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A CHARACTERIZATION OF $U_3(2^n)$ BY ITS SYLOW 2-SUBGROUP

BY

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ABSTRACT. We determine all the finite groups having a Sylow 2-subgroup isomorphic to that of $U_3(2^n)$, $n \geq 3$. In particular, the only such simple groups are the $U_3(2^n)$.

1. **Introduction.** Let N be the normalizer of a Sylow 2-subgroup in the projective special unitary group $U_3(2^n)$, $n \geq 3$. In [1], M. Collins proved

Theorem. *Suppose G is a finite simple group with Sylow 2-subgroup S. If $N_G(S)/O(N_G(S)) \cong N$, then $G \cong U_3(2^n)$.*

In this paper, we remove the hypothesis on the normalizer.

Theorem 1. *If G is a finite simple group with Sylow 2-subgroup isomorphic to that of $U_3(q)$, $q = 2^n$, $n \geq 3$, then $G \cong U_3(q)$.*

Theorem 2. *If G is a finite group with Sylow 2-subgroup isomorphic to that of $U_3(q)$, $q = 2^n$, $n \geq 3$, then either*

(i) *G is solvable of 2-length one*; *or*

(ii) *$G/O(G)$ has a normal subgroup of odd index isomorphic to $U_3(q)$.*

These results are a step in the general program of characterizing simple groups by their Sylow 2-subgroups. Using Collins' method as a skeleton for our proof, we analyze the possibilities for the action of $N_G(S)$ on S and $Z(S)$, where S is a Sylow 2-subgroup of G, then generalize certain of his arguments. For $q = 4$, the conclusion of Theorem 1 was obtained by R. Lyons [5]. Since the author proved these theorems, he has learned that M. Collins has obtained similar results. Collins' methods and our methods of proof differ significantly, however.

2. **Notation and assumed results.** Group theoretic notation is standard (e.g., see [3]). For a group X, $O(X)$ denotes the largest normal subgroup of odd order. $A_G(X)$ denotes $N_G(X)/C_G(X)$ for $X \subseteq G$. We use the bar convention for denoting

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homomorphic images. The 2-rank $m(X)$ of a group X is the minimal number of generators for an elementary abelian 2-subgroup of maximal order in X.

We need some information about the structure of S. Throughout this paper, $q = 2^n$, $n \geq 3$.

Lemma 1. (i) $\Omega_1(S) = Z(S) = S' = \Phi(S)$.

(ii) *If* $t \in S$, $t \notin Z(S)$, *then* $[t, S] = Z(S)$ *and* $C_S(t)$ *is abelian of order* q^2.

(iii) *If* $H < Z(S)$, *then* $Z(S/H) = Z(S)/H$; *for* t *above*, $C_{S/H}(tH)$ *has order* q^2 *as* $[tH, S/H] = Z(S/H)$.

Proof. The assertions follow from inspecting this presentation of S:

$$S = \{x(a, b) \mid a, b \in GF(q^2),\ aa^\sigma = b + b^\sigma \text{ where } \langle\sigma\rangle = \mathrm{Gal}(GF(q^2)/GF(q)) \text{ and } x(a, b)x(c, d) = x(a + c, b + d + ac^\sigma)\}.$$

We also use, sometimes without comment, the Feit-Thompson theorem on the solvability of groups of odd order [2], Walter's classification of groups with abelian Sylow 2-subgroups [8], Suzuki's classification of groups with 2-closed centralizers of involutions [7], and the following result of Gorenstein-Walter (see [4] for definitions).

Theorem C. *If* G *is a connected, balanced group, with* 2*-rank* $m(G) \geq 3$, $O(G) = 1$, *and the centralizer of every involution is* 2*-generated, then* $O(C_G(x)) = 1$, *for all involutions* $x \in G$.

Since Theorem 1 follows directly from Theorem 2, we may assume henceforth that G is a group satisfying the hypotheses of Theorem 2, and that $O(G) = 1$.

3. The proof.

Lemma 2. *Suppose* $\alpha \in A_G(S)$, $1 \neq |\alpha|$ *is odd, and* α *acts trivially on* $Z(S)$. *Then* α *acts fixed point freely on* $S/Z(S)$.

Proof. Write $\bar{S} = S/Z(S)$ and suppose $\bar{x} \in \bar{S}$, $\bar{x}^\alpha = \bar{x}$. Then, as $C_S(x) = C_S(y)$, for any $x, y \in \bar{x}$, α stabilizes $C_S(x)$, for $x \in x$. By Fitting's theorem, α is trivial on $C_S(x)$ since $C_S(x)$ is α-invariant, abelian, and $Z(S) = \Omega_1(C_S(x))$.

