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AN APPROXIMATION THEOREM FOR BIHOLOMORPHIC FUNCTIONS ON D^n

BY

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ABSTRACT. Let F be a biholomorphic mapping of the polydisk D^n into $\mathbf{C}^n$. We construct a sequence of polynomial mappings $\{P_j\}$ such that each P_j is subordinate to P_{j+1}, each P_j is subordinate to F and the P_j converge uniformly on compacta to F. The polynomials P_j are biholomorphic.

Introduction. Let D_r be the disk in the complex plane $\mathbf{C}$ with center at the origin and radius $r > 0$ ($D_1 \equiv D$). MacGregor [1] has shown that if f is a schlicht mapping of D into $\mathbf{C}$ then there exists a sequence of schlicht polynomials $\{P_j\}_{j=1}^{\infty}$, each P_j has degree j, such that P_j converges uniformly to f on compacta and such that P_j is subordinate to P_{j+1} for each $j = 1, 2, \cdots$. A second result is that if f is a convex schlicht mapping of D into $\mathbf{C}$, then the $\{P_j\}$ in the above result can be chosen to be convex schlicht polynomials. A close scrutiny of the proofs of these results show that they depend principally upon the facts that $\mathbf{C}$ is a normed linear space and the fact that f is a homeomorphism. We extend these results to the following case. Let D^n be the n fold product of D and assume F is a biholomorphic mapping of D^n into $\mathbf{C}^n$, $F(0) = 0$. Then there exists a sequence of polynomial mappings $\{P_j\}$, which are biholomorphic, and which converge uniformly to F on compacta. Further, each P_j is subordinate to P_{j+1} for $j = 1, 2, 3, \cdots$. Using a result of T. J. Suffridge we can also show that if $F(D^n)$ is convex in $\mathbf{C}^n$ then each P_j can be chosen so that $P_j(D^n)$ is convex.

Notation and definition. Let D_r denote the disk in the complex plane with center at the origin and radius $r > 0$, D_r^n is the n-fold product of such disks and $\overline{D}_r$ is the closure of such a disk. If $r = 1$ we omit the subscript. A point Z in $\mathbf{C}^n$ will be written as $Z = (z_1, \cdots, z_n)$, $z_j \in \mathbf{C}$, and a mapping F from D^n into $\mathbf{C}^n$ as $F(Z) = (f_1(Z), \cdots, f_n(Z)) = W$. If $f_j(Z)$ is holomorphic on D^n, then $f_j(Z) = \sum_{k=0}^{\infty} b_{jk}(Z)$ which $b_{j,k}$ are homogeneous polynomials of degree k. For N a positive integer we let $f_{j,N}(Z) = \sum_{k=0}^{N} b_{jk}(Z)$ and $F_N(Z) = (f_{1,N}(Z), f_{2,N}(Z), \cdots, f_{n,N}(Z))$. Whenever a sequence of mappings F_N converge uniformly on compacta to a mapping F we will write $F_N(Z) \Rightarrow F(Z)$. A mapping F (from D^n into $\mathbf{C}^n$)

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where each $f_j(Z)$ is a polynomial will be called a polynomial mapping, and the degree of F is $N(j) \equiv$ max degrees f_j $(1 \le j \le n)$. The norm of Z is written as $|Z| \equiv \max_{1 \le j \le n} |Z_j|$. The boundary of a set B is denoted by ∂B and the distance from A to B is $d(A, B)$. Finally, if $0 < r < 1$ and F is a mapping of D^n into $\mathbf{C}^n$ then $\Re F(r)$ denotes the image of the polydisk D_r^n under the mapping F. Given two biholomorphic mappings F and G with $F(0) = G(0)$ then F is subordinate to G if $\Re F \subset \Re G$ and this is written $F < G$.

Two lemmas. We shall need two lemmas. The proofs follow from standard properties of analytic functions and straightforward computation.

