$ is the sum of the others. As $G = P^G$, $\|\pi\| \geq 1$.

Throughout, let $t = [N : PZ]$. Let $m = \gamma(1)$ and $b = \rho(1)$. By [4, Lemma 2.11], $\pi$ vanishes on $N - (N_0 \cup Z)$. As in [4, Lemma 4.1],

$$l = \|\gamma\|^2 > (1/|G|)(|G| / |N|) \sum_{N_0} |\gamma(x)|^2$$

$$= -m^2/t|P| + (1/|N|) \sum_{N_0 \cup Z} (\pi(x) + \rho(x))(\overline{\pi(x)} + \overline{\rho(x)})$$

$$= -m^2/t|P| + (1/|N|) \left[ \sum_{N} (\pi(x)\overline{\pi(x)} + \pi(x)\overline{\rho(x)} + \rho(x)\overline{\pi(x)}) + \sum_{N_0 \cup Z} |\rho(x)|^2 \right]$$

$$\geq -m^2/t|P| + \|\pi\|^2 + (1/|N|) \sum_{N_0 \cup Z} |\rho(x)|^2$$

$$\geq -m^2/t|P| + \|\pi\|^2 + b^2/t.$$
When \( y = \chi \), throughout we let \( \alpha \) and \( \beta \) correspond to \( \pi \) and \( \rho \) respectively, \( q = \beta(1) \), and

\[
\chi|_N = \alpha + \beta,
\]

where we show that \( n \geq p + 1 \), \( \beta(1) \leq 1 \), \( \|\alpha\| = 1 \), \( \alpha|_P \) is a sum of \( \alpha(1) = n - q \) distinct nonprincipal linear characters of \( P \) permuted transitively by \( N \), \( |P| = p^2 \), \( P \) is elementary abelian, \( O_{\rho'}(G) = Z \), and \( C = PZ \).

Since \( |P| > p^2 \), \( O_{\rho'}(G) = (1) \), and \( O^2(G) \supseteq P^G = G \), by [6], we have \( n > p - 1 \) and \( n \geq p + 1 \) as \( n \neq p \) by hypothesis. If \( t = 1 \), then \( N = PZ \) is abelian and by (2), \( p < n \leq \|\alpha\|^2 + q < 1 + n^2/t|P| < 1 + (4p/3)^2/p^2 < 25/9 \), which is a contradiction. Therefore, \( t \geq 2 \) and by (2),

\[
\|\alpha\|^2 \leq 1 + n^2/t|P| < 1 + (4p/3)^2/2p^2 < 2,
\]

and \( \|\alpha\| = 1 \). Also, by (2), \( q = b \leq n/(|P|)^{1/2} \leq (4p/3)/p < 2 \), and \( q \leq 1 \). More generally, in (2),

\[
(\gamma|_P, 1_P) < \gamma(1)/(|P|)^{1/2}.
\]

If \( \alpha|_P \) has homogeneous constituents (Wedderburn components) of dimension \( e \), then by summing (4) over the irreducible constituents of \( \chi\bar{\chi} - 1_G \), we have

\[
(n - 1)e \leq [(n - q)/e]e^2 + q - 1 = ((\chi\bar{\chi} - 1_G)|_P, 1_P) < (n^2 - 1)/(|P|)^{1/2}.
\]

Then \( e \leq (n + 1)/p < (4p/3 + 1)/p < 2 \) and \( e = 1 \). As \( \|\alpha\| = 1 \), the distinct linear homogeneous spaces of \( \alpha|_P \) are permuted transitively by \( N \). Also, by (5),

\[
|P| < (n + 1)^2 < (4p/3 + 1)^2 < p^3 \quad \text{and} \quad |P| = p^2.
\]

If \( P \) is cyclic, \( [N: C] \parallel \text{Aut}(P) = p(p - 1) \) and \( (n - q)|_N < 1, \) contrary to \( n - q > p - 1 \). As \( \beta(1) \leq 1, \chi|_P \) has all distinct linear constituents.

Suppose \( O_{\rho'}(G) \supset Z \). Then by strong primitivity, \( \chi|_{O_{\rho'}(G)} \) is irreducible. Replacing elements \( x \) of \( X(P) \) by all unimodular scalar multiples of \( x \), we get a group \( P^* \) of exponent \( p \), since \( (n, p) = 1 \), with \( X(O_{\rho'}(G)) < P^*X(O_{\rho'}(G)) \). By [6, Lemma 8] for \( x \in P^* \) some scalar multiple \( y \) of \( x \) has all primitive \( p \)th roots of unity occurring equally often. As \( (n, p) = 1 \) and \( \det x = \det y, \ x = y \) and trace \( x \) is rational. Then by [9],

\[
p^2 = |P/P \cap Z| \leq |P^*| \leq p^{[n/(p-1)] + [n/p(p-1)] + \cdots} = p
\]

which is a contradiction. If \( C \supset PZ \), we may find a \( q \)-element \( \nu \in C - PZ \) for \( q \) a prime unequal to \( p \). By [2, proof of (3F)], since \( \chi|_P \) is a sum of distinct linear constituents, \( \nu \in O_{q'}(G) \subseteq O_{\rho'}(G) = Z \), which is a contradiction.

Throughout, we write

\[
\chi^2 = k1_G + \sum y_i
\]
where \( k \) equals 0 or 1 and the \( \gamma_i \) are irreducible nonprincipal characters of \( G \).

We now divide the proof into parts.

(A) \( G/Z \) is simple. If \( W \trianglelefteq G \) and \( W \not\subseteq Z \), then \( W = G \). Also, \( G = G' \). Irreducible nonprincipal characters \( \gamma \) of \( G \) have degree greater than \( p \). If \( U \) is a subgroup of \( Z \) and \( S \) is a subset of \( G \), let \( \overline{S} \) be the image of \( S \) in \( G/U \). Then \( \overline{N}_0 \) is the similarly defined \( N_0 \) of \( \overline{G} \), and \( \overline{N}_0 \) and \( \overline{P} \) are T.I. sets whose normalizer is \( \overline{N} \).