Let $L = L_1 \oplus L_2$ be the Lie algebra associated with S and let M be the image of $C_S(x)$ in L_1. Let $L_0 = L_{10} \oplus L_{20}$, M_0 be the above objects tensored with an algebraically closed field k of characteristic 2. In M_0, α has eigenvectors $\xi_1, \cdots, \xi_n$ for the eigenvalue 1. Let $\eta_1, \cdots, \eta_n$ be a complementary set of eigenvectors in L_{10} for the n remaining eigenvalues $\lambda_1, \cdots, \lambda_n \in k$. Since $[M_0, N_0] = 0$ in L_{20}, $k\eta_1 \oplus \cdots \oplus k\eta_n = N_0$ is an α-invariant complement to M_0 in L_{10} and the pairing $(n, m) \mapsto [n, m] \in L_{20}$, $n \in N_0$, $m \in M_0$ is nondegenerate, or else some element of $C_S(x)\backslash Z(S)$ has too large a centralizer in S.

So, L_{20} is spanned by all $[\eta_i, \xi_j]$, which are eigenvectors for the values $\lambda_i \cdot 1 = \lambda_i$. Since α is trivial on $Z(S) = L_2$, $\lambda_i = 1$. Thus, α is trivial on $S/Z(S) = L_1$. By 5.3.2 of [3], $\alpha = 1$. This proves the lemma.

Lemma 3. *$N_G(Z(S))$ is solvable of 2-length 1.*

Proof. It suffices to prove the statement for $C_G(Z(S))$, since $|N_G(Z(S))/C_G(Z(S))|$ is odd.

Set $D = \overline{C_G(Z(S))} = C_G(Z(S))/O(C_G(Z(S)))$. Then, $O(D) = 1$, $Z(\overline{S}) = Z(D)$. Set $E = D/Z(\overline{S})$; E has abelian Sylow 2-subgroups.

Suppose E is nonsolvable. Then E has a normal subgroup F of odd index, where F is the direct product of an elementary abelian 2-group, and at least one Janko group, group of Ree type, or $L_2(q)$ ($q \equiv 3, 5 \pmod 8$, $q \geq 5$, or $4|q$). Let N be the normalizer in F of a Sylow 2-subgroup S^*. By the Frattini argument, N is a quotient of $N_G(S) \cap C_G(Z(S))$. Lemma 2 then implies that any element of $A_F(S^*)$ acts fixed point freely on S^*. Hence the only possiblity is $F = L_2(q)$. If $\overline{F}$ is the preimage of F in D, $\overline{F}$ is a nonsplit perfect extension of F by $Z(\overline{S})$ because the induced extension $\overline{S}$ of S^* has $Z(\overline{S}) \subseteq \overline{S}'$. But $Z(\overline{S})$ is noncyclic while the multiplier of $L_2(q)$ is always cyclic [6], contradiction.

Thus, E is solvable, and so is $C_G(Z(S))$.

Definition. Choose $z \in Z(S)^{\#}$, and set $E_1 = C_G(z)/O(C_G(z))\langle z\rangle$. Let $\mathfrak{E}_1 = \{E_1\}$. Define families $\mathfrak{E}_i$, $i = 2, \cdots, n$, of sections of E_1 as follows: we say $E \in \mathfrak{E}_i$ if there is an $F \in \mathfrak{E}_{i-1}$ and an involution $\zeta \in Z(T)$, T a Sylow 2-subgroup of F, with $E = C_F(\zeta)/O(C_F(\zeta))\langle\zeta\rangle$.

E_i denotes a typical member of $\mathfrak{E}_i$ and S_i denotes the image of S in E_i under the obvious sequence of homomorphisms.

Proposition. *Each E_i is solvable and 2-closed.*

We use downward induction on i. The proof goes in a sequence of lemmas which are directed toward using Theorem C. In what follows, the ζ of the definition may be assumed to lie in S_{i-1}.

Lemma 4. *Let v be an involution in S_i not in $Z(S_i)$, for $i < n$. Then v is not conjugate in E_i to an element of $Z(S_i)$.*

Proof. By Lemma 1(iii), S_i is nonabelian. Suppose v is conjugate to $\zeta \in Z(S_i)$. By regarding E_i as a section of $C_G(Z)$, consider S_i as a quotient of S, and see that the preimage in S of $\langle v\rangle$ has exponent 4, while the preimage of $\langle\zeta\rangle$ is elementary, contradiction. The lemma follows.

