AN APPROXIMATION THEOREM
FOR BIHOLOMORPHIC FUNCTIONS ON $D^n$

BY
JOSEPH A. CIMA

ABSTRACT. Let $F$ be a biholomorphic mapping of the polydisk $D^n$ into $C^n$. We construct a sequence of polynomial mappings $\{P_j\}$ such that each $P_j$ is subordinate to $P_{j+1}$, each $P_j$ is subordinate to $F$ and the $P_j$ converge uniformly on compacta to $F$. The polynomials $P_j$ are biholomorphic.

Introduction. Let $D$ be the disk in the complex plane $C$ with center at the origin and radius $r > 0$ ($D_1 = D$). MacGregor [1] has shown that if $f$ is a schlicht mapping of $D$ into $C$ then there exists a sequence of schlicht polynomials $\{P_j\}_{j=1}^\infty$, each $P_j$ has degree $j$, such that $P_j$ converges uniformly to $f$ on compacta and such that $P_j$ is subordinate to $P_{j+1}$ for each $j = 1, 2, \ldots$. A second result is that if $f$ is a convex schlicht mapping of $D$ into $C$, then the $\{P_j\}$ in the above result can be chosen to be convex schlicht polynomials. A close scrutiny of the proofs of these results show that they depend principally upon the facts that $C$ is a normed linear space and the fact that $f$ is a homeomorphism. We extend these results to the following case. Let $D^n$ be the $n$-fold product of $D$ and assume $F$ is a biholomorphic mapping of $D^n$ into $C^n$, $F(0) = 0$. Then there exists a sequence of polynomial mappings $\{P_j\}$, which are biholomorphic, and which converge uniformly to $F$ on compacta. Further, each $P_j$ is subordinate to $P_{j+1}$ for $j = 1, 2, 3, \ldots$.

Using a result of T. J. Suffridge we can also show that if $F(D^n)$ is convex in $C^n$ then each $P_j$ can be chosen so that $P_j(D^n)$ is convex.

Notation and definition. Let $D_r$ denote the disk in the complex plane with center at the origin and radius $r > 0$, $D^n_r$ is the $n$-fold product of such disks and $\overline{D}_r$ is the closure of such a disk. If $r = 1$ we omit the subscript. A point $Z$ in $C^n$ will be written as $Z = (z_1, \ldots, z_n)$, $z_j \in C$, and a mapping $F$ from $D^n$ into $C^n$ as $F(Z) = (f_1(Z), \ldots, f_n(Z)) = W$. If $f_j(Z)$ is holomorphic on $D^n$, then $f_j(Z) = \sum_{k=0}^\infty b_{jk}(Z)$ which $b_{jk}$ are homogeneous polynomials of degree $k$. For $N$ a positive integer we let $f_{j,N}(Z) = \sum_{k=0}^N b_{jk}(Z)$ and $F_{N}(Z) = (f_{1,N}(Z), f_{2,N}(Z), \ldots, f_{n,N}(Z))$. Whenever a sequence of mappings $F_N$ converge uniformly on compacta to a mapping $F$ we will write $F_N(Z) \to F(Z)$. A mapping $F$ (from $D^n$ into $C^n$)
where each \( f_j(Z) \) is a polynomial will be called a polynomial mapping, and the degree of \( F \) is \( N(j) = \max \deg f_j \) \((1 \leq j \leq n)\). The norm of \( Z \) is written as \( |Z| = \max_{1 \leq j \leq n} |Z_j| \). The boundary of a set \( B \) is denoted by \( \partial B \) and the distance from \( A \) to \( B \) is \( d(A, B) \). Finally, if \( 0 < r < 1 \) and \( F \) is a mapping of \( D^n \) into \( \mathbb{C}^n \) then \( \mathcal{R} F(r) \) denotes the image of the polydisk \( D^n \) under the mapping \( F \). Given two biholomorphic mappings \( F \) and \( G \) with \( F(0) = G(0) \) then \( F \) is subordinate to \( G \) if \( \mathcal{R} F \subset \mathcal{R} G \) and this is written \( F < G \).