\textbf{Proof.} Let \( Z \trianglelefteq K \trianglelefteq G \). As \( P^G = G \) and \( O^p_p(G) = Z \), we have \( p^2 \mid |K| \) and \( p \mid |K| \), so \( K \) has a Sylow \( p \)-subgroup \( Q = P \cap K \) of order \( p \). As \( P \) is a T.I. set, \( P \) is a normal Sylow \( p \)-subgroup of \( C(Q) \). Then \( P \text{ char } C(Q) \subseteq N(Q) \). By a Sylow theorem \( G = KN(Q) \). Then \( N(Q) \) and \( N(Q)/K \cap N(Q) \cong G/K \) have normal Sylow \( p \)-subgroups. Then \( O^p_p(G/K) = O^p_p(G/K) = G/K \) since \( G = P^G \). Then \( G/K \) is a \( p \)-group and \( [G : K] = p \). As \( P \) is elementary abelian and \( ([N : P], p) = 1 \), by complete reducibility, \( Q \) has an \( N/C \) complement \( R \) in \( P \). Then \( (R, N) = (R, P(N \cap K)) = (R, N \cap K) \subseteq R \cap K \subseteq R \cap (P \cap K) = R \cap Q = 1 \). By (3) with \( a \) irreducible and \( a(1) > n - 1 \), \( G \) consists of homologies contrary to \[1, \text{Theorem 8, page 96}\] and quasiprimitivity. Therefore, \( G/Z \) is simple.

Let \( W \trianglelefteq G \) and \( W \not\subseteq Z \). Then \( W \) covers \( G/Z \). \( p^2 \mid |W| \), \( P \subseteq W \), and \( G = P^G \subseteq W \). As \( G' \) covers \( G/Z \), \( G' \not\subseteq Z \) and \( G' = G \). Let \( \gamma \) be an irreducible nonprincipal character of \( G \) of degree less than \( p + 1 \) and kernel \( W \). As \( \gamma \) covers \( W \), \( \gamma \) is a \( p \)-element of \( G \) with \( p \not\mid |\gamma| \). As \( \gamma \) is a \( p \)-element of \( G \), \( \gamma \) is a \( p \)-element of \( W \). For any group \( M \), let \( i_p(M) = [M : O^p_p(M)] \). As \( O^p_p(G/W) \geq O^p_p(G/K) \) = \( G/W \), by [6], \( i_p(G/W) \leq p \) if \( \gamma(1) < p \). If \( \gamma(1) = p \), then as \( P \) is abelian, by [2, (4A)], \( i_p(G/W) \leq p \). Then by [6, Lemma 1], \( p^2 = i_p(G/Z) \leq i_p(G/W) \leq p \), which is a contradiction.

Let \( U \subseteq Z \). Let \( \bar{x} \in \overline{G} \) centralize \( \bar{y} \in \overline{P}^p \). Then \( (x, y) \in Z \) is the quotient of the commuting \( p \)-elements \( y^x \) and \( y \). As \( |Z| = 1 \), \( (x, y) = 1 \). Then \( \overline{N}_0 \) is the \( N_0 \) for \( \overline{G} \). As \( N_0 \) consists only of entire cosets of \( U \), \( \overline{N}_0 \) is a T.I. set with normalizer \( \overline{N} \). As \( \overline{P}^p \subseteq \overline{N}_0 \), \( \overline{P} \) also is a T.I. set with normalizer \( \overline{N} \).

(B) \( N/C \cong L \) where \( L \) is a subgroup \( (L \) is fixed throughout this paper) of \( GL(2, p) \) of order \( t \). We view \( L \) as a 2-dimensional matrix group over \( GF(p) \). Also, column vectors and row vectors correspond to elements of \( P \) and linear characters of \( P \), respectively, on which \( L \) acts by matrix multiplication as \( N/C \) acts on \( P \) and characters of \( P \). If \( rc \) is the matrix product of a row vector \( r \) with a column vector \( c \), then \( e^{(2\pi i/p)rc} \) equals the value of the corresponding character of \( P \) at the corresponding element of \( P \).

\textbf{Proof.} As \( C \) is the kernel of the action of \( N \) on \( P \), \( N/C \) is isomorphic to a subgroup of \( \text{Aut}(P) \cong GL(2, p) \). As \( C = PZ \), \( |L| = t \).

(C) Let \( \theta \) be an irreducible character of \( N \) not having \( P \) in its kernel. Let
\( \phi \) be a linear constituent of \( \theta|_{\mathbb{P}} \) and \( H \) be the subgroup of \( N \) fixing \( \phi \). Then \( C \subseteq H \subseteq N_0 \cup Z \) and there exists a linear constituent \( \xi \) of \( \theta|_{H} \) with \( \xi|_{\mathbb{P}} = \phi \) and \( \theta = \xi^N \), the induced character. Also, \( \theta \) vanishes on \( N - (N_0 \cup Z) \).

**Proof.** Let \( \theta \) be the character of the representation \( R \) on the space \( W \). Let \( U \) be the homogeneous space (Wedderburn component) for \( R \) restricted to \( P \) such that \((\dim U)\phi\) is the character of the representation of \( P \) on \( U \). Let \( H \) be the subgroup of \( N \) fixing \( \phi \). Then \( H \) is also the subgroup of \( N \) fixing \( U \). Let \( \xi \) be the character of the representation of \( H \) on \( U \). Then by Frobenius reciprocity, \((\xi^N, \theta) = (\xi, \theta|_{H}) \geq 1\). Since \( \theta \) is irreducible and \( \theta(1) = \dim W = [N:H] \dim U = [N:H] \xi(1) = \xi^N(1), \theta = \xi^N \). Also, \( \xi \) is irreducible since \( \xi^N = \theta \) is irreducible.

Now, \( H/C \) corresponds to a \( p' \)-subgroup of the subgroup \( M \) of \( \text{GL}(2, p) \approx \text{Aut}(P) \) fixing a nonprincipal character of \( P \). As \( M \) is isomorphic to a normal extension of a group of order \( p \) by a cyclic group of order \( p - 1 \), \( H/C \) is cyclic. As elements of \( C = PZ \) are represented by scalars on \( U \) and \( \xi \) is irreducible, \( \xi \) is linear. As any element of \( H \) fixes the nonprincipal linear character \( \xi^P \) of \( P \), by the permutation lemma, it also fixes a nonidentity element of \( P \) and lies in \( N_0 \cup Z \). This proves (C) since \( \xi^N \) vanishes on \( N - (N_0 \cup Z) \) since \( N = MN_{N_0} \) and \( WC N_0 UZ \).