Lemma 1. *Let f be a holomorphic mapping of D^n into $\mathbf{C}$, $f(Z) = \sum_{k=0}^{\infty} P_k(Z)$, where the P_k are homogeneous polynomials. The partial sums of f are $f_N \equiv \sum_{k=0}^{N} P_k(Z)$. Let $\{C_j\}$ be a strictly increasing sequence of positive integers and $0 < r_j \nearrow 1$. Then $f_{C_j}(r_jZ)$ tends uniformly to f on compacta, written $f_{C_j}(r_jZ) \Rightarrow f(Z)$.*

Before stating Lemma 2 we need a definition. Assume F is a holomorphic mapping on D^n into $\mathbf{C}^n$. Let Z and W be in $\mathbf{C}^n$. Define the matrix $A_F(Z, W) = (a_{ij}(Z, W))$ as follows:

$$a_{ij}(Z, W) = (\partial f_i/\partial z_j)(z_1, \cdots, z_j, W_{j+1}, \cdots, W_n) \quad \text{if } z_j = W_j$$

$$= \frac{[f_i(z_1, \cdots, z_j, W_{j+1}, \cdots, W_n) - f_i(z_1, \cdots, z_{j-1}, W_j, \cdots, W_n)]}{[z_j - W_j]} \quad \text{if } z_j \neq W_j,$$

and define the nonnegative function

$$\phi_F(Z, W) = |\det(a_{i,j}(Z, W))| + \sum_{j=1}^{n} |f_j(Z) - f_j(W)|.$$

The Jacobian of F is written $J_F(Z)$.

Lemma 2. *Assume F is a holomorphic mapping of D^n into $\mathbf{C}^n$, $F(Z) = (f_1(Z), \cdots, f_n(Z))$ and assume that J_F is nonsingular for $Z \in D^n$. Then $\phi_F(Z, W) = 0$ if and only if $F(Z) = F(W)$, $Z \neq W$.*

The subordination theorems in $\mathbf{C}^n$. We will use the following lemmas to prove our results.

Lemma 3. *Let F be a biholomorphic mapping of D^n into $\mathbf{C}^n$, $F(0) = 0$ and let $0 < r_0 < 1$ be given. There exists an integer $N_1(r_0)$ such that F_N is a biholomorphic mapping on $\overline{D_{r_0}^n}$ if only $N \ge N_1(r_0)$.*

Proof. For any $r < 1$ define the nonnegative function

$$m_F(r) = \inf \Phi_F(Z, W), \qquad |Z| \le r, \ |W| \le r.$$

Let $f_i(Z)$ have an expansion in terms of its homogeneous polynomials $f_i(Z) = \sum_{k=1}^{\infty} A_k^i(Z)$ and then

$$F_N(Z) = (f_{1,N}(Z), \cdots, f_{n,N}(Z)) = \left(\sum_{k=1}^{N} A_k^1(Z), \cdots, \sum_{k=1}^{N} A_k^n(z) \right)$$

We claim that the entries of the matrix for $A_{F_N}(Z, W)$ tend uniformly to the entries of the matrix $A_F(Z, W)$ if $|Z| \le r$ and $|W| \le r$. For the terms of the form $\partial f_{i,N}/\partial z_j$ this is a consequence of the Cauchy integral formula (and the convergence is uniform on compacta). Also if $|z_j - W_j| \ge \eta > 0$ it is clear that the terms

$$[f_{i,N}(z_1, \cdots, z_j, W_{j+1}, \cdots, W_n) - f_{i,N}(z_1, \cdots, z_{j-1}, W_j \cdots)]/[z_j - W_j]$$

can be made uniformly close to the corresponding term in the A_F matrix. For r fixed and $\rho = (r+1)/2$ we have a positive constant P such that $|F_N(Z)| \le P$ for all $|Z| \le \rho$ and all $N = 1, 2, 3, \cdots$. An application of the Cauchy integral theorem shows that if $0 < |z_j - W_j| \le \eta = (1-r)/4$ we have another constant M (depending on r but not N) such that

$$\left| \frac{f_{i,N}(z_1, \cdots, z_j, W_{j+1}, \cdots, W_n) - f_{i,N}(z_1, \cdots, z_{j-1}, W_j, \cdots, W_n)}{z_j - W_j} - \frac{\partial f_{i,N}}{\partial z_j}(z_1, \cdots, z_j, W_{j+1}, \cdots, W_n) \right| \le |z_j - W_j| \cdot M.$$

Hence, an application of the triangle inequality will yield that for (i, j) fixed there is an N such that the (i, j) entry of $A_{F_N}(Z, W)$ is within ϵ of the (i, j) entry of the $A_F(Z, W)$ matrix on $|Z| \le r$, $|W| \le r$. This means for $r_0 < 1$ and $0 < \epsilon < m_F(r_0)/2$ we can find an integer N_0 so that

$$|m_{F_N}(r) - m_F(r)| < \epsilon \tag{1}$$

for $0 \le r \le r_0$. Also the entries in the Jacobian matrix for the F_N converge uniformly on $(|Z| \le r_0)$ to the entries in the Jacobian matrix for F. Applying Lemma 2 to the F_N mappings we can find an $N_1(r_0)$ so that the result of Lemma 3 will hold on $|Z| \le r_0$ for $N \ge N_1(r_0)$.

