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BOUNDEDLY COMPLETE M-BASES AND COMPLEMENTED SUBSPACES IN BANACH SPACES

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ABSTRACT. Subsequences of boundedly complete M-bases need not be boundedly complete. An example of a somewhat reflexive space is given whose dual and one of whose factors fail to be somewhat reflexive. A geometric description of boundedly complete M-bases is given which is equivalent to the definitions of V. D. Milman and W. B. Johnson. Finally, certain M-bases for separable spaces give rise to proper complemented subspaces.

A sequence (x_n) in a Banach space is called an *M-basis* (or *Markushevich* basis) of E if (x_n) is complete in E (i.e., the closed linear span $[x_n]$ of (x_n) is the whole space E) and if there exists a total sequence of functionals $(f_n) \subset E^*$ (i.e., $\{x \in E \mid f_n(x) = 0 \ (n = 1, 2, \cdots)\} = \{0\}$) such that (x_n, f_n) is a biorthogonal system (i.e., $f_i(x_j) = \delta_{ij}$ for $i, j = 1, 2, \cdots$); obviously, (f_n) is uniquely determined. The sequence (x_n) is called a basis of E if for every $x \in E$ there exists a unique sequence of scalars (α_n) such that

$$x = \sum_{i=1}^{\infty} \alpha_i x_i.$$

It is known [2] that every basis is an M-basis, with $f_n(x) = \alpha_n$ $(x = \sum_{i=1}^{\infty} \alpha_i x_i \in E$, $n = 1, 2, \cdots)$, which are called the *coefficient functionals*.

A basis (x_n) of a Banach space E is called *boundedly complete* if the relation

$$\sup_n \left\| \sum_{i=1}^{n} \alpha_i x_i \right\| < \infty$$

implies that $\sum_{i=1}^{\infty} \alpha_i x_i$ converges in E. It is known ([18], [3]) that (x_n) is a boundedly complete basis of E, with the coefficient functionals (f_n), iff the canonical mapping ϕ of E into $[f_n]^*$ (i.e., the mapping defined by $[\phi(x)](f) = f(x)$ for all $x \in E$, $f \in [f_n]^*$) is an isomorphism of E onto $[f_n]^*$. The notion of boundedly completeness has been extended to M-bases in several different ways, most of them equivalent to this property of the canonical mapping ϕ: $E \longrightarrow [f_n]^*$.

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We shall say that *an M-basis*, $(x_n; f_n)$, *of* E *is boundedly complete if* ϕ *is an isomorphism onto* $[f_n]^*$. In §1 of the present note we shall disprove a claim of V. D. Milman [21] on boundedly complete M-bases, which, if it had been true, would have provided an affirmative answer to a problem of [4] on boundedly completeness. In §2 we consider the existence of reflexive subspaces in Banach spaces. We also answer some questions concerning the *somewhat reflexive* spaces of Herman and Whitley [10]. In §3 we introduce a new geometric characterization of boundedly complete M-bases. Finally, in §4 we examine complemented subspaces in separable Banach spaces arising from certain M-bases.

1. The existence of boundedly complete basic sequences. In [4] the question was raised *whether every separable conjugate Banach space* E^* *contains a subspace with a boundedly complete basis* (with no assumption of separability on E^* the question was raised in [25, Problem 3.8] and [22, Problem 3]); by subspace we shall mean, throughout this note, closed linear subspace. Since the first version of this paper, Johnson and Rosenthal [16] have given an affirmative answer to the problem. Earlier, V. D. Milman [21] claimed that the answer to this question is affirmative, via the following argument: Since E^* is separable, there exists, by a result of Gapoškin and Kadec [7], a biorthogonal system $(x_n; f_n)$ such that $[x_n] = E$ and $[f_n] = E^*$. Then (f_n) is a boundedly complete M-basis of E^*. Since $[x_n] = E$, (f_n) contains ([17], [21]) a subsequence (f_{n_j}) which is a basic sequence (i.e., a basis of its closed linear span $[f_{n_j}]$). Now, Milman claims [21, Proposition 3.4b] that every subsequence of a boundedly complete M-basis is a boundedly complete M-basis of its closed linear span and hence $[f_{n_j}]$ is a subspace of E^* with the desired property, i.e., having a boundedly complete basis (f_{n_j}). Firstly, we want to remark that this claim is false, even when (f_{n_j}) is a basic sequence, as shown by