(D) Let \( N/C \approx L \subseteq \text{GL}(2, p) \) as in (B). Then either

I. \( L/Z(L) \approx A_5 \) and \( |L| | 60(p - 1) \),

II. \( L/Z(L) \approx A_4 \) and \( |L| | 12(p - 1) \),

III. \( L/Z(L) \approx S_4 \) and \( |L| | 24(p - 1) \),

IV. \( L \) is monomial, \( |L| | 2(p - 1)^2 \), and \( L \) contains a diagonal subgroup \( A \)

with \( [L : A] \leq 2 \), or

V. \( L \) can be written as a monomial group in \( \text{GL}(2, p^2) \) where \( L \) contains a subgroup

\[
A = \left\langle \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^p \end{pmatrix} \right\rangle \quad \text{for some} \quad \zeta \in \text{GF}(p^2) \quad \text{and} \quad [L : A] \leq 2.
\]

Here, \( |L| | 2(p^2 - 1) \).

In cases I, II, and III, \( L \) is irreducible, \( |Z(L)| | (p - 1) \), and \( -I_2 \in Z(L) \).

**Proof.** We may consider \( L \) to be a faithful 2-dimensional representation of a \( p' \)-group over a field of characteristic \( p \). In the case of \( p' \)-groups, complex irreducible representations have a one to one correspondence with \( p \)-modular representations. Then \( L \) may be obtained from a finite complex \( p \)-integral 2-dimensional group by taking coefficients modulo an ideal dividing \( (p) \). By the classification in [1] of 2-dimensional complex linear groups, we have I, II, or III or \( L \) may be taken as monomial when written over a larger field. In the monomial cases there exists \( A \), an abelian subgroup of index 1 or 2 in \( L \). The character of the representation of \( A \) by our linear group \( L \) is a sum of two linear
characters $\sigma$ and $\tau$. If $\sigma$ and $\tau$ lie in $\text{GF}(p)$, we have case IV. If they do not, since $\sigma + \tau$ lies in $\text{GF}(p)$, $\sigma$ and $\tau$ lie in $\text{GF}(p^2)$ and are algebraic conjugates over $\text{GF}(p)$. Then $\sigma = \tau^p$.

(E) Let $\theta$, $\phi$, and $H$ be as in (C). Then except in case IV, $[H : C] \leq 5$. If $\theta = \alpha$, throughout this paper let $b = [H : C]$. Then $t = (n - q)b$. Suppose we are in case IV or V. Let $\theta = \alpha$, let $A$ be as in (D), and let $M \subseteq L$ correspond to $H/C \subseteq N/C$. Then $M \cap A = \{1\}$ and $[H : C] \leq 2$.

Proof. As in the proof of (C), $H/C$ corresponds to the cyclic subgroup $M$ of $L \subseteq \text{GL}(2, p)$ of order dividing $p - 1$ and fixing the character $\phi$. As elements of $M$ have an eigenvalue 1, $M \cap \text{ZGL}(2, p) = \{1\}$. Then since $M$ is cyclic, in case I, II, and III, $[H : C]$ divides the order of some element in $A_5$, $A_4$, or $S_4$, so $[H : C] \leq 5$. In case V, $I_2$ is the only element of $A$ with an eigenvalue 1, so $M \cap A = \{1\}$ and $[H : C] \leq 2$.

Suppose $\theta = \alpha$. By (C), $n - q = \alpha(1) = [N : H] = t/b$. Suppose further that we have case IV. Let $A$ be the diagonal subgroup of $L$ of index at most 2. Then if $M \cap A = \{1\}$, $[H : C] \leq 2$. Therefore, suppose that $\text{diag}(\sigma, \tau)$ lies in $(M \cap A)^\#$. Then, as elements of $M$ have an eigenvalue 1, $\sigma$ or $\tau = 1$, say $\sigma = 1$. Then $\phi$ corresponds to a multiple of $(1, 0)$. Let $\zeta$ be the homomorphism from $A$ to $\text{GF}(p)^\#$ with $\zeta(\text{diag}(\pi, \rho)) = \pi$. Then $M \cap A$ is the kernel of $\zeta$ and $[A : M \cap A] = |\zeta(A)|/(p - 1)$. Also, $[L : M] = [L : A][A : M \cap A]/[M : M \cap A]2(p - 1)$. Therefore, $\alpha(1) = [N : H] = [L : M]2(p - 1)$. Then, since $\alpha(1) \leq n < 4p/3 < 2(p - 1)$, $\alpha(1) < p - 1$. Furthermore, $n = \alpha(1) + q \leq \alpha(1) + 1 < p$, contrary to (A).

(F) Let $\zeta$ and $\xi$ be the characters of the symmetric and skew-symmetric tensors of $\chi^2$, respectively. Then, if $-I_2 \in L$, $(n + q)/2 = (1_p, \zeta|_p)$ and $(n - q)/2 = (1_p, \xi|_p)$.

Proof. As multiplying a row vector by $-I_2$ corresponds to taking the complex conjugate of the corresponding character, if $-I_2 \in L$ then all complex conjugates of constituents of $\alpha|_p$ also occur in $\alpha|_p$. By linearity of the homogeneous constituents of $\chi|_p$ this concludes the proof of (F).

(G) $(\chi_2^2|_p, 1_p)$ equals $q$ or $n$. If $(\chi_2^2|_p, 1_p) = n$ then $n - q$ is even.

Proof. If $(\chi_2^2|_p, 1_p) \neq q$, then $\alpha|_p$ contains a pair of complex conjugate linear characters. Then $\alpha|_p$ consists entirely of pairs of complex linear characters since the action of $N$ is transitive on the linear constituents of $\alpha|_p$ and commutes with scalar multiplication (by $-1$ in particular and by elements of $\text{GF}(p)$ in general) of the linear characters of $P$. This proves (G) since $\chi|_p$ has only linear homogeneous components.