Lemma 4. *Let* $r_0 < 1$ *and assuming that* F *is as in Lemma* 3 *define* $d(r_0) \equiv d(\Re F(r_0), \partial \Re F(1))$. *Then* $0 < d(r_0) < \infty$ *and there is an* $N_2(r_0)$ *such that* $d(\Re F_N(r_0), \partial \Re F(1)) > 0$ *whenever* $N \ge N_2(r_0)$.

Proof. F is assumed to be a homeomorphism, hence $d(r_0) > 0$. If $d(r_0) = \infty$ this would imply that $\Re F(1) = \mathbf{C}^n$. Let $G(W) = F^{-1}(W)$ be the inverse mapping to

F from $\mathbf{C}^n$ into D^n. Using one variable theorems one shows that $g_i(W_1, 0, \cdots, 0) = g_i(0, \cdots, 0) = 0$. For W_1 fixed (say $W_1 = W_1^0$) we have

$$g_i(W_1^0, W_2, 0, \cdots, 0) = g_i(W_1^0, 0, \cdots, 0) = g_i(0, \cdots, 0) = 0.$$

This will show each $g_i(Z) = 0$ and this implies F is not biholomorphic. Since $F_N \Rightarrow F$ we can find $N_2 = N_2(r_0)$ so that if $N \geq N_2$ then $|F_N(Z) - F(Z)| < d(r_0)/2$ on $|Z| \leq r_0$. This yields the result of our lemma.

Having established the above lemmas we can proceed to the principal theorem.

Theorem 1. *Assume F is a biholomorphic mapping of D^n into $\mathbf{C}^n$ ($F(0) = 0$), $F(Z) = (f_1(Z), \cdots, f_n(Z))$. Then there exist polynomial mappings $P_k(Z)$ which are biholomorphic (on D^n) and of degree k ($k = 1, 2, 3, \cdots$) such that $P_1 < P_2 < P_3 \cdots$ ($P_k < F$) and $P_k(Z) \Rightarrow F(Z)$.*

Proof. We assume first that some component of F (i.e. some f_i) is not a polynomial. Let then $0 < r_0 < 1$ be given. For this r_0 we select an $N_1(r_0)$ as in Lemma 3 and an $N_2(r_0)$ as in Lemma 4 and we set $N(r_0) = N_1(r_0) + N_2(r_0)$. Choose an $N_1 > N(r_0)$ so that F_{N_1} has degree N_1. Let $E_N(r)$ be the preimage of $\Re F_N(r)$ under the mapping F, $E_N(r) = F^{-1}(\Re F_N(r))$. By the openness of the mapping F we can find a number $1 > \rho_1 > r_0$ so that $E_N(r_0) \subseteq D^n_{\rho_1}$. We continue now by setting $1 > r_1 > (1 + \rho_1)/2$. We can select an $N(r_1)$ as in Lemma 3. Let a be the number which is the minimum of $d(r_1)$ and $d(\Re F_{N_1}(r_0), \partial \Re F(r_1))$. Again uniform convergence of F_N to F on compacta allows us to find $N_2(r_1)$ so that $|F(Z) - F_N(Z)| < a/2$ whenever $N \geq N_2(r_1)$ and $|Z| \leq r_1$. These last comments imply that

$$d(\Re F_{N_1}(r_0), \partial \Re F_N(r_1)) > 0 \tag{1}$$

if $N \geq N_2(r_1)$. We have arcwise connected sets $\Re F_{N_1}(r_0)$ and $\Re F_N(r_1)$ with a common point and (1) implies $\Re F_N(r_1) \supseteq \Re F_{N_1}(r_0)$ if $N \geq N_2(r_1)$. We can set $N = N_1(r_1) + N_2(r_1) + N_1$ and choose $N_2 \geq N$ so that F_{N_2} has degree N_2. We define the polynomial mappings

$$P_{N_1}(Z) = F_{N_1}(r_0 Z), \qquad P_{N_2}(Z) = F_{N_2}(r_1 Z).$$

P_{N_1} and P_{N_2} are biholomorphic on $\overline{D^n}$ and are polynomial maps of degree N_1 and N_2 respectively. They are also subordinate. One can now proceed to construct polynomial mappings $P_{N_j}(Z)$ ($N_1 < N_2 < \cdots$) which are biholomorphic on $\overline{D^n}$ and satisfy $P_{N_1} < P_{N_2} < P_{N_3} \cdots$. An application of Lemma 1 shows that $P_{N_j}(Z) \Rightarrow F(Z)$.

