A CHARACTERIZATION OF $U_3(2^n)$
BY ITS SYLOW 2-SUBGROUP

BY

ROBERT L. GRIESS, JR.

ABSTRACT. We determine all the finite groups having a Sylow 2-subgroup
isomorphic to that of $U_3(2^n)$, $n \geq 3$. In particular, the only such simple groups
are the $U_3(2^n)$.

1. Introduction. Let $N$ be the normalizer of a Sylow 2-subgroup in the pro-
jective special unitary group $U_3(2^n)$, $n \geq 3$. In [1], M. Collins proved

**Theorem.** Suppose $G$ is a finite simple group with Sylow 2-subgroup $S$. If
$N_G(S)/O(N_G(S)) \cong N$, then $G \cong U_3(2^n)$.

In this paper, we remove the hypothesis on the normalizer.

**Theorem 1.** If $G$ is a finite simple group with Sylow 2-subgroup isomorphic
to that of $U_3(q)$, $q = 2^n$, $n \geq 3$, then $G \cong U_3(q)$.

**Theorem 2.** If $G$ is a finite group with Sylow 2-subgroup isomorphic to that
of $U_3(q)$, $q = 2^n$, $n \geq 3$, then either

(i) $G$ is solvable of 2-length one; or

(ii) $G/O(G)$ has a normal subgroup of odd index isomorphic to $U_3(q)$.

These results are a step in the general program of characterizing simple
groups by their Sylow 2-subgroups. Using Collins' method as a skeleton for our
proof, we analyze the possibilities for the action of $N_G(S)$ on $S$ and $Z(S)$, where
$S$ is a Sylow 2-subgroup of $G$, then generalize certain of his arguments. For
$q = 4$, the conclusion of Theorem 1 was obtained by R. Lyons [5]. Since the
author proved these theorems, he has learned that M. Collins has obtained sim-
ilar results. Collins' methods and our methods of proof differ significantly,
however.

2. Notation and assumed results. Group theoretic notation is standard (e.g.,
see [3]). For a group $X$, $O(X)$ denotes the largest normal subgroup of odd order.
$A_G(X)$ denotes $N_G(X)/C_G(X)$ for $X \subseteq G$. We use the bar convention for denoting
homomorphic images. The 2-rank \( m(X) \) of a group \( X \) is the minimal number of generators for an elementary abelian 2-subgroup of maximal order in \( X \).

We need some information about the structure of \( S \). Throughout this paper, \( q = 2^n, n \geq 3 \).

**Lemma 1.** (i) \( \Omega_1(S) = Z(S) = S' = \Phi(S) \).
(ii) If \( t \in S, t \notin Z(S) \), then \([t, S] = Z(S)\) and \( C_S(t) \) is abelian of order \( q^2 \).
(iii) If \( H < Z(S) \), then \( Z(S/H) = Z(S)/H \); for \( t \) above, \( C_{S/H}(tH) \) has order \( q^2 \) as \([tH, S/H] = Z(S/H)\).

**Proof.** The assertions follow from inspecting this presentation of \( S \):
\[
S = \{x(a, b) \mid a, b \in GF(q^2), \quad aa^\sigma = b + ba \text{ where } (a) = \text{Gal}(GF(q^2)/GF(q))
\]
and \( x(a, b)x(c, d) = x(a + c, b + d + ac^\sigma) \}.

We also use, sometimes without comment, the Feit-Thompson theorem on the solvability of groups of odd order \( 2 \), Walter’s classification of groups with abelian Sylow 2-subgroups \( 8 \), Suzuki’s classification of groups with 2-closed centralizers of involutions \( 7 \), and the following result of Gorenstein-Walter (see \( 4 \) for definitions).

**Theorem C.** If \( G \) is a connected, balanced group, with 2-rank \( m(G) \geq 3 \), \( O(G) = 1 \), and the centralizer of every involution is 2-generated, then \( O(C_G(x)) = 1 \), for all involutions \( x \in G \).

Since Theorem 1 follows directly from Theorem 2, we may assume henceforth that \( G \) is a group satisfying the hypotheses of Theorem 2, and that \( O(G) = 1 \).