Example 1. Let $E = c_0$ and let

$$x_{2n-1} = e_{2n-1} - 2^n e_{2n} + 2^{n+1} e_{2n+2} \qquad (n = 1, 2, \cdots),$$

$$x_{2n} = 2^n e_{2n} \qquad (n = 1, 2, \cdots),$$

$$f_{2n-1} = h_{2n-1} \qquad (n = 1, 2, \cdots),$$

$$f_2 = h_1 + \tfrac{1}{2} h_2, \quad f_{2n} = -h_{2n-3} + h_{2n-1} + (1/2^n) h_{2n} \qquad (n = 2, 3, \cdots),$$

where (e_n) is the unit vector basis of c_0 (i.e., $e_n = \{\underbrace{0, \cdots, 0}_{n-1}, 1, 0, \cdots\}$) and (h_n) the sequence of coordinate functionals on $E = c_0$ (i.e., $h_n(x) = \xi_n$ for all $x = (\xi_j) \in E$), hence $h_i(e_j) = \delta_{ij}$ $(i, j = 1, 2, \cdots)$. Then $(x_n; f_n)$ is a biorthogonal system. Furthermore, since $e_{2n} = (1/2^n) x_{2n}$, $e_{2n-1} = x_{2n-1} + x_{2n} - x_{2n+2}$ $(n = 1, 2, \cdots)$ and, since $[e_n] = E$, we have $[x_n] = E$. Similarly, since $h_{2n-1} =$

f_{2n-1}, $h_2 = 2(f_2 - f_1)$, $h_{2n} = 2^n(f_{2n} - f_{2n-1} + f_{2n-3})$ $(n = 2, 3, \cdots)$ and, since $[h_n] = E^*$, we have $[f_n] = E^*$, making (f_n) boundedly complete. However, the subsequence (f_{2n}) of (f_n) is a basic sequence which is not boundedly complete. Indeed, for any scalars $\alpha_1, \cdots, \alpha_n$ we have

$$\left\|\sum_{i=1}^{n} \alpha_i f_{2i}\right\| = \left\|\alpha_1(h_1 + \tfrac{1}{2}h_2) + \sum_{i=2}^{n} \alpha_i\left(-h_{2i-3} + h_{2i-1} + \frac{1}{2^i}h_{2i}\right)\right\|$$

$$= \left\|\sum_{i=1}^{n-1} (\alpha_i - \alpha_{i+1})h_{2i-1} + \alpha_n h_{2n-1} + \sum_{i=1}^{n}\frac{1}{2^i}\alpha_i h_{2i}\right\|$$

$$= \sum_{i=1}^{n-1} |\alpha_i - \alpha_{i+1}| + |\alpha_n| + \sum_{i=1}^{n}\frac{1}{2^i}|\alpha_i|,$$

whence, for any scalars $\alpha_1, \cdots, \alpha_{n+1}$,

$$\left\|\sum_{i=1}^{n} \alpha_i f_{2i}\right\| \leq \sum_{i=1}^{n} |\alpha_i - \alpha_{i+1}| + |\alpha_{n+1}| + \sum_{i=1}^{n+1}\frac{1}{2^i}|\alpha_i| = \left\|\sum_{i=1}^{n+1} \alpha_i f_{2i}\right\|,$$

and therefore, (f_{2n}) is a "monotone" basic sequence [3]. Finally, for $\alpha_1 = \cdots = \alpha_n = 1$, we have $\|\sum_{i=1}^n f_{2i}\| = 1 + \sum_{i=1}^n (1/2^i)$ $(n = 1, 2, \cdots)$, whence $\sup_n \|\sum_{i=1}^n f_{2i}\| = 2$, but $\sum_{i=1}^\infty f_{2i}$ is not convergent, since $\|f_{2n}\| > 1$ $(n = 1, 2, \cdots)$, and thus (f_{2n}) is not boundedly complete.