The remaining part of this proof consists of filling in the polynomial mappings of appropriate degrees between the P_{n_k} and $P_{n_{k+1}}$. Assume then $k < j$ and that P and Q are polynomial mappings of degrees k and j respectively. Further we have

P and Q biholomorphic on open sets containing $\overline{D^n}$ and $P < Q$ on D^n. There is an $\Re > 1$ so that P is a biholomorphic polynomial mapping on $D^n_{\Re}$. Choose $1 < \rho < \Re$ and define P^* on D^n as $P^*(Z) = P(\rho Z)$. Note that ρ can be chosen so that $d(\Re P(1), \partial\Re P^*(1)) > 0$ and $d(\Re P^*(1), \partial \Re Q) > 0$, and P^* is a biholomorphic polynomial mapping on D^n of degree k. Let $(b) = (b_1, \cdots, b_n) \in \mathbf{C}^n$ and define

$$Q^*(Z) = P^*(Z) + (b_1 z_1^{k+1}, b_2 z_2^{k+1}, \cdots, b_n z_n^{k+1}).$$

It is clear that Q^* can be made uniformly close to P^* on D^n if only $|b|$ is small. Hence $|\det J_{Q^*}(Z)| \neq 0$ on D^n and since ϕ_{Q^*} can be made arbitrarily close to ϕ_{P^*} by a suitable choice of b we conclude from Lemma 2 that there is an $\eta > 0$ such that if $|b| < \eta$ then Q^* is a biholomorphic polynomial of degree $k+1$. Since $d(\Re P^*, \partial \Re Q^*)$ can be made small with $|b|$ and since $P < P^* < Q$ we have for $0 < \eta$ sufficiently small that $P < Q^* < Q$.

The pair Q^*, Q are now biholomorphic polynomial mappings on an open set containing $\overline{D^n}$ of degrees $k+1$ and j respectively and so we can find such polynomial mappings for $k+1, \cdots, k+j-(k+1)$. We have now the chain of polynomial, biholomorphic mappings P_j of degree j which satisfy the subordination relation $P_1 < P_2 < P_3 < \cdots, P_n < F$, and such that $P_{n_j} \Rightarrow F$. Assume then that P_{n_j} and $P_{n_{j+1}}$ are successive members of the sequence $\{P_{n_j}\}$ and the degree of P_{n_j} is m and the degree of $P_{m_{j+1}}$ is k. We have chosen numbers $1 < \rho_\nu$ and $b^\nu \in \mathbf{C}^n$, $\nu = m+1, m+2, \cdots, k-1$, and successively defined polynomials $P_{\nu+1}(Z) = P_\nu(P_\nu Z) + (b_1^{\nu+1} z_1^{\nu+1}, b_2^{\nu+1} z_2^{\nu+1}, \cdots, b_n^{\nu+1} z_n^{\nu+1})$. Since $P_{n_j}(Z) \equiv P_m(Z)$ is biholomorphic on $\overline{D^n}$ we can choose the $1 < \rho_\nu$ so close to one that $P_m(P_m P_{m+1}, \cdots, P_{\nu-1} Z)$ is arbitrarily close to $P_m(Z)$ on all of D^n. The recursive definition of the P_ν will yield the estimate

$$\begin{aligned} &|P_\nu(Z) - P_m(\rho_m \rho_{m+1} \cdots \rho_{\nu-1} Z)| \\ &\quad = \Bigg| \sum_{j=1}^{\nu-m} (b_1^{m+j}(\rho_{\nu-m+1}\rho_{\nu-m+2} \cdots \rho_{\nu-m-(j-1)})^{m+j} z_1^{m+j}, \\ &\qquad\qquad \cdots, b_n^{m+j}(\rho_{\nu-m+1} \cdots \rho_{\nu-m-(j-1)})^{m+j} z_n^{m+j}) \Bigg| \\ &\quad \leq |(\rho_{m+1} \cdots \rho_{k-1})|^{k-m} \sum_{j=1}^{k-m-1} |b^{m+j}|. \end{aligned}$$