In the next four lemmas, when $i < n$, v has the above meaning, and when $i = n$, v is any involution of S_n. We may drop the subscript and write E for E_i when confusion is unlikely.

Lemma 5. *A Sylow 2-subgroup of $C_E(v)$ is contained in any Sylow 2-subgroup of E in which v lies.*

Proof. Use Lemmas 1(iii) and 4.

Lemma 6. *$C_E(v)$ is solvable of 2-length 1.*

Proof. Express $K = C_E(v)/O(C_E(v))$ as a section $K = A/M$ of $C_G(z)$. The lemma will follow once we show that A/M is covered by a subgroup of $N_G(Z(S))$, by Lemma 3.

Let $w \in A$ represent v with $w^2 = t \in Z(S)^{\#}$. Let T be a Sylow 2-subgroup of M, $T \subseteq Z(S)$. Let $(\zeta_0, \cdots, \zeta_{i-1})$ be the sequence of involutions defining E, i.e., $\zeta_0 = z, \zeta_1 \in E_1, \cdots$, etc. We may choose an involution $z_j \in T$ representing ζ_j.

We claim $N_A(T)$ acts trivially on $T = \langle z_0, \cdots, z_{i-1}\rangle$. For $i-1 = 0$, this is obvious, so assume $i-1 > 0$. Consider K as a quotient of a subgroup H of $C_{E_{i-1}}(\zeta_{i-1})$, and write H as a quotient A/B of $C_G(z)$, with $M \supseteq B$. By the Frattini argument, $A = B \cdot N_A(T)$. Since $z_{i-1} \in T$ maps to an element of $Z(A/B)$, $N_A(T)$ stabilizes the normal series $T \supset T_0 \supset 1$, where $T_0 = T \cap B$ is a Sylow 2-subgroup of B. Now $N_A(T)$ acts trivially on $T/T_0 \cong Z_2$, and, by induction, is trivial on T_0. So, $N_A(T)$ induces a 2-group of automorphisms on T, by 5.3.2 of [3]. But T is contained in a Sylow 2-center of $C_G(z)$. Hence, $N_A(T)$ acts trivially, i.e., $N_A(T) = C_A(T)$.

Now, set $C = C_{C_A(T)}(w)$, $C_j = \{x \in C_A(T) | [w, x] \in \langle z_0, \cdots, z_j\rangle\}$ for $j = 0, \cdots, i-1$. Then, C_{i-1} covers A/M, $|C_j : C_{j-1}| = 2$, for $j = 1, \cdots, i-1$, and $|C_0 : C| = 2$; also, C and $C_0, \cdots, C_{i-1}$ are all normal in C_{i-1}, and these groups have common core $O(C)$. Now, $T \subset C$ and $U = C_S(w)$ is abelian of exponent 4, order q^2. So, $T \subseteq Z(S) = \Phi(U)$. Since U is abelian and $\Omega_1(U) = \Phi(U)$, Walter's classification implies $\overline{U} \lhd \overline{C} \subseteq \overline{C}_{i-1} = C_{i-1}/O(C)$; hence $Z(\overline{S}) \lhd \overline{C}$. Now, $Z(\overline{S}) \lhd \overline{S \cap C_{i-1}}$ and $\overline{C}(\overline{S \cap C_{i-1}}) = \overline{C}_{i-1}$. Therefore, $Z(\overline{S}) \lhd \overline{C}_{i-1}$. This means A/M is covered by subgroup of $N_G(Z(S))$, which is solvable of 2-length one. This proves the lemma.

Lemma 7. *$C_E(v)$ is 2-generated.*

Proof. Set $\Gamma = \Gamma_{C_i, 2}$, where $C_i = C_{S_i}(v)$. For all i, C_i contains a four-group disjoint from $\langle v\rangle$. So, $O(C_E(v)) \subseteq \Gamma$. But $X = C_E(v)/O(C_E(v))$ is 2-closed, and $O_2(X)$ contains a four-group. Hence, the Frattini argument implies that $C_E(v)$ is 2-generated.

Lemma 8. *If t is an involution in $C_E(v)$, then $O(C_E(t)) \cap C_E(v) \subseteq O(C_E(v))$.*

Proof. Let $D = C_E(v)/O(C_E(v))$ and let $\overline{t} \in \overline{S}_i$ be the image in D of $t \in S_i$.

Suppose $i < n$. $O_2(D)$ is a Sylow 2-subgroup of D and any nonidentity element of odd order in D acts nontrivially on $Z(O_2(D))$, by Lemmas 2, 3, 6. Thus, $Z(O_2(D))$ normalizes no subgroup of odd order in D. Since $Z(O_2(D)) \subseteq C_E(\bar{t}) = 1$, $O(C_E(E)) = 1$, which implies the lemma.