(H) Let $\gamma = \gamma_i$ be a nonprincipal constituent of $\chi^2$. Let $\gamma|_N = \pi + \rho$ as in (I) with $P$ in the kernel of $\rho$ but not in the kernel of any constituent of $\pi$. Then $||\pi|| = 1$. 

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Proof. Let $m = \gamma(1)$, $b = \rho(1)$, and $h = [H : C]$ with $H$ being the $H$ in (C) when $\theta = \alpha$. Let $\zeta$ and $\xi$ be the characters of the symmetric and skew-symmetric tensors of $\chi^2$, respectively. We may assume that $\|\pi\|^2 \geq 2$, otherwise (H) holds. By (C) irreducible characters of $N$ without $P$ in the kernel have degree no larger than $[N : C] = t$. Therefore, $\|\pi\|^2 \geq \lceil \pi(1)/t \rceil = \lceil (m - b)/t \rceil$ where $\lfloor x \rfloor$ is the smallest integer greater than or equal to $x$. Then fixing $p$ and using (2), we use the following to define various functions:

$$\begin{align*}
(7) & \quad \frac{m^2}{tp^2} + \max (2, \lceil (m - b)/t \rceil) + b^2/t < 1, \\
(8) & \quad b(t, m, b) = -\frac{m^2}{tp^2} + \max (2, \frac{m}{t} + b^2/t - b/t) < 1, \quad \text{and} \\
(9) & \quad g(t, m) = -\frac{m^2}{tp^2} + \max (2, \frac{m}{t}) < 1.
\end{align*}$$

For at least one of $\mu = \zeta$ or $\xi$ we have $\mu = s1_G + \gamma + \nu$ where $\nu$ is a sum of nonprincipal irreducible characters $\psi_i$ of $G$ and $s \leq k \leq 1$. By (4), $\psi_i(1) > (\psi_i, p, 1_p)p$. Summing this over $i$, we have

$$\mu(1) - s - m = \nu(1) \geq (\nu, p, 1_p)p = [(\mu, p, 1_p) - s - b]p.$$  

Suppose that $-l_2 \in L$. Then by (F) and the above inequality, if $\mu = \zeta$ we have

$$(10) \quad m(n + 1)/2 - s - m \geq [(n + q)/2 - s - b]p$$

and if $\mu = \xi$ we have

$$(11) \quad m(n - 1)/2 - s - m \geq [(n - q)/2 - s - b]p.$$  

In either case, we define $f(n, m)$. For fixed $m$, $f(n, m)$ decreases as a function of $n \geq (p - 1)/2$. Since $p < n < 4p/3$ by (A),

$$(12) \quad b > (n - 1)/2 - 1 - m(n + 1)/2p + m/p = f(n, m)$$

$$\begin{align*}
(13) & \quad b \geq f(n, m) \geq f(4p/3, m) = m/p - (2/9)p - 13/6. \\
(14) & \quad b \geq f(n, m) \geq f(4p/3, m) = m/p - (2/9)p - 13/6.
\end{align*}$$  

We no longer assume that $-l_2 \in L$. The cases $p = 7, 11, 13, 17, 19, \text{ and } 23$ were examined by computer for $n = p + 1, \cdots, [4p/3]; \quad q = 0 \text{ or } 1; \quad s = 0 \text{ or } 1; \quad m = p + 1, \cdots, -s + n(n + 1)/2; \quad b = 1, \cdots, 5; \text{ and } b = 0, \cdots, -s + (n + q)/2$. The computer ruled out cases for any of the following reasons:

1. Inequality (7) failed.
2. $n - q$ and $p$ failed to satisfy any of the following two propositions:
\( P_1: (n - q) \) divides \( 60(p - 1) \) or \( 24(p - 1) \),
\( P_2: (n - q) \) divides \( 2(p - 1)^2 \) or \( 2(p^2 - 1) \). (This uses (D).)

3. \( P_1 \) fails and \( b > 2 \). (This uses (D) and (E).)

4. If \((b > 2 \) or \( P_2 \) fails) and \( P_3 \) holds where \( P_3 \) is the proposition:
\[
P_3: \text{Inequality (10) fails and } [n(n - 1)/2 - s - m < 0 \) or \( (11) \) fails].
\]
(Here, if \(-l_2 \in L \), then if \( \mu = \zeta \), we have (10), and if \( \mu = \xi \), we have \( s + m \leq n(n - 1)/2 \) and (11). Therefore, \(-l_2 \in L \) implies \( P_3 \) fails. However, if \( b > 2 \) or \( P_2 \) fails, then by (D) and (E), \(-l_2 \in L \).)

The only cases left by the computer satisfied one of the following:
1. \( p = 7 \), \( n - q = 8 \), and \( P_3 \) holds.
2. \( p = 7 \), \( n - q = 9 \), \( b \leq 3 \), \( m \geq 30 \), and the left side of (7) is greater than \( A \).
3. \( p = 13 \), \( n - q = 16 \), \( b = 2 \), and \( P_3 \) holds.
4. \( p = 19 \), \( n - q = 24 \), \( b = 2 \), and \( P_3 \) holds.

In cases 1, 3, and 4, \( P_3 \) holds, so \(-l_2 \notin L \). Then we have case IV or V in (D). Let \( A \) be the diagonal subgroup of \( L \) of index at most 2 (in case V, \( A \) may have to be written in \( GF(2, p^2) \)). As \( 2 \mid b(n - q)/2 = t/2 \mid |A| \), in case V we have \(-l_2 = \text{diag}(-1, (-1)^p) \in A \), which is a contradiction. Therefore, we have case IV.

Let \( D \) be the group of all nonsingular diagonal matrices in \( GL(2, p) \). Then \( A \subseteq D \) and \( |D| = (p - 1)^2 \). In all cases 1, 3, and 4, two divides \( b(n - q)/2 \) and \( |A| \) to at least as high a power as it divides \((p - 1)^2 \) and \(|D| \). Then a Sylow 2-subgroup of \( A \) is a Sylow 2-subgroup of \( D \) and contains \(-l_2 \), which is a contradiction.