3. The proof.

**Lemma 2.** Suppose \( \alpha \in A_G(S), 1 \neq |\alpha| \) is odd, and \( \alpha \) acts trivially on \( Z(S) \). Then \( \alpha \) acts fixed point freely on \( S/Z(S) \).

**Proof.** Write \( S = S/Z(S) \) and suppose \( x \in S, \alpha x = x \). Then, as \( C_S(x) = C_S(y) \), for any \( x, y \in S, \alpha \) stabilizes \( C_S(x) \), for \( x \in x \). By Fitting’s theorem, \( \alpha \) is trivial on \( C_S(x) \) since \( C_S(x) \) is \( \alpha \)-invariant, abelian, and \( Z(S) = \Omega_1(C_S(x)) \).

Let \( L = L_1 \oplus L_2 \) be the Lie algebra associated with \( S \) and let \( M \) be the image of \( C_S(x) \) in \( L_1 \). Let \( L_0 = L_{10} \oplus L_{20} \), \( M_0 \) be the above objects tensored with an algebraically closed field \( k \) of characteristic 2. In \( M_0 \), \( \alpha \) has eigenvectors \( \xi_1, \ldots, \xi_n \) for the eigenvalue 1. Let \( \eta_1, \ldots, \eta_n \) be a complementary set of eigenvectors in \( L_{10} \) for the \( n \) remaining eigenvalues \( \lambda_1, \ldots, \lambda_n \in k \).

Since \([M_0, N_0] = 0\) in \( L_{20} \), \( k\eta_1 \oplus \cdots \oplus k\eta_n = N_0 \) is an \( \alpha \)-invariant complement to \( M_0 \) in \( L_{10} \) and the pairing \( (n, m) \mapsto [n, m] \in L_{20} \), \( n \in N_0, m \in M_0 \) is non-degenerate, or else some element of \( C_S(x) \setminus Z(S) \) has too large a centralizer in \( S \).
So, $L_2$ is spanned by all $[\eta_i, \xi_j]$, which are eigenvectors for the values $\lambda_i \cdot 1 = \lambda_i$. Since $\alpha$ is trivial on $Z(S) = L_2$, $\lambda_i = 1$. Thus, $\alpha$ is trivial on $S/Z(S) = L_1$. By 5.3.2 of [3], $\alpha = 1$. This proves the lemma.

Lemma 3. $N_G(Z(S))$ is solvable of 2-length 1.

Proof. It suffices to prove the statement for $C_G(Z(S))$, since $|N_G(Z(S))/C_G(Z(S))|$ is odd.

Set $D = C_G(Z(S)) = C_G(Z(S))/O(C_G(Z(S)))$. Then, $O(D) = 1$, $Z(S) = Z(D)$. Set $E = D/Z(S)$; $E$ has abelian Sylow 2-subgroups.

Suppose $E$ is nonsolvable. Then $E$ has a normal subgroup $F$ of odd index, where $F$ is the direct product of an elementary abelian 2-group, and at least one Janko group, group of Ree type, or $L_2(q)$, $q = 3, 5 \mod 8$, $q > 5$ or $4|q)$. Let $N$ be the normalizer in $F$ of a Sylow 2-subgroup $S^*$. By the Frattini argument, $N$ is a quotient of $N_G(Z(S)) \cap C_G(Z(S))$. Lemma 2 then implies that any element of $A_F(S^*)$ acts fixed point freely on $S^*$. Hence the only possibility is $F = L_2(q)$. If $F$ is the preimage of $F$ in $D$, $F$ is a nonsplit perfect extension of $F$ by $Z(S)$ because the induced extension $\bar{S}$ of $S^*$ has $Z(S) \subseteq \bar{S}$. But $Z(S)$ is noncyclic while the multiplier of $L_2(q)$ is always cyclic [6], contradiction.

Thus, $E$ is solvable, and so is $C_G(Z(S))$.

