BOUNDLY COMPLETE M-BASES AND COMPLEMENTED SUBSPACES IN BANACH SPACES

BY

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ABSTRACT. Subsequences of boundedly complete M-bases need not be boundedly complete. An example of a somewhat reflexive space is given whose dual and one of whose factors fail to be somewhat reflexive. A geometric description of boundedly complete M-bases is given which is equivalent to the definitions of V. D. Milman and W. B. Johnson. Finally, certain M-bases for separable spaces give rise to proper complemented subspaces.

A sequence \((x_n)\) in a Banach space is called an M-basis (or Markushevich basis) of \(E\) if \((x_n)\) is complete in \(E\) (i.e., the closed linear span \([x_n]\) of \((x_n)\) is the whole space \(E\)) and if there exists a total sequence of functionals \((f_n) \subset E^*\) (i.e., \(\{x \in E \mid f_n(x) = 0 \ (n = 1, 2, \ldots)\} = \{0\}\)) such that \((x_n, f_n)\) is a biorthogonal system (i.e., \(f_i(x_j) = \delta_{ij}\) for \(i, j = 1, 2, \ldots\)); obviously, \((f_n)\) is uniquely determined. The sequence \((x_n)\) is called a basis of \(E\) if for every \(x \in E\) there exists a unique sequence of scalars \((\alpha_n)\) such that

\[
x = \sum_{i=1}^{\infty} \alpha_i x_i.
\]

It is known [2] that every basis is an M-basis, with \(f_n(x) = \alpha_n (x = \sum_{i=1}^{\infty} \alpha_i x_i) \in E, \ n = 1, 2, \ldots\), which are called the coefficient functionals.

A basis \((x_n)\) of a Banach space \(E\) is called boundedly complete if the relation

\[
\sup_n \left\| \sum_{i=1}^{n} \alpha_i x_i \right\| < \infty
\]

implies that \(\sum_{i=1}^{\infty} \alpha_i x_i\) converges in \(E\). It is known ([18], [3]) that \((x_n)\) is a boundedly complete basis of \(E\), with the coefficient functionals \((f_n)\), if the canonical mapping \(\phi\) of \(E\) into \([f_n]^*\) (i.e., the mapping defined by \([\phi(x)](f) = f(x)\) for all \(x \in E, f \in [f_n]^*\)) is an isomorphism of \(E\) onto \([f_n]^*\). The notion of boundedly completeness has been extended to M-bases in several different ways; most of them equivalent to this property of the canonical mapping \(\phi: E \rightarrow [f_n]^*\).

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We shall say that an M-basis, \((x_n; f_n)\), of \(E\) is boundedly complete if \(\phi\) is an isomorphism onto \([f_n]^*\). In §1 of the present note we shall disprove a claim of V. D. Milman [21] on boundedly complete M-bases, which, if it had been true, would have provided an affirmative answer to a problem of [4] on boundedly completeness. In §2 we consider the existence of reflexive subspaces in Banach spaces. We also answer some questions concerning the somewhat reflexive spaces of Herman and Whitley [10]. In §3 we introduce a new geometric characterization of boundedly complete M-bases. Finally, in §4 we examine complemented subspaces in separable Banach spaces arising from certain M-bases.

1. The existence of boundedly complete basic sequences. In [4] the question was raised whether every separable conjugate Banach space \(E^*\) contains a subspace with a boundedly complete basis (with no assumption of separability on \(E^*\) the question was raised in [25, Problem 3.8] and [22, Problem 3]); by subspace we shall mean, throughout this note, closed linear subspace. Since the first version of this paper, Johnson and Rosenthal [16] have given an affirmative answer to the problem. Earlier, V. D. Milman [21] claimed that the answer to this question is affirmative, via the following argument: Since \(E^*\) is separable, there exists, by a result of Gapaškin and Kadec [7], a biorthogonal system \((x_n; f_n)\) such that \([x_n] = E\) and \([f_n] = E^*\). Then \((f_n)\) is a boundedly complete M-basis of \(E^*\). Since \([x_n] = E\), \((f_n)\) contains \((17), [21]) a subsequence \((f_{n_j})\) which is a basic sequence (i.e., a basis of its closed linear span \([f_{n_j}]\)). Now, Milman claims [21, Proposition 3.4b] that every subsequence of a boundedly complete M-basis is a boundedly complete M-basis of its closed linear span and hence \([f_{n_j}]\) is a subspace of \(E^*\) with the desired property, i.e., having a boundedly complete basis \((f_{n_j})\). Firstly, we want to remark that this claim is false, even when \((f_{n_j})\) is a basic sequence, as shown by Example 1. Let \(E = c_0\) and let