**Two lemmas.** We shall need two lemmas. The proofs follow from standard properties of analytic functions and straightforward computation.

**Lemma 1.** Let \( f \) be a holomorphic mapping of \( D^n \) into \( \mathbb{C} \), \( f(Z) = \sum_{k=0}^{\infty} P_k(Z) \), where the \( P_k \) are homogeneous polynomials. The partial sums of \( f \) are \( f_N = \sum_{k=0}^{N} P_k(Z) \). Let \( \{C_j\} \) be a strictly increasing sequence of positive integers and \( 0 < r_j < 1 \). Then \( f_{C_j}(r_j Z) \) tends uniformly to \( f \) on compacta, written \( f_{C_j}(r_j Z) \to f(Z) \).

Before stating Lemma 2 we need a definition. Assume \( F \) is a holomorphic mapping on \( D^n \) into \( \mathbb{C}^n \). Let \( Z \) and \( W \) be in \( \mathbb{C}^n \). Define the matrix \( A_F(Z, W) = (a_{ij}(Z, W)) \) as follows:

\[
a_{ij}(Z, W) = \left( \frac{\partial f_i}{\partial z_j} \right)(z_1, \ldots, z_j, W_{j+1}, \ldots, W_n) \quad \text{if } z_j = W_j
\]

\[
= \frac{[f_i(z_1, \ldots, z_{j-1}, W_{j+1}, \ldots, W_n) - f_i(z_1, \ldots, z_{j-1}, W_j, \ldots, W_n)]}{[z_j - W_j]} \quad \text{if } z_j \neq W_j,
\]

and define the nonnegative function

\[
\phi_F(Z, W) = |\det(a_{ij}(Z, W))| + \sum_{j=1}^{n} |f_j(Z) - f_j(W)|.
\]

The Jacobian of \( F \) is written \( J_F(Z) \).

**Lemma 2.** Assume \( F \) is a holomorphic mapping of \( D^n \) into \( \mathbb{C}^n \), \( F(Z) = (f_1(Z), \ldots, f_n(Z)) \) and assume that \( J_F \) is nonsingular for \( Z \in D^n \). Then \( \phi_F(Z, W) = 0 \) if and only if \( F(Z) = F(W) \), \( Z \neq W \).

**The subordination theorems in \( \mathbb{C}^n \).** We will use the following lemmas to prove our results.

**Lemma 3.** Let \( F \) be a biholomorphic mapping of \( D^n \) into \( \mathbb{C}^n \), \( F(0) = 0 \) and let \( 0 < r_0 < 1 \) be given. There exists an integer \( N_1(r_0) \) such that \( F_N \) is a biholomorphic mapping on \( D^n_{r_0} \) if only \( N \geq N_1(r_0) \).

**Proof.** For any \( r < 1 \) define the nonnegative function
Let $f_i(Z)$ have an expansion in terms of its homogeneous polynomials $f_i(Z) = \sum_{k=1}^{\infty} A_i^k(Z)$ and then

$$F_N(Z) = (f_{1,N}(Z), \ldots, f_{n,N}(Z)) = \left( \sum_{k=1}^{N} A_1^k(z), \ldots, \sum_{k=1}^{N} A_n^k(z) \right)$$

We claim that the entries of the matrix for $A_{F_N}^r(Z, W)$ tend uniformly to the entries of the matrix $A_{F}^r(Z, W)$ if $|Z| < r$ and $|W| < r$. For the terms of the form $\partial f_i^N/\partial z_j$, this is a consequence of the Cauchy integral formula (and the convergence is uniform on compacta). Also if $|z_j - W_j| > \eta > 0$ it is clear that the terms

$$[f_i^N(z_1, \ldots, z_j, W_j+1, \ldots, W_n) - f_i^N(z_1, \ldots, z_j-1, W_j, \ldots)]/[z_j - W_j]$$

can be made uniformly close to the corresponding term in the $A_{F}^r$ matrix. For $r$ fixed and $\rho = (r + 1)/2$ we have a positive constant $P$ such that $|F_N(Z)| \leq P$ for all $|Z| \leq \rho$ and all $N = 1, 2, 3, \ldots$. An application of the Cauchy integral theorem shows that if $0 < |z_j - W_j| \leq \eta = (1 - r)/4$ we have another constant $M$ (depending on $r$ but not $N$) such that

$$|f_i^N(z_1, \ldots, z_j, W_j+1, \ldots, W_n) - f_i^N(z_1, \ldots, z_j-1, W_j, \ldots)| \leq |z_j - W_j| \cdot M.$$