Definition. Choose $z \in Z(S)^w$, and set $E_1 = C_G(z)/O(C_G(z))(z)$. Let $\xi_1 = \{E_1\}$. Define families $\xi_i$, $i = 2, \ldots, n$, of sections of $E_1$ as follows: we say $E \in \xi_i$ if there is an $F \in \xi_{i-1}$ and an involution $\zeta \in Z(T)$, $T$ a Sylow 2-subgroup of $F$, with $E = C_F(\zeta)/O(C_F(\zeta))(\zeta)$.

$E_i$ denotes a typical member of $\xi_i$, and $S_i$ denotes the image of $S$ in $E_i$ under the obvious sequence of homomorphisms.

Proposition. Each $E_i$ is solvable and 2-closed.

We use downward induction on $i$. The proof goes in a sequence of lemmas which are directed toward using Theorem C. In what follows, the $\zeta$ of the definition may be assumed to lie in $S_{i-1}$.

Lemma 4. Let $\nu$ be an involution in $S_i$ not in $Z(S_i)$, for $i < n$. Then $\nu$ is not conjugate in $E_i$ to an element of $Z(S_i)$.

Proof. By Lemma 1(iii), $S_i$ is nonabelian. Suppose $\nu$ is conjugate to $\zeta \in Z(S_i)$. By regarding $E_i$ as a section of $C_G(Z)$, consider $S_i$ as a quotient of $S$, and see that the preimage in $S$ of $\langle \nu \rangle$ has exponent 4, while the preimage of $\langle \zeta \rangle$ is elementary, contradiction. The lemma follows.

In the next four lemmas, when $i < n$, $\nu$ has the above meaning, and when $i = n$, $\nu$ is any involution of $S_n$. We may drop the subscript and write $E$ for $E_i$ when confusion is unlikely.
Lemma 5. A Sylow 2-subgroup of $C_E(v)$ is contained in any Sylow 2-subgroup of $E$ in which $v$ lies.

Proof. Use Lemmas 1(iii) and 4.

Lemma 6. $C_E(v)$ is solvable of 2-length 1.

Proof. Express $K = C_E(v)/O(C_E(v))$ as a section $K = A/M$ of $C_G(z)$. The lemma will follow once we show that $A/M$ is covered by a subgroup of $N_G(Z(S))$, by Lemma 3.

Let $w \in A$ represent $v$ with $w^2 = t \in Z(S)^A$. Let $T$ be a Sylow 2-subgroup of $M$, $T \subseteq Z(S)$. Let $(\zeta_0, \ldots, \zeta_{i-1})$ be the sequence of involutions defining $E$, i.e., $\zeta_0 = z$, $\zeta_1 \in E_1$, $\ldots$, etc. We may choose an involution $z_j \in T$ representing $\zeta_j$.

We claim $N_A(T)$ acts trivially on $T = \langle z_0, \ldots, z_{i-1} \rangle$. For $i - 1 = 0$, this is obvious, so assume $i - 1 > 0$. Consider $K$ as a quotient of a subgroup $H$ of $C_{E_{i-1}}(\zeta_{i-1})$, and write $H$ as a quotient $A/B$ of $C_G(z)$, with $M \supseteq B$. By the Frattini argument, $A = B \cdot N_A(T)$. Since $z_{i-1} \in T$ maps to an element of $Z(A/B)$, $N_A(T)$ stabilizes the normal series $T \supset T_0 \supset 1$, where $T_0 = T \cap B$ is a Sylow 2-subgroup of $B$. Now $N_A(T)$ acts trivially on $T/T_0 \cong Z_2$, and, by induction, is trivial on $T_0$. So, $N_A(T)$ induces a 2-group of automorphisms on $T$, by 5.3.2 of [3]. But $T$ is contained in a Sylow 2-center of $C_G(z)$. Hence, $N_A(T)$ acts trivially, i.e., $N_A(T) = C_A(T)$.