\[
\begin{align*}
x_{2n-1} &= e_{2n-1} - 2^n e_{2n} + 2^{n+1} e_{2n+2} & (n = 1, 2, \ldots), \\
x_{2n} &= 2^n e_{2n} & (n = 1, 2, \ldots), \\
f_{2n-1} &= b_{2n-1} & (n = 1, 2, \ldots), \\
f_2 &= b_1 + \frac{1}{2} b_2, & f_{2n} = -b_{2n-3} + b_{2n-1} + (1/2^n)b_{2n} & (n = 2, 3, \ldots),
\end{align*}
\]

where \((e_n)\) is the unit vector basis of \(c_0\) (i.e., \(e_n = \{0, \ldots, 0, 1, 0, \ldots\}\)) and \((b_n)\) the sequence of coordinate functionals on \(E = c_0\) (i.e., \(b_n(x) = \xi_n\) for all \(x = (\xi_i) \in E\)), hence \(b_i(e_j) = \delta_{ij}\) (i, j = 1, 2, \ldots). Then \((x_n; f_n)\) is a biorthogonal system. Furthermore, since \(e_{2n} = (1/2^n)x_{2n}\), \(e_{2n-1} = x_{2n-1} + x_{2n} - x_{2n+2}\) \((n = 1, 2, \ldots)\) and, since \([e_n] = E\), we have \([x_n] = E\). Similarly, since \(b_{2n-1} = \)
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\[ b_{2n} = 2^n (f_{2n} - f_{2n-1} + f_{2n-3}) \quad (n = 2, 3, \ldots) \]
and, since \[ [f_n] = E^*, \] we have \[ [f_n] = E^*, \] making \[ \langle f_n \rangle \] boundedly complete. However, the subsequence \[ \langle f_{2n} \rangle \] of \[ \langle f_n \rangle \] is a basic sequence which is not boundedly complete. Indeed, for any scalars \( a_1, \ldots, a_n \) we have

\[
\left\| \sum_{i=1}^{n} a_i f_{2i} \right\| = \left\| a_1 (b_1 + 1/2 b_2) + \sum_{i=2}^{n} a_i \left( -b_{2i-3} + b_{2i-1} + 1/2 i b_{2i} \right) \right\| \\
= \left\| \sum_{i=1}^{n-1} (a_i - a_{i+1}) b_{2i-1} + a_n b_{2n-1} + \sum_{i=1}^{n} \frac{1}{2^i} a_i b_{2i} \right\| \\
= \sum_{i=1}^{n-1} |a_i - a_{i+1}| + |a_n| + \sum_{i=1}^{n} \frac{1}{2^i} |a_i|,
\]

whence, for any scalars \( a_1, \ldots, a_{n+1} \),

\[
\left\| \sum_{i=1}^{n} a_i f_{2i} \right\| \leq \sum_{i=1}^{n} |a_i - a_{i+1}| + |a_{n+1}| + \sum_{i=1}^{n+1} \frac{1}{2^i} |a_i| = \left\| \sum_{i=1}^{n+1} a_i f_{2i} \right\|,
\]

and therefore, \( \langle f_{2n} \rangle \) is a "monotone" basic sequence [3]. Finally, for \( a_1 = \ldots = a_n = 1 \), we have \( \left\| \sum_{i=1}^{n} f_{2i} \right\| = 1 + \sum_{i=1}^{n} (1/2^i) \) \( (n = 1, 2, \ldots) \), whence \( \sup_n \left\| \sum_{i=1}^{n} f_{2i} \right\| = 2 \), but \( \sum_{i=1}^{n} f_{2i} \) is not convergent, since \( \|f_{2n}\| > 1 \) \( (n = 1, 2, \ldots) \), and thus \( \langle f_{2n} \rangle \) is not boundedly complete.

It is interesting to see what conditions on a biorthogonal system \( \langle x_n; f_n \rangle \) such that \( [x_n] = E, [f_n] = E^* \) will guarantee that every basic subsequence \( \langle f_{n_j} \rangle \) of \( \langle f_n \rangle \) is boundedly complete. Standard arguments show that, if \( \langle f_{n_j} \rangle \) is \( \sigma(E^*, E) \)-closed, then \( \langle f_{n_j} \rangle \) is boundedly complete (if \( [f_{n_j}] = E^* \)). Hence, since clearly \( [f_{n_j}] \subset [x_n; i \neq n_1, n_2, \ldots]^1 \), and since the latter is \( w^* \)-closed, a rather natural sufficient condition is \( [f_{n_j}] = \{ x_i; i \leq n_1, n_2, \ldots \} \). This condition is violated in Example 1 since \( \sum_{i=1}^{\infty} 2^{-i} b_{2i} \) is in \( [x_{2n-1}]^1 \) but is not in \( [f_{2n}] \). These considerations lead to the following problem.