Hence, an application of the triangle inequality will yield that for $(i, j)$ fixed there is an $N$ such that the $(i, j)$ entry of $A_{F_N}^r(Z, W)$ is within $\epsilon$ of the $(i, j)$ entry of the $A_{F}^r(Z, W)$ matrix on $|Z| \leq r$, $|W| \leq r$. This means for $r_0 < 1$ and $0 < \epsilon < m_F(r_0)/2$ we can find an integer $N_0$ so that

$$|m_{F_N}^r(r) - m_F(r)| < \epsilon$$

for $0 \leq r \leq r_0$. Also the entries in the Jacobian matrix for the $F_N$ converge uniformly on $|Z| \leq r_0$ to the entries in the Jacobian matrix for $F$. Applying Lemma 2 to the $F_N$ mappings we can find an $N_1(r_0)$ so that the result of Lemma 3 will hold on $|Z| \leq r_0$ for $N \geq N_1(r_0)$.

**Lemma 4.** Let $r_0 < 1$ and assuming that $F$ is as in Lemma 3 define $d(r_0) = d(\partial F(r_0), \partial F(1))$. Then $0 < d(r_0) < \infty$ and there is an $N_2(r_0)$ such that $d(\partial F_N(r_0), \partial F(1)) > 0$ whenever $N \geq N_2(r_0)$.

**Proof.** $F$ is assumed to be a homeomorphism, hence $d(r_0) > 0$. If $d(r_0) = \infty$ this would imply that $\partial F(1) = C^n$. Let $G(W) = F^{-1}(W)$ be the inverse mapping to
From $C^n$ into $D^n$. Using one variable theorems one shows that $g_i(W_1, 0, \ldots, 0) = g_i(0, \ldots, 0) = 0$. For $W_1$ fixed (say $W_1 = W_0$) we have

$$g_i(W_0^1, W_2, 0, \ldots, 0) = g_i(W_1^0, 0, \ldots, 0) = g_i(0, \ldots, 0) = 0.$$  

This will show each $g_i(Z) = 0$ and this implies $F$ is not biholomorphic. Since $F_N \Rightarrow F$ we can find $N_2 = N_2(r_0)$ so that if $N \geq N_2$ then $|F_N(Z) - F(Z)| < d(r_0)/2$ on $|Z| \leq r_0$. This yields the result of our lemma.

Having established the above lemmas we can proceed to the principal theorem.

Theorem 1. Assume $F$ is a biholomorphic mapping of $D^n$ into $C^n$ ($F(0) = 0$), $F(Z) = (f_1(Z), \ldots, f_n(Z))$. Then there exist polynomial mappings $P_k(Z)$ which are biholomorphic (on $D^n$) and of degree $k$ ($k = 1, 2, 3, \ldots$) such that $P_1 < P_2 < P_3 \ldots$ ($P_k < F$) and $P_k(Z) \Rightarrow F(Z)$.