Now if we are given $\epsilon > 0$ we choose the $\{\rho_j\}$ so close to one that $|P_m(Z) - P_m(\rho_m \rho_{m+1} - \rho_{\nu-1} Z)| < \epsilon/2$ for $\nu = m+1, \cdots, k-1$ and so that if $|b^{m+j}| < \eta$ for $j = 1, 2, \cdots, k-(m-1)$ then

$$|P_\nu(Z) - P_m(\rho_m \rho_{m+1} \cdots \rho_{\nu-1} Z)| < \epsilon/2.$$

Hence, $|P_\nu(Z) - P_m(Z)| < \epsilon$ for $z \in D^n$ and $\nu = m+1, \cdots, k-1$. This proves that one can choose the full sequence $\{P_N(Z)\}$ so that it satisfies the subordination chain relation and so that $P_N(Z) \Rightarrow F(Z)$.

It remains to consider only the case where F is a biholomorphic polynomial mapping of D^n of say degree N. Choose $0 < r_j \nearrow 1$ and define $F_j(Z) = F(r_j Z)$ for $j = 1, 2, 3, \cdots$. We can choose $b^1 = (b_1^1, \cdots, b_n^1) \in \mathbf{C}^n$ of small norm so that

$$(b_1^1 z_1, \cdots, b_1^n z_n) = P_1(Z) < F_1(Z).$$

Now the first part of the proof allows us to assert the existence of biholomorphic polynomials $P_1, \cdots, P_N$ satisfying $P_1 < P_2 < \cdots < P_N = F_1$, where each P_j has degree j and the degree of F_1 is N. Again choose a $b^2 \in \mathbf{C}^n$ with small modulus so that

$$P_{N+1}(N) = F_2(Z) + (b_1^2 z_1^{N+1}, \cdots, b_n^2 z_1^{N+1})$$

is univalent in D^n and so that the image of $\overline{D^n}$ under the mapping P_N is properly contained in the image of D^n under the mapping P_{N+1} which in turn is properly contained in the image of D^n under the mapping F_3. The proof is now finished.

Theorem 2. *Let $F(Z)$ be a biholomorphic mapping $(F(0) = 0)$ of D^n into $\mathbf{C}^n$ such that the image of D^n under F is a convex domain. Then there exists a sequence of biholomorphic polynomial mappings P_k, of degree k, from D^n into $\mathbf{C}^n$ with convex range and such that $P_k < P_{k+1}$ for $n = 1, 2, 3, \cdots$ and $P_k \Rightarrow F$.*

Proof. The proof is a sandwiching together of two known results. The first is that of T. J. Suffridge [2] which states that if F satisfies the hypothesis of Theorem 2 then there exist convex univalent mappings $g_j \colon D \to \mathbf{C}^1$ and a nonsingular linear transformation T of $\mathbf{C}^n$ into $\mathbf{C}^n$ such that F has a decomposition

$$F(Z) = T \circ G(Z) = T(g_1(z_1), g_2(z_2), \cdots, g_n(z_n)).$$

The second result is that of T. H. MacGregor [1] which states that there exist convex univalent polynomials $P_{k,j}(z_j)$ so that $P_{k,j}(z_j) < P_{k+1,j}(z_j)$ for $k = 1, 2, 3, \cdots$, and such that $P_{k,j}(z_j) \Rightarrow g_j(z_j)$, on D. The degree of $P_{k,j}$ is k. Defining the functions

$$T_k(Z) = T(P_{k,1}(z_1), \cdots, P_{k,n}(z_n)) \equiv T \circ P_k(Z)$$

we see that T_k is a polynomial mapping of degree k and that T_k is biholomorphic. If $T_k(Z) \equiv T \circ P_k(Z) = T(W) = X$, then $W = P_k(Z) = (P_{k,1}(z_1), \cdots, P_{k,n}(z_n))$. But $P_{k,j} < P_{k+1,j}$ implies the existence of a $Z^1 = (z_1^1, \cdots, z_n^1) \in D^n$ such that $P_{k+1,j}(z_j^1) = P_{k,j}(z_j)$. This implies that $T_{k+1}(Z^1) = T_k(Z)$ and so $T_k < T_{k+1}$ for all $k = 1, 2, \cdots$. If now $Z \in D^n$ we have

$$|F(Z) - T_k(Z)| = |T \circ G(Z) - T \circ P_k(Z)| \le \|T\| \, |G(Z) - P_k(Z)|$$

and so $T_k(Z) \Rightarrow F(Z)$.

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