**Problem 1.** If \( E^* \) is separable, is there an M-basis \( \langle x_n; f_n \rangle \) for which \( [f_n] = E^* \) and such that, for every subsequence of the integers, \( [f_{n_j}] \) is \( \sigma(E^*, E) \)-closed?

**Problem 2.** Does every separable Banach space have an M-basis for which \( [x_n; i \neq n_1, n_2, \ldots]^1 \) for every subsequence \( \langle n_i \rangle \) of the integers?

An M-basis satisfies the condition of Problem 2 if and only if for every \( x \) in \( E \) there exist multipliers \( \lambda_{n,i}(x) \) \( 1 \leq i \leq n < \infty \) such that \( x = \lim_{n \to \infty} \sum_{i=1}^{n} \lambda_{n,i}(x) f_{i}(x) \). Such a condition is somewhat weaker than strongly series summable M-bases ([23], [14]) in which the multipliers do not depend on \( x \). Such M-bases exist in separable complex Banach spaces whose duals have the bounded approximation property [14].
2. On somewhat reflexive spaces. Milman [21] has also used the erroneous
claim (see §1) that subsequences of boundedly complete M-bases are boundedly
complete in his proof of the following interesting result: \textit{If }E^{**}\textit{ is separable then
both }E\textit{ and }E^{*}\textit{ contain infinite dimensional reflexive subspaces.}

W. B. Johnson and H. P. Rosenthal [16] have proven that the above statement
is true. Below we still include a proof of this statement in the case that }E^{**}\textit{ has
the approximation property, which we obtained independently. These results are
used in the example concerning somewhat reflexive spaces in the weaker form we
present.

Herman and Whitley [10] have called a space }E\textit{ somewhat reflexive if every
infinite dimensional subspace of }E\textit{ contains an infinite dimensional reflexive sub-
space. They show, for example, that every quasi-reflexive space (i.e.,
\( \dim E^{**}/E < \infty \)) is somewhat reflexive and note that the space \( (J \times J \times \cdots)_2 \)
(with \( J \) the quasi-reflexive space of James [11]) is somewhat reflexive. It should
be noted that a proof of the Milman statement (above) will force }E\textit{ (having }E^{**}\textit{ separable)
to be somewhat reflexive: If }X\textit{ is an infinite dimensional subspace of }
E,\textit{ then }X^{**} \sim X^{\perp}\textit{ makes }X^{**}\textit{ separable (where }\sim\textit{ stands for “isomorphic to”),
so that }X\textit{ would contain an infinite dimensional reflexive subspace.

In what follows, we make frequent use of the following theorem of James [12]
and Lindenstrauss [19]: \textit{For any separable B-space, }E,\textit{ there is a B-space }Y\textit{ with
a shrinking basis such that }E\textit{ is a continuous image of }Y^{*}\textit{ and such that }Y^{**} \sim \textit{Y \times E^{*}.}

\textbf{Theorem 1.} \textit{If }E^{**}\textit{ is separable and has the approximation property, then
both }E\textit{ and }E^{*}\textit{ are somewhat reflexive.}

\textbf{Proof.} Let }Y\textit{ be the space of James and Lindenstrauss above which has
}Y^{**} \sim Y \times E^{*}.\textit{ Then }Y^{**}\textit{ has the approximation property, and hence a boundedly
complete basis by Theorem 1.4 of [15]. By Corollary 5 of [4], every infinite
dimensional subspace }G\textit{ of }E\textit{ (i.e., }E^{**} \subset Y^{**}\textit{) contains a boundedly complete basic
sequence, say }\{x_{n}\}\textit{. By Proposition 2 of [4], some block basic sequence }\{y_{n}\}\textit{ with
respect to }\{x_{n}\}\textit{ is shrinking. It follows that }\{y_{n}\} \subset G\textit{ is reflexive as desired.
The same argument can now be applied to }E^{*}\textit{ since, by [8], }E^{*}\textit{ has the approxima-
tion property.