Proof. We assume first that some component of $F$ (i.e. some $f_j$) is not a polynomial. Let then $0 < r_0 < 1$ be given. For this $r_0$ we select an $N_1(r_0)$ as in Lemma 3 and an $N_2(r_0)$ as in Lemma 4 and we set $N(r_0) = N_1(r_0) + N_2(r_0)$. Choose an $N_1 > N(r_0)$ so that $F_{N_1}$ has degree $N_1$. Let $E_N(r)$ be the preimage of $RF_N(r)$ under the mapping $F$, $E_N(r) = F^{-1}(RF_N(r))$. By the openness of the mapping $F$ we can find a number $1 > r_1 > r_0$ so that $E_N(r_1)$ is a common point and (1) implies $E_{N_1}(r_1) \subseteq D^n$. We continue now by setting $1 > r_1 > (1 + \rho_1)/2$. We can select an $N(r_1)$ as in Lemma 3. Let $a$ be the number which is the minimum of $d(r_1)$ and $d(\partial RF_{N_1}(r_0), \partial RF_{N_1}(r_1))$. Again uniform convergence of $F_N$ to $F$ on compacta allows us to find $N_2(r_1)$ so that $|F(Z) - F_N(Z)| < a/2$ whenever $N \geq N_2(r_1)$ and $|Z| \leq r_1$. These last comments imply that

$$d(\partial RF_{N_1}(r_0), \partial RF_{N_1}(r_1)) > 0$$

if $N \geq N_2(r_1)$. We have arcwise connected sets $\partial RF_{N_1}(r_0)$ and $\partial RF_{N_2}(r_1)$ with a common point and (1) implies $\partial RF_{N_1}(r_1) \supseteq \partial RF_{N_2}(r_0)$ if $N \geq N_2(r_1)$. We can set $N = N_1(r_1) + N_2(r_1) + N_1$ and choose $N_2 \geq N$ so that $F_{N_2}$ has degree $N_2$. We define the polynomial mappings

$$P_{N_1}(Z) = F_{N_1}(r_0 Z), \quad P_{N_2}(Z) = F_{N_2}(r_1 Z).$$

$P_{N_1}$ and $P_{N_2}$ are biholomorphic on $D^n$ and are polynomial maps of degree $N_1$ and $N_2$ respectively. They are also subordinate. One can now proceed to construct polynomial mappings $P_n(Z)$ ($N_1 < N_2 < \ldots$) which are biholomorphic on $D^n$ and satisfy $P_{N_1} < P_{N_2} < P_{N_3} \ldots$. An application of Lemma 1 shows that $P_{N_j}(Z) \Rightarrow F(Z)$.

The remaining part of this proof consists of filling in the polynomial mappings of appropriate degrees between the $P_{n_k}$ and $P_{n_{k+1}}$. Assume then $k < j$ and that $P$ and $Q$ are polynomial mappings of degrees $k$ and $j$ respectively. Further we have
P and Q biholomorphic on open sets containing $\overline{D^n}$ and $P < Q$ on $D^n$. There is an $\mathfrak{R} > 1$ so that $P$ is a biholomorphic polynomial mapping on $D^n_\mathfrak{R}$. Choose $1 < \rho < \mathfrak{R}$ and define $P^*$ on $D^n$ as $P^*(Z) = P(\rho Z)$. Note that $\rho$ can be chosen so that $d(\mathfrak{R} P(1), \partial \mathfrak{R} P^*(1)) < 0$ and $d(\mathfrak{R} P^*(1), \partial \mathfrak{R} Q) > 0$, and $P^*$ is a biholomorphic polynomial mapping on $D^n$ of degree $k$. Let $(b) = (b_1, \ldots, b_n) \in \mathbb{C}^n$ and define

$$Q^*(Z) = P^*(Z) + (b_1 z_1^{k+1}, b_2 z_2^{k+1}, \ldots, b_n z_n^{k+1}).$$

It is clear that $Q^*$ can be made uniformly close to $P^*$ on $D^n$ if only $|b|$ is small. Hence $|\det J_{Q^*}(Z)| \neq 0$ on $D^n$ and since $\phi_{Q^*}$ can be made arbitrarily close to $\phi_{P^*}$ by a suitable choice of $b$ we conclude from Lemma 2 that there is an $\eta > 0$ such that if $|b| < \eta$ then $Q^*$ is a biholomorphic mapping of degree $k + 1$. Since $d(\mathfrak{R} P^*, \partial \mathfrak{R} Q^*)$ can be made small with $|b|$ and since $P < P^* < Q$ we have for $0 < \eta$ sufficiently small that $P < Q^* < Q$.