It is interesting to see what conditions on a biorthogonal system $(x_n; f_n)$ such that $[x_n] = E$, $[f_n] = E^*$ will guarantee that every basic subsequence (f_{n_j}) of (f_n) is boundedly complete. Standard arguments show that, *if* $[f_{n_j}]$ *is* $\sigma(E^*, E)$*-closed, then* (f_{n_j}) *is boundedly complete* (if $[f_n] = E^*$). Hence, since clearly $[f_{n_j}] \subset [x_i|_{i \neq n_1, n_2, \ldots}]^\perp$, and since the latter is w^*-closed, a rather natural sufficient condition is $[f_{n_j}] = [x_i|_{i \neq n_1, n_2, \ldots}]^\perp$. This condition is violated in Example 1 since $\sum_{i=1}^\infty 2^{-i}h_{2i}$ is in $[x_{2n-1}]^\perp$ but is not in $[f_{2n}]$. These considerations lead to the following problem.

Problem 1. If E^* is separable, is there an M-basis $(x_n; f_n)$ for which $[f_n] = E^*$ and such that, for every subsequence of the integers, $[f_{n_j}]$ is $\sigma(E^*, E)$ closed?

Problem 2. Does every separable Banach space have an M-basis for which $[x_{n_j}] = [f_i|_{i \neq n_1, n_2, \ldots}]_\perp$ for every subsequence (n_j) of the integers?

An M-basis satisfies the condition of Problem 2 if and only if for every x in E there exist multipliers $(\lambda_{n,i}(x)|\ 1 \leq i \leq n < \infty)$ such that $x = \lim_{n\to\infty} \sum_{i=1}^n \lambda_{n,i}(x) f_i(x) x_i$. Such a condition is somewhat weaker than strongly series summable M-bases ([23], [14]) in which the multipliers do not depend on x. Such M-bases exist in separable complex Banach spaces whose duals have the bounded approximation property [14].

2. **On somewhat reflexive spaces.** Milman [21] has also used the erroneous claim (see §1) that subsequences of boundedly complete M-bases are boundedly complete in his proof of the following interesting result: *If E^{**} is separable then both E and E^* contain infinite dimensional reflexive subspaces.*

W. B. Johnson and H. P. Rosenthal [16] have proven that the above statement is true. Below we still include a proof of this statement in the case that E^{**} has the approximation property, which we obtained independently. These results are used in the example concerning somewhat reflexive spaces in the weaker form we present.

Herman and Whitley [10] have called a space E *somewhat reflexive* if every infinite dimensional subspace of E contains an infinite dimensional reflexive subspace. They show, for example, that every quasi-reflexive space (i.e., $\dim E^{**}/E < \infty$) is somewhat reflexive and note that the space $(J \times J \times \cdots)_{l_2}$ (with J the quasi-reflexive space of James [11]) is somewhat reflexive. It should be noted that a proof of the Milman statement (above) will force E (having E^{**} separable) to be somewhat reflexive: If X is an infinite dimensional subspace of E, then $X^{**} \sim X^{\perp\perp}$ makes X^{**} separable (where $\sim$ stands for "isomorphic to"), so that X would contain an infinite dimensional reflexive subspace.

In what follows, we make frequent use of the following theorem of James [12] and Lindenstrauss [19]: *For any separable B-space, E, there is a B-space Y with a shrinking basis such that E is a continuous image of Y^* and such that $Y^{**} \sim Y \times E^*$.*

Theorem 1. *If E^{**} is separable and has the approximation property, then both E and E^* are somewhat reflexive.*

Proof. Let Y be the space of James and Lindenstrauss above which has $Y^{**} \sim Y \times E^*$. Then Y^{**} has the approximation property, and hence a boundedly complete basis by Theorem 1.4 of [15]. By Corollary 5 of [4], every infinite dimensional subspace G of E $(\subset E^{**} \subset Y^{**})$ contains a boundedly complete basic sequence, say (x_n). By Proposition 2 of [4], some block basic sequence (y_n) with respect to (x_n) is shrinking. It follows that $[y_n] \subset G$ is reflexive as desired. The same argument can now be applied to E^* since, by [8], E^* has the approximation property.