Now, set $C = C_{C_A(T)}(w)$, $C_j = \{x \in C_A(T)[w, x] \in \langle z_0, \ldots, z_j \rangle\}$ for $j = 0, \ldots, i - 1$. Then, $C_{i-1}$ covers $A/M$, $|C_j : C_{j-1}| = 2$, for $j = 1, \ldots, i - 1$, and $|C_0 : C| = 2$; also, $C$ and $C_0, \ldots, C_{i-1}$ are all normal in $C_{i-1}$, and these groups have common core $O(C)$. Now, $T \subseteq C$ and $U = C\langle w \rangle$ is abelian of exponent 4, order $q^2$. So, $T \subseteq Z(S) = \Phi(U)$. Since $U$ is abelian and $\Omega_1(U) = \Phi(U)$, Walter's classification implies $U \leq C \leq C_{i-1} = C_{i-1}/O(C)$; hence $Z(S) \leq C$. Now, $Z(S) \leq S \cap C_{i-1}$ and $C(S \cap C_{i-1}) = C_{i-1}$. Therefore, $Z(S) \leq C_{i-1}$. This means $A/M$ is covered by subgroup of $N_G(Z(S))$, which is solvable of 2-length one. This proves the lemma.

Lemma 7. $C_E(v)$ is 2-generated.

Proof. Set $\Gamma = \Gamma_{C_{i+1}^2}$, where $C_i = C_{S_i^2}(v)$. For all $i$, $C_i$ contains a four-group disjoint from $\langle v \rangle$. So, $O(C_E(v)) \subseteq \Gamma$. But $X = C_E(v)/O(C_E(v))$ is 2-closed, and $O_2(X)$ contains a four-group. Hence, the Frattini argument implies that $C_E(v)$ is 2-generated.

Lemma 8. If $t$ is an involution in $C_E(v)$, then $O(C_E(t)) \subseteq O(C_E(v))$.

Proof. Let $D = C_E(v)/O(C_E(v))$ and let $\tilde{t} \in S_i$ be the image in $D$ of $t \in S_i$.
Suppose $i < n$. $O_2(D)$ is a Sylow 2-subgroup of $D$ and any nonidentity element of odd order in $D$ acts nontrivially on $Z(O_2(D))$, by Lemmas 2, 3, 6. Thus, $Z(O_2(D))$ normalizes no subgroup of odd order in $D$. Since $Z(O_2(D)) \subset C_E(t) = 1$, $O(C_E(t)) = 1$, which implies the lemma.

Suppose $i = n$. Then $O_2(D)$ is abelian and every element of odd order in $D$ acts fixed point freely on $O_2(D)$. So, the centralizer in $D$ of any $t$ is a 2-group. Again, the lemma holds.

**Lemma 9.** The proposition holds.

**Proof.** By construction, each $O(E_i) = 1$. We argue by downward induction on $i$.

Let $i = n$. Then $E_n$ has abelian Sylow 2-subgroups and is a section of $C_G(Z(S))$ by the proof of Lemma 6 (take $T = Z(S)$ in that notation). Hence, $E_n$ is solvable by Lemma 3 and $E_n = O_{2,2}(E_n)$. So, for $i = n$, the lemma holds.

Now, let $F \in E_i$, $i < n$. For any $\zeta \in Z(S_i)$, $E = C_F(\zeta)\langle O(C_F(\zeta))\rangle \in E_{i+1}$ is solvable and 2-closed by induction. For an involution $v$ outside a Sylow 2-center, $C_F(v)$ is solvable of length 1, by Lemma 6.

We wish to show balance holds in $F$. By Lemma 8, it suffices to prove, for $t \in S_i$, that the image of $O(C_F(t))$ in $C = C_F(\zeta)\langle O(C_F(\zeta))\rangle$ is 1. Now, $E = C_F(\zeta)\langle O(C_F(\zeta))\rangle$ is solvable and 2-closed. Imitating the argument that $N_A(T) = C_A(T)$ in the proof of Lemma 6, we get that $C$ is isomorphic to a subgroup $C^*$ of odd index in $N^* = (N_G(S) \cap C_G(T))O(N_G(S))/O(N_G(S))T$, where $T \subset Z(S)$. If $S^*$ is the image of $S$ in $N^*$, we will have $O(C_{S^*}(t)) = 1$ for any involution $t \in C^*$, provided we show $C_{S^*}(t)$ normalizes no subgroup of odd order in $C^*$.