The pair $Q^*, Q$ are now biholomorphic polynomial mappings on an open set containing $\overline{D^n}$ of degrees $k + 1$ and $j$ respectively, and so we can find such polynomial mappings for $k + 1, \ldots, k + j - (k + 1)$. We have now the chain of polynomial, biholomorphic mappings $P_j$ of degree $j$ which satisfy the subordination relation $P_1 < P_2 < P_3 < \ldots, P_n < F$, and such that $P_n \Rightarrow F$. Assume then that $P_{n_j}$ and $P_{n_j+1}$ are successive members of the sequence $\{P_{n_j}\}$ and the degree of $P_{n_j}$ is $m$ and the degree of $P_{n_j+1}$ is $k$. We have chosen numbers $1 < \nu$ and $b^\nu \in \mathbb{C}_n$, $\nu = m + 1, m + 2, \ldots, k - 1$, and successively defined polynomials $P_{\nu+1}(Z) = P_{\nu}(P_{\nu} Z) + (b_1^{\nu+1} z_1^{\nu+1}, b_2^{\nu+1} z_2^{\nu+1}, \ldots, b_n^{\nu+1} z_n^{\nu+1})$. Since $P_{n_j}(Z) = P_m(Z)$ is biholomorphic on $D^n$ we can choose the $1 < \nu$ so close to one that $P_m(P_m P_{m+1} \ldots, P_{\nu-1} Z)$ is arbitrarily close to $P_m(Z)$ on all of $D^n$. The recursive definition of the $P_{\nu}$ will yield the estimate

$$|P_{\nu}(Z) - P_m(b_m^{\nu+1} \ldots b_{\nu-1}^{\nu+1})|$$

$$= \left| \sum_{j=1}^{\nu-m} \left( b_1^{\nu-m+j} (b_{\nu-m+1} \ldots b_{\nu-m-2} \ldots b_{\nu-m-(j-1)})^{m+j} z_1^{m+j} z_2^{m+j} \ldots b_n^{\nu+1} z_n^{\nu+1} \right) \right|$$

$$\leq \left| (\rho_{m+1} \ldots \rho_{k-1}) \right| \sum_{j=1}^{k-m} |b_m^{j}|.$$

Now if we are given $\epsilon > 0$ we choose the $\{|\rho|^j\}$ so close to one that $|P_m(Z) - P_m(b_m^{\nu+1} - \rho_{\nu-1} Z)| < \epsilon/2$ for $m = 1, \ldots, k - 1$ and so that if $|b_m^{\nu+1}| < \eta$ for $j = 1, 2, \ldots, k - (m - 1)$ then

$$|P_{\nu}(Z) - P_m(b_m^{\nu+1} \ldots b_{\nu-1}^{\nu+1})| < \epsilon/2.$$

Hence, $|P_{\nu}(Z) - P_m(Z)| < \epsilon$ for $z \in D^n$ and $\nu = m + 1, \ldots, k - 1$. This proves that
one can choose the full sequence \( \{ P_n(Z) \} \) so that it satisfies the subordination chain relation and so that \( P_n(Z) \Rightarrow F(Z) \).

It remains to consider only the case where \( F \) is a biholomorphic polynomial mapping of \( D^n \) of say degree \( N \). Choose \( 0 < r_j \neq 1 \) and define \( F_j(Z) = F(r_j Z) \) for \( j = 1, 2, 3, \ldots \). We can choose \( b^1 = (b^1_1, \ldots, b^1_n) \in \mathbb{C}^n \) of small norm so that

\[
(b^1_1 z_1, \ldots, b^1_n z_n) = P^1_1(Z) < F_1(Z).
\]

Now the first part of the proof allows us to assert the existence of biholomorphic polynomials \( P_1, \ldots, P_N \) satisfying \( P_1 < P_2 < \cdots < P_N = F_1 \), where each \( P_j \) has degree \( j \) and the degree of \( F_1 \) is \( N \). Again choose a \( b^2 \in \mathbb{C}^n \) with small modulus so that

\[
(b^2_1 z_1^{N+1}, \ldots, b^2_n z_1^{N+1}) = P^1_{N+1}(Z).
\]

is univalent in \( D^n \) and so that the image of \( D^n \) under the mapping \( P_N \) is properly contained in the image of \( D^n \) under the mapping \( P^1_{N+1} \) which in turn is properly contained in the image of \( D^n \) under the mapping \( F_1 \). The proof is now finished.