We now turn to some questions raised in [10] concerning somewhat reflexive
spaces. There the authors remarked that they did not know whether or not duals
and/or quotients of somewhat reflexive spaces are somewhat reflexive. This ex-
ample answers both questions in the negative:

\textbf{Example 2.} Let }Y\textit{ be the space of James and Lindenstrauss above such that
}c_{0}\textit{ is a quotient of }Y^{*}\textit{ and }Y^{**} \sim Y \times l_{1}.\textit{ }Y\textit{ and }l_{1}\textit{ both have bases, so by}
Theorem 1, \( Y^* \) is somewhat reflexive. However, neither \( l_1 \) nor \( c_0 \) contains an infinite dimensional reflexive subspace so that this space is somewhat reflexive and has both a quotient and dual which are not somewhat reflexive. A somewhat more remarkable feature of this space \( (Y^*) \) is that although it is separable, contains no copy of \( c_0 \) or \( l_1 \), its second dual \( Y^{**} \cong Y^* \times m \) is nonseparable.

Consideration of this example has led the authors to the following question.

**Problem 3.** If \( E \) is a separable somewhat reflexive space, is \( E^* \) separable?

### 3. Boundedly complete M-bases

Many definitions of boundedly complete M-bases appear in the literature. All of these (known to the authors) reduce in the basis situation to the concept studied by Dunford and Morse [6], Alaoglu [1], Karlin [18] and James [11] (the name "boundedly complete" seems to be due to Day [3]).

The original definition asserts that a basis is boundedly complete if the boundedness of the partial sums \( \sum_{i=1}^{n} a_i x_i \) forces the convergence of the series \( \sum_{i=1}^{\infty} a_i x_i \). The main motivation for such a definition (at least at this point in time) is the fact that boundedly complete bases span dual spaces. The second author (in [25]) has shown that such "boundedness implies convergence" conditions are much too strong for use with M-bases. Therefore, for M-bases, the definitions in the literature are constructed so that boundedly complete M-bases span separable duals ([13], [21]). In this section we present a geometric definition of boundedly complete M-basis which is equivalent to the known "soft" definitions in the literature.

**Definition.** Let \( (x_n; f_n) \) be an M-basis for \( E \). We shall call it norm-boundedly-complete if

\[
\sup_n \inf_{S_n u = 0} \left\| \sum_{i=1}^{n} a_i x_i + u \right\| < \infty
\]

implies the existence of \( x \) in \( E \) with \( f_n(x) = a_n \) for all \( n \). (Here \( S_n u = \sum_{i=1}^{n} f_i(u)x_i \).)

The above definition is strongly related to the norming characteristic of the subspace \( [f_n] \) of \( E^* \) through the following considerations: For any \( x \) in \( E \) let

\[
|x| = \sup_{f \in [f_n]; \|f\| \leq 1} f(x).
\]

This always defines a norm on \( E \), and in case it is equivalent to the original norm, we say that \( [f_n] \) is norming (or of positive characteristic [5]). If \( (x_n) \) is a basis for \( E \), it follows that \( [f_n] \) is norming [24]. If \( (x_n; f_n) \) is an M-basis for \( E \), then it can be shown that
\[ |x| = \sup_n \inf_{S_n u=0} \|(S_n x) + u\|. \]

In what follows, \( \phi: E \to [f_n]^* \) is to denote the natural map defined by \( (\phi(x))(f) = f(x) \). It is well known (e.g. [5]) that \([f_n]^*\) is norming if and only if \( \phi \) is an isomorphism of \( E \) into \([f_n]^*\). We call an \( M \)-basis \((x_n; f_n)\) of \( E \) boundedly complete if \( \phi \) is an isomorphism of \( E \) into \([f_n]^*\) (equivalently, \( \phi(E) = [f_n]^* \)).

**Theorem 2.** An \( M \)-basis \((x_n; f_n)\) for \( E \) is boundedly complete if and only if it is norm-boundedly complete.