If $t \in Z(S^*)\langle \rangle$, clearly $\delta^* = C_{S^*}(t)$ normalizes no subgroup of odd order in $N^*$. If $t$ is an involution in $S^*$ not in $Z(S^*)$ with $O(C_{N^*}(t)) \neq 1$, then $Z(S^*)$ normalizes, hence centralizes (as $N^*$ is 2-closed), a nontrivial subgroup of odd order. Let $x \in O(C_{N^*}(t))\langle \rangle$ and let $y \in N_G(S) \cap C_G(T)$ represent $x$, $|y|$ odd. Lemma 2 implies that $y$ acts nontrivially on $Z(S)$ since $y$ is nontrivial on $S$ and fixes the coset of $Z(S)$ in $S$ corresponding to $t$. But $y$ centralizes $T$, hence must act nontrivially on $Z(S)/T \cong Z(S^*)$, as $|y|$ is odd. So, $x$ acts nontrivially on $Z(S^*)$, a contradiction. This gives $O(C_{S^*}(t)) = 1$ in all cases. Therefore, balance holds in $F$.

Next, we show that $C_F(t)$ is 2-generated for every involution $t$ of $F$. If $t$ is not 2-central, this is proven in Lemma 7. Let $t$ be 2-central. There is a four-group in $C_F(t)$ disjoint from $(t)$. So, $O(C_F(t)) \subset \Gamma = \Gamma_{S_i,2}$. Consider $C_F(t) / O(C_F(t))$. Since this group is 2-closed, it is 2-generated because, for $i < n - 1$, a Sylow 2-center has rank at least 2, and, for $i = n - 1$, a Sylow 2-subgroup is extra special of order $2^{2n+1}$, hence contains a four-group. The Frattini argument now shows that $C_F(t)$ is 2-generated.
Now, we show $F$ is connected. For $i < n - 1$, a Sylow 2-center is noncyclic, whence connectivity. For $i = n - 1$, the extra special Sylow 2-subgroup contains an elementary abelian normal subgroup of order $2^n \geq 2^3$. By the remark on p. 4 of [4], $F$ is connected in this case, too.

Since $O(F) = 1$, $m(F) \geq 3$, Theorem C implies $O(C_F(t)) = 1$ for every involution $t \in F$. Our previous arguments then imply that every involution of $F$ has 2-closed centralizer. By Suzuki's classification, $S_i$ does not occur as a Sylow 2-subgroup of a simple group. So, $F$ is not simple. We want to show $F$ solvable.

If $O_2(F) \neq 1$, then $W = Z(S_i) \cap O_2(F) \neq 1$. Lemma 4 implies that $W$ is strongly closed in $O_2(F)$ with respect to $F$. Hence $W < F$. Since $|F:C_F(W)|$ is odd, $F$ is solvable and 2-closed because $C_F(W)/\langle \zeta \rangle$, $\zeta \in W^H$, is contained in some $E \in \mathcal{E}_{i+1}$ and $E$ is solvable and 2-closed by induction. If $O_2(F) = 1$, then Theorem 2 of [7] implies that $F$ has cyclic, quaternion, or semidihedral Sylow 2-subgroups, contradicting $m(F) \geq 3$. Thus, $F$ is solvable, and the proposition is proven.

**Lemma 10.** Let $z$ be an involution of $S$. Then $C_G(z) \subseteq N_G(S)$.

**Proof.** We know $C_G(z)$ is solvable of 2-length 1. Since $m(S) \geq 3$, Theorem C implies $O(C_G(z)) = 1$ as $O(G) = 1$. Thus $O_G(z)$ is 2-closed, i.e., $C_G(z) \subseteq N_G(S)$.

**Lemma 11.** Theorem 2 holds.

**Proof.** For each involution $z$ of $G$, $C_G(z)$ is 2-closed. Thus, Suzuki's classification implies Theorem 2.

**REFERENCES**


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MICHIGAN

48104