Therefore, we have case 2. As \( n - q = 9 \) \( 4 \) \( 2(p^2 - 1) \) and neither 60, 12, nor 24 divide \( 9b \), in (D) we have case IV. Then by (E), \( M \cap A = \{1\} \) and \( b = 1 \) or 2 where \( M \subseteq L \) corresponds to \( H/C \subseteq N/C \). Changing coordinates, we take
\[
A = \{\text{diag}(x, y)| x, y \in GF(7), x, y = 1, 2, 4\}.
\]
If \( b = 2 \), we have \( \{0 0\} \in M \) for some \( \psi \). Changing coordinates by \( v \to (\text{diag}(\psi, 1)v \) for \( v \in \{\text{column vectors}\} \cong P \), we take \( \psi = 1 \). Further changing coordinates by a scalar matrix, or by a diagonal matrix if \( b = 1 \), we take \((1, 1) \) to correspond to a linear constituent of \( \alpha | p \). Then \( \{x(y)| x, y = 1, 2, 4\} \) is the set of linear constituents of \( \alpha | p \) and \( \{x(y)| x, y = 1, 2, 3, 4, 5, 6, 9\} \) is the set of 81 constituents of \( \chi^2 | p \) with multiplicities. If \( b = 1 \), then under the action of \( N \), the constituents of \( \mu | p \) lie in four orbits of length 9: \( O_1, O_2, O_3, \) and \( O_4 \), or five orbits of length 9: \( O_1, O_2, O_2, O_3, \) and \( O_4 \) if \( \mu \) corresponds to the skew-symmetric or symmetric tensors, respectively, of \( \chi^2 \). Here \( O_1 \) is represented by \((2, 2), O_2 \) by \((3, 3), O_3 \) by \((2, 3), \) and \( O_4 \) by \((3, 2) \). If \( b = 2 \), then \( O_3 \) and \( O_4 \) are joined into an orbit \( O_{34} \) of length 18. As \( 9 \mid |O_i| \) for all \( i \), \( 9 \mid m \) and \( m = 36 \) or 45. If \( \|n\|^2 \max(2, |(m - b)/2|) = m/9b \), then the right side of (2) exceeds 1.4, which is a contradiction. By (C), orbits of linear constituents of \( \chi^2 | p \) correspond to irreduc-
ible constituents of $X_{|N|}^2$. If $b = 2$, then $O_{34}$ corresponds to the only possible irreducible constituents of $\pi$ of degree 18. As $m \geq 36$, $\pi$ has at least two irreducible constituents of degree 9 and $\|\pi\|^2 > \lfloor \frac{m}{18} \rfloor$, which is a contradiction. Therefore, $b = 1$. Then in application of (C) to any irreducible constituents of $X_{|N|}^2$, $H = PZ = C$. If $\nu$ and $\mu$ both correspond to $O_{2}$, then by (C), $\nu$ and $\mu$ are induced from linear constituents $\phi$ and $\theta$ of $\nu|_C$ and $\mu|_C$, respectively, with $\phi|_P = \theta|_P$. As $X_{|Z|}^2$ is a multiple of $\phi|_Z$ and $\theta|_Z$, $\phi|_Z = \theta|_Z$, $\phi = \theta$, and $\nu = \mu$. As $\|\pi\|^2 = \lfloor \frac{m}{9} \rfloor$, $\pi$ cannot contain both constituents corresponding to $O_{2}$ as they are identical. Then $m = 36$, a case not reported by the computer as (7) fails.

We may now assume that $p \geq 29$. As $n < \frac{4p}{3}$ and $t = (n - q)b \leq 5n, t \leq \frac{20p}{3}$ and $2t \leq \frac{40p}{3} < \frac{p}{2}$. For fixed $t$, $g(t, m)$ is a decreasing function of $m$ for $0 < m < 2t$, increasing for $2t < m < \frac{p}{2}$, and decreasing for $\frac{p}{2} < m$. For $m < \frac{p}{2}$, (15) $g(t, m) \geq g(t, 2t) = (\frac{4}{9})^2 + \frac{2}{p} > 1 + 2 - \frac{80}{3}p > 1$. Then by (9), $m \geq \frac{p}{2}$. Also, $f(\frac{4p}{3}, m) = m/p - (2/9)p - 13/6 \geq \frac{p}{2} - (2/9)p - 13/6$ (16) $= (5/18)p - 13/6 > (5/18)29 - 13/6 > 1$.

Suppose that $-I_2 \in L$. Then (14) holds. For $b \geq 1$ and $t$ and $m$ fixed, $b(t, m, b)$ is an increasing function of $b$. By (16) we may combine (8) and (14) and define $k(t, m)$:

(17) $k(t, m) = b(t, m, f(\frac{4p}{3}, m)) \leq b(t, m, b) < 1$.

As $m \geq \frac{p}{2} > 2t$, $b(t, m, b) = -m^2/tp^2 + m/t + b^2/t - b/t$. The $m^2$ terms cancel in $k(t, m)$. The coefficient of $m$ in $k(t, m)$ is $1/t + (2/9)(1/p)[(2/9)p - 13/6] - (1/9)(1/p) = (5/9 - 16/3p)/t > 0$. Therefore, for $m \geq \frac{p}{2}$, by (16) and (15) $k(t, m) \geq k(t, \frac{p}{2}) \geq b(t, \frac{p}{2}, 1) = g(t, \frac{p}{2}) > 1$. This contradicts (17).

Therefore, we have $p \geq 29, m \geq \frac{p}{2}$, and $-I_2 \notin L$. By (D) we then have case IV or V. Let $x = (4p - 1)/3$. By (E), $b \leq 2$ and $t = (n - q)b \leq 2n \leq 2x$. Also, $m \leq n(n + 1)/2 \leq x(x + 1)/2 < 2p(4p + 3)/9 < p^2$. As $g(t, m)$ is decreasing in $m$ for $m \geq \frac{p}{2}$, we have by (9),
\[ g(t, m) = \frac{m}{t}(1 - \frac{m}{p^2}) \]
\[ \geq \frac{m}{2s}(1 - \frac{m}{p^2}) \geq \frac{(x(x + 1)/2)/2x}{1 - x(x + 1)/2p^2} = \frac{(x + 1)(2p^2 - x^2 - x)/8p^2}{4p + 2)(18p^2 - 16p^2 + 8p - 1 - 12p + 3)/216p^2} \]
\[ \geq (4/29 + 2)(1/108)(1 - 2/29) = (3186)/3132 > 1, \]
which is a contradiction.