**Theorem 2.** Let \( F(Z) \) be a biholomorphic mapping \( (F(0) = 0) \) of \( D^n \) into \( \mathbb{C}^n \) such that the image of \( D^n \) under \( F \) is a convex domain. Then there exists a sequence of biholomorphic polynomial mappings \( P, \ldots, P_N \) of degree \( k \), from \( D^n \) into \( \mathbb{C}^n \) with convex range and such that \( P^1 < P^2 < \cdots < P_N = F \), where each \( P^j \) has degree \( j \) and the degree of \( F \) is \( N \). Again choose a \( b^2 \in \mathbb{C}^n \) with small modulus so that

\[
(b^2_1 z_1^{N+1}, \ldots, b^2_n z_1^{N+1}) = P^1_{N+1}(Z) < P^2_{N+1}(Z).
\]

The second result is that of T. J. Suffridge [2] which states that if \( F \) satisfies the hypothesis of Theorem 2 then there exist convex univalent mappings \( g_1: D \rightarrow \mathbb{C}^1 \) and a nonsingular linear transformation \( T \) of \( \mathbb{C}^n \) into \( \mathbb{C}^n \) such that \( F \) has a decomposition

\[
F(Z) = T \circ G(Z) = T(g_1(z_1), g_2(z_2), \ldots, g_n(z_n)).
\]

The proof is a sandwiching together of two known results. The first is that of T. J. Suffridge [2] which states that if \( F \) satisfies the hypothesis of Theorem 2 then there exist convex univalent mappings \( g_1, \ldots, g_n: D \rightarrow \mathbb{C}^1 \) and a nonsingular linear transformation \( T \) of \( \mathbb{C}^n \) into \( \mathbb{C}^n \) such that \( F \) has a decomposition

\[
F(Z) = T \circ G(Z) = T(g_1(z_1), g_2(z_2), \ldots, g_n(z_n)).
\]

The second result is that of T. H. MacGregor [1] which states that there exist convex univalent polynomials \( P^j(z_j) \) so that \( P^1_j(z_j) < P^k_{j+1}(z_j) \) for \( k = 1, 2, 3, \ldots \), and such that \( P^j_n(z_j) \Rightarrow g_j(z_j) \) on \( D \). The degree of \( P^j_k \) is \( k \). Defining the functions

\[
T_k(Z) = T(P^j_{k,1}(z_1), \ldots, P^j_{k,n}(z_n)) = T \circ P^j_k(Z)
\]

we see that \( T_k \) is a polynomial mapping of degree \( k \) and that \( T_k \) is biholomorphic. If \( T_k(Z) = T \circ P^j_k(Z) = T(W) = X \), then \( W = P^j_k(Z) = (P^j_{k,1}(z_1), \ldots, P^j_{k,n}(z_n)) \). But \( P^j_{k,i} < P^k_{j+1,i} \) implies the existence of a \( \mathbb{Z} \) such that \( P^k_{j+1,i} < P^j_{k,i} \). This implies that \( T^k_{j+1}(Z) = T_k(Z) \) and so \( T^k < T^k_{j+1} \) for all \( k = 1, 2, \ldots \). If now \( Z \in D^n \) we have

\[
|F(Z) - T_k(Z)| = |T \circ G(Z) - T \circ P^j_k(Z)| \leq \|T\||G(Z) - P^j_k(Z)|
\]

and so \( T_k(Z) \Rightarrow F(Z) \).
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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KENTUCKY, LEXINGTON, KENTUCKY 40506

Current address: Department of Mathematics, University of North Carolina, Chapel Hill, North Carolina 27514