We now turn to some questions raised in [10] concerning somewhat reflexive spaces. There the authors remarked that they did not know whether or not duals and/or quotients of somewhat reflexive spaces are somewhat reflexive. This example answers both questions in the negative:

Example 2. Let Y be the space of James and Lindenstrauss above such that c_0 is a quotient of Y^* and $Y^{**} \sim Y \times l_1$. Y and l_1 both have bases, so by

Theorem 1, Y^* is somewhat reflexive. However, neither l_1 nor c_0 contains an infinite dimensional reflexive subspace so that this space is somewhat reflexive and has both a quotient and dual which are not somewhat reflexive. A somewhat more remarkable feature of this space (Y^*) is that although it is separable, contains no copy of c_0 or l_1, its second dual $Y^{***} \sim Y^* \times m$ is nonseparable.

Consideration of this example has led the authors to the following question.

Problem 3. If E is a separable somewhat reflexive space, is E^* separable?

3. **Boundedly complete M-bases.** Many definitions of boundedly complete M-bases appear in the literature. All of these (known to the authors) reduce in the basis situation to the concept studied by Dunford and Morse [6], Alaoglu [1], Karlin [18] and James [11] (the name "boundedly complete" seems to be due to Day [3]).

The original definition asserts that a basis is boundedly complete if the boundedness of the partial sums $\Sigma_{i=1}^n a_i x_i$ forces the convergence of the series $\Sigma_{i=1}^\infty a_i x_i$. The main motivation for such a definition (at least at this point in time) is the fact that boundedly complete bases span dual spaces. The second author (in [25]) has shown that such "boundedness implies convergence" conditions are much too strong for use with M-bases. Therefore, for M-bases, the definitions in the literature are constructed so that boundedly complete M-bases span separable duals ([13], [21]). In this section we present a geometric definition of boundedly complete M-basis which is equivalent to the known "soft" definitions in the literature.

Definition. Let $(x_n; f_n)$ be an M-basis for E. We shall call it *norm-boundedly-complete* if

$$\sup_n \inf_{S_n u = 0} \left\| \sum_{i=1}^n a_i x_i + u \right\| < \infty$$

implies the existence of x in E with $f_n(x) = a_n$ for all n. (Here $S_n u = \Sigma_{i=1}^n f_i(u) x_i$.)

The above definition is strongly related to the norming characteristic of the subspace $[f_n]$ of E^* through the following considerations: For any x in E let

$$|x| = \sup_{f \in [f_n];\ \|f\| \le 1} f(x).$$

This always defines a norm on E, and in case it is equivalent to the original norm, we say that $[f_n]$ is *norming* (or *of positive characteristic* [5]). If (x_n) is a basis for E, it follows that $[f_n]$ is norming [24]. If $(x_n; f_n)$ is an M-basis for E, then it can be shown that

$$|x| = \sup_n \inf_{S_n u=0} \|(S_n x) + u\|.$$

In what follows, $\phi: E \longrightarrow [f_n]^*$ is to denote the natural map defined by $(\phi(x))f = f(x)$. It is well known (e.g. [5]) that $[f_n]$ is norming if and only if ϕ is an isomorphism of E into $[f_n]^*$. We recall (see the introduction) that we call an M-basis $(x_n; f_n)$ of E *boundedly complete* if ϕ is an isomorphism of E into $[f_n]^*$(equivalently, $\phi(E) = [f_n]^*$).

Theorem 2. *An M-basis $(x_n; f_n)$ for E is boundedly complete if and only if it is norm-boundedly complete.*

Proof. Assume that $\phi(E) = [f_n]^*$ and let (a_j) be a sequence of scalars with

$$\inf_{S_n u=0} \left\| \sum_{i=1}^{n} a_i x_i + u \right\|$$

bounded in n. Since $[f_n]$ is separable, some subsequence of $\phi(\Sigma_{i=1}^n a_i x_i + u_n)$ converges weak* to some F in $[f_n]^*$. ((u_n) has been chosen to keep $(\Sigma_{i=1}^n a_i x_i + u_n)$ a bounded sequence.) Then $F = \phi(x)$ for some $x \in E$ whence

$$f(x) = (\phi(x))(f) = F(f) = \lim_k \left(\phi\left(\sum_{i=1}^{n_k} a_i x_i + u_{n_k}\right)\right) f = \lim_k f\left(\sum_{i=1}^{n_k} a_i x_i + u_{n_k}\right)$$