(I) Let \( \gamma \) be a nonprincipal irreducible character of \( G \) with \( \gamma|_N = \pi + \rho \) as in (1). Let \( \|\sigma\| = 1 \) and \( \pi(1) = \alpha(1) \). Then \( \rho(1) \leq 1 \). For the remainder of the proof, let \( \omega = \gamma_1 \) be a fixed nonprincipal irreducible constituent of \( \chi^2 \) with \( \omega|_\mathbb{P} \) containing \( \phi^2 \) where \( \phi \) is a fixed linear constituent of \( \alpha|_\mathbb{P} \). Write

\[ \omega|_N = \sigma + \tau \]
as in (1). Then \( \|\sigma\| = 1 \), \( \alpha(1) = \alpha(1) \), and \( q \leq (r(1) \leq 1 \).

Proof. By (4), \( pp(1) \leq \pi(1) + \rho(1) \) and \( \rho(1) \leq \pi(1)/(p - 1) < 4p/3(p - 1) < 2. \)
As \( \phi^2 \) is a constituent of \( \chi^2|_\mathbb{P} \), \( \omega = \gamma_1 \) exists. By (H), \( \|\sigma\| = 1 \). As \( \phi^2 \) and \( \phi \) are fixed by the same subgroup \( H \) of \( N \), by (C) we have \( \sigma(1) = [N: H] = \alpha(1) \).
Letting \( \gamma = \omega \), we have \( r(1) \leq 1 \). If \( q = 1 \) and \( r(1) = 0 \), then by (A), \( p < \omega(1) = \sigma(1) = \alpha(1) < \chi(1) \), contrary to (A) and minimality of \( \chi(1) \) for a counterexample since by (A) the kernel of \( \omega \) lies in \( Z \).

(J) Let \( \gamma \) be a nonprincipal irreducible character of \( G \) with \( \gamma|_N = \pi + \rho \) as (1). Let \( \|\pi\| = 1 \). Let \( \gamma|_Z \) and \( \omega|_Z \) be multiples of the same linear character of \( Z \). Let \( s = \sigma(1) \) and \( r = \pi(1) \). Then \( s\gamma - r\omega = (s\pi - r\sigma)^G \), the induced character, and vanishes outside of \( N_0^G = \bigcup_{g \in G} g^{-1}N_0 g \). Also, \( sp(1) = r(1) \). Furthermore, if \( \pi = \sigma \), then \( \gamma = \omega \).

Proof. Suppose that \( \pi \neq \sigma \). By hypothesis, \( s\pi - r\sigma \) vanishes on \( Z \). By (C), \( \pi \) and \( \sigma \) vanish on \( N - (N_0 \cup Z) \) and \( s\pi - r\sigma \) vanishes on \( N - N_0 \). As \( N_0 \) is a T.I. set, by [3, Lemma 38.15], \( \|(s\pi - r\sigma)^G\| = \|s\pi - r\sigma\| = s^2 + r^2 \). By Frobenius reciprocity,

\[ \|(s\pi - r\sigma)^G, s\gamma - r\omega = (s\pi - r\sigma, (s\pi - r\sigma)|_N) = (s\pi - r\sigma, s\pi + sp - r\sigma - r\tau) = s^2 + r^2 \]

Then \( \|(s\pi - r\sigma)^G, s\gamma - r\omega, \|s\gamma - r\omega\| \) all equal \( s^2 + r^2 \),
so by the Cauchy-Schwarz inequality, \( s\gamma - r\omega = (s\pi - r\sigma)^G \) which has support on \( N_0^G \), since \( s\pi - r\sigma \) has support on \( N_0 \). As \( 1 \neq N_0^G \),

\[ 0 = s\gamma(1) - r\omega(1) = s\pi(1) + sp(1) - r\sigma(1) - r\tau(1) = sp(1) - r(1) \]

Suppose that \( \pi = \sigma \) and \( \omega \neq \gamma \). Then \( \pi(1) = \sigma(1) = \alpha(1) \) and by (I), \( \rho(1) \leq 1 \).

Let \( D\sigma \) be the determinant of the corresponding representation. As \( G = G' \), \( \gamma \) and \( \omega \) are unimodular. Then if \( r(1) = \rho(1) = 1 \), we have for \( x \in N \),
As \( r(1) \) and \( \rho(1) \leq 1 \), in any event \( \rho(x)r(x) \) is real and nonnegative. Let \( e = (1/|G|) \sum_{x \in N_0^G} |\gamma(x)|^2 \) and \( f = (1/|G|) \sum_{x \in N_0^G} |\omega(x)|^2 \). Then since \( \pi \) and \( \sigma \) vanish off \( N_0 \cup Z \) by (C), we have

\[
\begin{align*}
(1/|G|) \sum_{x \in N_0^G} \gamma(x)\overline{\omega(x)} &= (1/|G|)(|G|/|N|) \sum_{x \in N_0^G} (\pi(x) + \rho(x))(\sigma(x) + r(x)) \\
&= (1/|N|) \sum_{x \in N_0 \cup Z} (\pi(x)\overline{\sigma(x)} + \pi(x)r(x) + \rho(x)\overline{\sigma(x)} + \rho(x)r(x)) - \gamma(1)\omega(1)/p^2 t \\
&= (\pi, \sigma) + (\pi, r) + (\rho, \sigma) + \sum_{x \in N_0 \cup Z} \rho(x)\overline{r(x)} - \gamma(1)\omega(1)/p^2 t \\
&\geq 1 + 0 + 0 - (t + 1)(4p/3 + 1)/p^2 t > 1/2.
\end{align*}
\]