**Proof.** Assume that \( \phi(E) = [f_n]^* \) and let \((a_i)\) be a sequence of scalars with
\[
\inf_{S_n u=0} \left\| \sum_{i=1}^{n} a_i x_i + u \right\|
\]
bounded in \( n \). Since \([f_n]^*\) is separable, some subsequence of \( \phi(\sum_{i=1}^{n} a_i x_i + u_n) \) converges weak* to some \( F \) in \([f_n]^*\). \((u_n)\) has been chosen to keep \((\sum_{i=1}^{n} a_i x_i + u_n)\) a bounded sequence.) Then \( F = \phi(x) \) for some \( x \in E \) whence
\[
f(x) = (\phi(x))(f) = F(f) = \lim_k \left( \phi \left( \sum_{i=1}^{n_k} a_i x_i + u_{n_k} \right) \right) f = \lim_k \left( \sum_{i=1}^{n_k} a_i x_i + u_{n_k} \right)
\]
for all \( f \in [f_n] \) and hence \( f_i(x) = a_i \) \((i = 1, 2, \ldots)\). For the other direction suppose that \((y_n)\) is a bounded sequence in \( E \) with \( \lim_j f_i(y_j) = a_i \) (existing for each \( n \)). It follows that \( S_n(y_j) \) converges strongly to \( \sum_{i=1}^{n} a_i x_i \) for each \( n \). Thus, one can choose \((y_{i,n})\) and
\[
z_n = \sum_{i=1}^{n} a_i x_i + (I - S_n)y_{i,n}
\]
in such a way that \( \|z_n - y_{i,n}\| < 2^{-n} \) for each \( n \). Since
\[
\inf_{S_n u=0} \left\| \sum_{i=1}^{n} a_i x_i + u \right\| \leq \|z_n\| \leq \|y_{i,n}\| + 2^{-n},
\]
there is \( x \) in \( E \) with \( f_i(x) = a_i \) for all \( i \). This gives the result by Theorem II.5 of [13].

4. \( M \)-bases and complemented subspaces. Milman's Proposition 3.5, Theorem 3.7, and Theorem 3.8 of [21] give conditions which guarantee the complementation in \( E \) of \([x_n]\) where \((x_n)\) is a boundedly complete basic sequence in the
separable Banach space $E$. For example, $[x_n]$ is complemented if there are coefficient functionals $(f_n) \subset E^*$ with $[f_n]$ norming $[x_n]$. In Theorem 3 below, $(x_{nk})$ need not be basic or boundedly complete, but must be part of an $M$-basis for $E$. Extensions of $M$-bases of subspaces to $M$-bases of $E$ are treated in [9] and [26].

**Theorem 3.** Let $(x_n; f_n)$ be an $M$-basis for $E$, let $(n_k)$ be an infinite subsequence of the integers and let $(n'_k)$ denote the complementary subsequence. If $[f_{nk}]$ is norming over $[x_{nk}]$, then $E = [x_{nk}] \oplus [x_{nk}]$. If $[f_n]$ is norming over $E$, the converse holds.

**Proof.** Consider the quotient map $q: E \to E/[x_{nk}]$. Then by the norming hypothesis and by $[f_{nk}] \subset [x_{nk}]$, there exists $\mu > 0$ such that, for $u \in [x_{nk}]$,

$$\mu \|u\| \leq \sup_{g \in [f_{nk}]; \|g\| \leq 1} |g(u)| \leq \|q(u)\| \leq \|u\|,$$

whence $q|[x_{nk}]$ is an isomorphism. Further $\text{sp}(x_n)$ is dense in $E$, so that $\text{sp}(q(x_{nk}))$ is dense in $E/[x_{nk}]$. Thus, $[x_{nk}]$ is isomorphic to $E/[x_{nk}]$. It is standard (and readily verified) that a projection of $E$ onto $[x_{nk}]$ along $[x_{nk}]$ is given by $P = (q|[x_{nk}])^{-1} q$.

For the second assertion, let $[f_n]$ be norming over $E = [x_{nk}] \oplus [x_{nk}]$. Let $u \in [x_{nk}]$ and $g \in \text{sp}(f_n)$ with $\|g\| \leq 1$ such that $g(u) > \mu \|u\|$. If $P$ is the projection of $E$ onto $[x_{nk}]$ along $[x_{nk}]$, $P^*g \in \text{sp}(f_{nk})$ (because $P^*f_{nk} = f_{nk}$, $P^*f_{nk} = 0$) and $\|P^*g\| (P^*g/\|P^*g\|)(u) = (P^*g)(u) = g(Pu) = g(u) > \mu \|u\|$. Thus,

$$(P^*g/\|P\|)(u) \geq (\mu/\|P\|) \|u\|,$$

so that $[f_{nk}]$ norms $[x_{nk}]$.

**Corollary.** Let $(x_n; f_n)$ be an $M$-basis for $E$ such that for every subsequence $(n_k)$ of the integers, $[f_{nk}]$ norms $[x_{nk}]$. Then $(x_n)$ is an unconditional basis of $E$.

**Proof.** For every $(n_k)$, by Theorem 3, $E = [x_{nk}] \oplus [x_{nk}]$. The result follows from a result of Lorch [20].

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