for all $f \in [f_n]$ and hence $f_i(x) = a_i$ $(i = 1, 2, \cdots)$. For the other direction suppose that (y_j) is a bounded sequence in E with $\lim_j f_n(y_j) = a_n$ (existing for each n). It follows that $S_n(y_j)$ converges strongly to $\Sigma_{i=1}^n a_i x_i$ for each n. Thus, one can choose (y_{j_n}) and

$$z_n = \sum_{i=1}^{n} a_i x_i + (I - S_n) y_{j_n}$$

in such a way that $\|z_n - y_{j_n}\| < 2^{-n}$ for each n. Since

$$\inf_{S_n u=0} \left\| \sum_{i=1}^{n} a_i x_i + u \right\| \leq \|z_n\| \leq \|y_{j_n}\| + 2^{-n},$$

there is x in E with $f_i(x) = a_i$ for all i. This gives the result by Theorem II.5 of [13].

4. M**-bases and complemented subspaces.** Milman's Proposition 3.5, Theorem 3.7, and Theorem 3.8 of [21] give conditions which guarantee the complementation in E of $[x_n]$ where (x_n) is a boundedly complete basic sequence in the

separable Banach space E. For example, $[x_n]$ is complemented if there are coefficient functionals $(f_n) \subset E^*$ with $[f_n]$ norming $[x_n]$. In Theorem 3 below, (x_{n_k}) need not be basic or boundedly complete, but must be part of an M-basis for E. Extensions of M-bases of subspaces to M-bases of E are treated in [9] and [26].

Theorem 3. *Let $(x_n; f_n)$ be an M-basis for E, let (n_k) be an infinite subsequence of the integers and let (n_k') denote the complementary subsequence. If $[f_{n_k}]$ is norming over $[x_{n_k}]$, then $E = [x_{n_k}] \oplus [x_{n_k'}]$. If $[f_n]$ is norming over E, the converse holds.*

Proof. Consider the quotient map $q\colon E \longrightarrow E/[x_{n_k'}]$. Then by the norming hypothesis and by $[f_{n_k}] \subset [x_{n_k'}]^\perp$, there exists $\mu > 0$ such that, for $u \in [x_{n_k}]$,

$$\mu\|u\| \le \sup_{g \in [f_{n_k}];\, \|g\| \le 1} |g(u)| \le \|q(u)\| \le \|u\|,$$

whence $q \mid [x_{n_k}]$ is an isomorphism. Further $\mathrm{sp}(x_n)$ is dense in E, so that $\mathrm{sp}\,(q(x_{n_k}))$ is dense in $E/[x_{n_k'}]$. Thus, $[x_{n_k}]$ is isomorphic to $E/[x_{n_k'}]$. It is standard (and readily verified) that a projection of E onto $[x_{n_k}]$ along $[x_{n_k'}]$ is given by $P = (q \mid [x_{n_k}])^{-1}q$.

For the second assertion, let $[f_n]$ be norming over $E = [x_{n_k}] \oplus [x_{n_k'}]$. Let $u \in [x_{n_k}]$ and $g \in \mathrm{sp}\,(f_n)$ with $\|g\| \le 1$ such that $g(u) \ge \mu\|u\|$. If P is the projection of E onto $[x_{n_k}]$ along $[x_{n_k'}]$, $P^*g \in \mathrm{sp}\,(f_{n_k})$ (because $P^*f_{n_k} = f_{n_k}$, $P^*f_{n_k'} = 0$) and $\|P^*\|(P^*g/\|P^*\|)(u) = (P^*g)(u) = g(Pu) = g(u) \ge \mu\|u\|$. Thus, $(P^*g/\|P\|)(u) \ge (\mu/\|P\|)\|u\|$, so that $[f_{n_k}]$ norms $[x_{n_k}]$.

Corollary. *Let $(x_n; f_n)$ be an M-basis for E such that for every subsequence (n_k) of the integers, $[f_{n_k}]$ norms $[x_{n_k}]$. Then (x_n) is an unconditional basis of E.*

Proof. For every (n_k), by Theorem 3, $E = [x_{n_k}] \oplus [x_{n_k'}]$. The result follows from a result of Lorch [20].

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