As

\[
\begin{align*}
(1/|G|) \sum_{x \in G \setminus N_0^G} \gamma(x)\overline{\omega(x)} &= -(1/|G|) \sum_{x \in N_0^G} \gamma(x)\overline{\omega(x)},
\end{align*}
\]

we have by the Cauchy-Schwarz inequality

\[
\frac{1}{2} < \left( \frac{1}{|G|} \sum_{x \in N_0^G} |\gamma(x)||\omega(x)| \right)^{1/2} \leq \left[ \left( \frac{1}{|G|} \sum_{x \in N_0^G} |\gamma(x)|^2 (1/|G|) \sum_{x \in N_0^G} |\omega(x)|^2 \right) \right]^{1/2} = [ef]^{1/2}
\]

and

\[
\frac{1}{2} < \left( \frac{1}{|G|} \sum_{x \in G \setminus N_0^G} |\gamma(x)||\omega(x)| \right)^{1/2} \leq \left[ \left( \frac{1}{|G|} \sum_{x \in G \setminus N_0^G} |\gamma(x)|^2 (1/|G|) \sum_{x \in G \setminus N_0^G} |\omega(x)|^2 \right) \right]^{1/2} = [(1 - e)(1 - f)]^{1/2}.
\]

Multiplying the last two equations together, we have

\[
\frac{1}{4} < [ef]^{1/2}[(1 - e)(1 - f)]^{1/2} = [e(1 - e)]^{1/2}[(1 - f)]^{1/2} \leq [1/4]^{1/2} = 1/4,
\]

which is a contradiction.

(K) We have \( r(1) = q = k = 0 \) and \( \chi^2(x) = \chi(1)\omega(x) \) for all \( x \not\in N_0^G \).

\textbf{Proof.} By (I), \( \sigma(1) = \alpha(1) = n - q \). By (H), we may apply (J) to the \( \gamma_i \) in (6). Letting \( \gamma_i = \pi_i + \rho_i \) as in (1), \( d_i = \pi_i(1) \), and \( D = \Sigma d_i - \Sigma \pi_i(1) = n^2 - k - \Sigma \rho_i(1) = n^2 - (\chi^2_{\Sigma p, 1 p}) \), we have \( \rho_i(1) = d_i r(1)/(n - q) \) and for \( u \not\in N_0^G \),
Summing the last two equations over \( i \), we have
\[
(19) \quad (\chi_p^2, 1_p) - k = \sum \gamma_i^p(1) = \sum d_i r(1)/(n - q) = D r(1)/(n - q)
\]
and, for \( u \not\in N_0^G \),
\[
(20) \quad \chi^2(u) - k = \sum \gamma^p_i(u) = \sum d_i \omega(u)/(n - q) = D \omega(u)/(n - q).
\]

Suppose that \( r(1) = 1 \) and \( N_0 \not\subseteq PZ \). Then for some \( v \in P^\#: |N_0 \cup Z| \geq |C(V)| \geq 2|PZ| \). As \( r(1) = 1 \), \( (1/|N|) \sum_{N_0 \cup Z} |r(x)|^2 = |N_0 \cup Z|/|N| \). By the second to last line of (2), for \( \gamma = \omega \),
\[
(21) \quad \frac{(4p - 1)/3 + 1}{2} \geq (n - q + 1)^2 / 2p^2 \geq \omega(1)^2 / 2p^2 > |N_0 \cup Z| / |N| \geq 2/p.
\]

Then \( p < 11, \ p = 7, \ q = 0, \) and \( n = 9 \). Even then the 2 in \( 2/t \) in (21) cannot be replaced by a larger integer or (21) would fail. As \( N_0 \cup Z \) contains only entire cosets of \( PZ \), \( |N_0 \cup Z| = 2|PZ| = |C(V)| \). As \( N(N_0) = N \), \( C(v) = N_0 \cup Z \) corresponds to a normal subgroup \( E \) of order 2 of \( L \) where the element \( x \) of \( E^\# \) has an eigenvalue 1. As \( E \subseteq Z(L) \), by diagonalizing \( x \), we have \( L \subseteq D \) where \( D \) is the group of all nonsingular diagonal matrices in \( GL(2, p) \). As \( 9 = (n - q) \mid |L| \), \( L \) contains \( |\text{diag}(y, w)|, \ y, w = 1, 2, 4 \), a Sylow 3-subgroup of \( D \). Then \( L \) contains at least five elements with an eigenvalue 1, and \( |N_0 \cup Z| \geq 5|PZ| \), contrary to (21).

Still suppose that \( r(1) = 1 \). Then \( N_0 \subseteq PZ \) and only \( p \)-singular elements lie in \( N_0 \) and \( N_0^G \). There exists some \( y \in N - PZ \). If \( y \in N_0^G \), then \( [y]^p \in P^\# \) and \( y \in C([y]^p) \subseteq N_0 \cup Z \subseteq PZ \), which is a contradiction. Therefore, \( y \in N - (N_0^G \cup Z) \).

By (C), \( \omega(y) = r(y) \) and \( \chi(y) = \beta(y) \). Then by (20) and (G),
\[
2 \geq \beta^2(y) - k = |\chi^2(y) - k| = D |\omega(y)|/(n - q) = D |r(y)|/(n - q) \geq (n^2 - n)/(n - q) \geq n - 1,
\]
which is a contradiction.

Therefore, \( r(1) \neq 1 \). By (I), \( q \leq r(1) \leq 1 \), so \( q = r(1) = 0 \). Then by (19),
\[
((\chi^2_p, 1_p) = k \leq 1 \). By (G), \( 0 = q = (\chi^2_p, 1_p) = k \). Then for \( u \not\in N_0^G \), by (20)
\[
\chi^2(u) = D \omega(u)/(n - q) = n^2 \omega(u)/n = \chi(1) \omega(u).
\]

Proof. The group \( G = G/\Omega_1^1(\Omega_2^1(Z)) \) has the faithful representation \( Y \) corresponding to \( \omega \) since by (A) the kernel of \( \omega \) lies in \( Z \). By (A), \( Y(G) \) is a counterexample to the theorem. As \( \omega(1) = \chi(1) \) by (K), by minimality of \( |G|, |\Omega_1^1(Z)|, 2 \) = 1. Let \( \zeta \) be a primitive \( |G| \)th root of unity, \( Q \) be the rationals, and \( K = Q(\zeta) \). Let \( \mu \) be the automorphism of \( K \) taking \( \{\zeta\}_2 \), and \( \{\zeta\}_2 \) (that is, the 2-part of \( \zeta \) where \( \zeta = \{\zeta\}_2, \{\zeta\}_2 \) to \( \{\zeta\}_2, \{\zeta\}_2 \), respectively. Then \( \omega \mid Z = \chi \mid Z \) and by (J), for \( u \not\in N_0^G \), \( \omega(u) = \chi^\mu(u) \). Then by (K), for \( u \not\in N_0^G \),
Let $u \not\in N_0^G$ and $\chi(u) \neq 0$, and let $\nu$ be any automorphism of $K$. There exists an integer $e$ such that $\zeta^\nu = \zeta^e$ and $(e, |G|) = 1$. Then $u^e \not\in N_0^G$, otherwise, $u^e \not\in Z$, $(u) = (u^e)$ centralizes a $p$-singular element and $u \in N_0^G$. Then by (22),

$$(23) \chi^\nu(u)^2 = \chi(u^e)^2 = \chi(1)\chi^\mu(u) = \chi(1)\chi^\mu(\nu).$$

As we let $\nu$ run over all automorphisms of $K$, $\nu \mu$ runs over all automorphisms of $K$. Taking the product of (23) as $\nu$ ranges over all automorphisms of $K$,

$$(24) \prod_{\nu \in \text{aut } K} \chi^\nu(u) = \chi(1)^{|\text{aut } K|}.$$ 

As $|\chi^\nu(u)| \leq \chi(1)$ for all $\nu \in \text{aut } K$, by (24) this is always equality. Then $|\chi(u)| = \chi(1)$ and $u \in Z$.

We are now able to complete the proof of the theorem. By (K), $q = 0$, and $\chi_N = \alpha$, so by (L),

$$1 = \|\chi\|^2 = (1/|G|) \sum_{x \in G} |\chi(x)|^2$$

$$= (1/|G|) \left[ \sum_{x \in Z} |\chi(x)|^2 + \sum_{x \in N_0^G} |\chi(\bar{x})|^2 \right] = (|Z|/|G|)n^2 + (1/|G|)(|G|/|N|) \sum_{x \in N_0} |\chi(x)|^2$$

$$< (|Z|/|N|)n^2 + (1/|N|) \sum_{x \in N_0} |\chi(x)|^2 = (1/|N|) \sum_{x \in Z \cup N_0} |\chi(x)|^2 \leq \|\alpha\|^2 = 1,$$

which is a contradiction.

4. The case $p \equiv -1 \pmod{4}$. The following theorem combines Theorem 2 and [7, Theorem 4].

**Theorem 3.** Let $p$ be prime greater than 5 with $p \equiv -1 \pmod{4}$. Let $G$ be a finite group with a faithful, quasiprimitive, complex representation $X$ with character $\chi$ of dimension $n < 4p/3$, $n \neq p$, and if $p = 7$, $n \leq 8$. Then $p^2 \not| |G/Z(G)|$.

The hypothesis $p > 5$ of Theorem 3 is unnecessary by the classification of 2-dimensional groups in the case $p = 3$. Given Theorem 2 it is easier to prove the combination Theorem 3 and [7, Theorems 2, 3, and 4] than it is to prove [7, Theorems 2, 3, and 4]. Since the stronger induction hypothesis allowed in proving the former combination eliminates some of the cases that had to be studied in the proof of [7, Theorems 2, 3, and 4]. We now prove Theorem 3.

**Proof.** We may replace elements $x$ of $X(G)$ by all unimodular scalar multiples...
of x. This does not affect quasiprimivity or change $G/Z(G)$ within isomorphism. Therefore, we may assume that $X(G)$ is unimodular. Let $P$ be a Sylow $p$-subgroup of $G$. Then $P$ is abelian, otherwise some element in $(P' \cap Z(P))^\#$ contradicts [1, Theorem 8, p. 96] and quasiprimivity. Then as $(n, p) = 1$, $(|Z(G)|, p) = 1$.

We may assume that $X(G)$ is a counterexample to our theorem, so $|P| > p$. Then by [7, Theorem 4], $P$ is a T.I. set and there exists a subgroup $E \subseteq G$ with $F = C(E)$ (possibly equal to $G$) having the following properties: $P \subseteq F$ and $X|F$ has a strongly primitive constituent $Y$ of degree larger than $p$. Also, $Y$ is faithful on $P$ and $Y(P) \subseteq Y(F)$ satisfy [4, Hypothesis 4.1]. Furthermore, all homogeneous subspaces of $Y(P)$ are linear and $C_{Y(F)}(Y(P)) = Y(P)Z(Y(F))$. Also, $(|Z(Y(F))|, p) = 1$. Finally, $N_{Y(F)}(Y(P))$ acts transitively on the nonprincipal homogeneous subspaces of $Y(P)$.

By the above, $Y$ has degree unequal to $p$. $Y$ is quasiprimitive, $|Y(P)| = |P| > p$, $(|Z(Y(F))|, p) = 1$, and $Y(P) \subseteq Y(F)$ satisfy [4, Hypothesis 4.1]. Therefore, $Y(F)$ contradicts Theorem 2. This completes the proof.

We now prove Theorem 1 from the abstract. Let $G$ be a linear group corresponding to $\overline{G}$ satisfying the hypothesis of Theorem 1. By [7, Theorem 1, (b)], $\overline{P}$ is a trivial intersection set. By Theorem 3, $p \equiv 1 \pmod{4}$. By replacing elements $x$ of $G$ by all unimodular scalar multiples, we may assume that $G$ is unimodular. As $p \nmid n$, $(|Z(G)|, p) = 1$. Suppose that for all $x \in G$, $C(x)$ has a normal Sylow $p$-subgroup. As in the proof of Theorem 3, $P$ is abelian. Suppose that $N_0$ is defined as in Theorem 2, and for some $y, g \in G$, $y \in N_0 \cap N_0^g$. Let $H$ be the preimage of $C(y)$. Then $C(y) \subseteq H$ and $C(y)$ and $H$ have normal Sylow $p$-subgroups, so $C(y)$ has a normal Sylow $p$-subgroup $Q$ which we may assume is contained in a Sylow $p$-subgroup $P^b$ of $G$. Now, $y$ centralizes some nonidentity elements $u$ and $v$ of $P$ and $P^g$, respectively. As $u, v \in Q \subseteq P^b$ and $P$ is a T.I. set, $P = P^b = P^g$ and $P$ satisfies [4, Hypothesis 4.1]. Then by Theorem 2, $|P| < p$, which is a contradiction.

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