Suppose $i = n$. Then $O_2(D)$ is abelian and every element of odd order in D acts fixed point freely on $O_2(D)$. So, the centralizer in D of any $\bar{t}$ is a 2-group. Again, the lemma holds.

Lemma 9. *The proposition holds.*

Proof. By construction, each $O(E_i) = 1$. We argue by downward induction on i.

Let $i = n$. Then E_n has abelian Sylow 2-subgroups and is a section of $C_G(Z(S))$ by the proof of Lemma 6 (take $T = Z(S)$ in that notation). Hence, E_n is solvable by Lemma 3 and $E_n = O_{2,2'}(E_n)$. So, for $i = n$, the lemma holds.

Now, let $F \in \mathfrak{E}_i$, $i < n$. For any $\zeta \in Z(S_i)$, $E = C_F(\zeta)/O(C_F(\zeta))\langle\zeta\rangle \in \mathfrak{E}_{i+1}$ is solvable and 2-closed by induction. For an involution v outside a Sylow 2-center, $C_F(v)$ is solvable of 2-length 1, by Lemma 6.

We wish to show balance holds in F. By Lemma 8, it suffices to prove, for $t \in S_i$, that the image of $O(C_F(t))$ in $C = C_F(\zeta)/O(C_F(\zeta))$ is 1. Now, $E = C_F(\zeta)/O(C_F(\zeta))\langle\zeta\rangle$ is solvable and 2-closed. Imitating the argument that $N_A(T) = C_A(T)$ in the proof of Lemma 6, we get that C is isomorphic to a subgroup C^* of odd index in $N^* = (N_G(S) \cap C_G(T))O(N_G(S))/O(N_G(S))T$, where $T \subset Z(S)$. If S^* is the image of S in N^*, we will have $O(C_{C^*}(t)) = 1$ for any involution $t \in C^*$, provided we show $C_{S^*}(t)$ normalizes no subgroup of odd order in C^*.

If $t \in Z(S^*)^{\#}$, clearly $S^* = C_{S^*}(t)$ normalizes no subgroup of odd order in N^*. If t is an involution in S^* not in $Z(S^*)$ with $O(C_{N^*}(t)) \neq 1$, then $Z(S^*)$ normalizes, hence centralizes (as N^* is 2-closed), a nontrivial subgroup of odd order. Let $x \in O(C_{N^*}(t))^{\#}$ and let $y \in N_G(S) \cap C_G(T)$ represent x, $|y|$ odd. Lemma 2 implies that y acts nontrivially on $Z(S)$ since y is nontrivial on S and fixes the coset of $Z(S)$ in S corresponding to t. But y centralizes T, hence must act nontrivially on $Z(S)/T \cong Z(S^*)$, as $|y|$ is odd. So, x acts nontrivially on $Z(S^*)$, a contradiction. This gives $O(C_{C^*}(t)) = 1$ in all cases. Therefore, balance holds in F.

Next, we show that $C_F(t)$ is 2-generated for every involution t of F. If t is not 2-central, this is proven in Lemma 7. Let t be 2-central. There is a four-group in $C_F(t)$ disjoint from $\langle t\rangle$. So, $O(C_F(t)) \subseteq \Gamma = \Gamma_{S_i,2}$. Consider $C_F(t)/O(C_F(t))$. Since this group is 2-closed, it is 2-generated because, for $i < n-1$, a Sylow 2-center has rank at least 2, and, for $i = n-1$, a Sylow 2-subgroup is extra special of order 2^{2n+1}, hence contains a four-group. The Frattini argument now shows that $C_F(t)$ is 2-generated.

Now, we show F is connected. For $i < n - 1$, a Sylow 2-center is noncyclic, whence connectivity. For $i = n - 1$, the extra special Sylow 2-subgroup contains an elementary abelian normal subgroup of order $2^n \geq 2^3$. By the remark on p. 4 of [4], F is connected in this case, too.

Since $O(F) = 1$, $m(F) \geq 3$, Theorem C implies $O(C_F(t)) = 1$ for every involution $t \in F$. Our previous arguments then imply that every involution of F has 2-closed centralizer. By Suzuki's classification, S_i does not occur as a Sylow 2-subgroup of a simple group. So, F is not simple. We want to show F solvable.

If $O_2(F) \neq 1$, then $W = Z(S_i) \cap O_2(F) \neq 1$. Lemma 4 implies that W is strongly closed in $O_2(F)$ with respect to F. Hence $W \lhd F$. Since $|F: C_F(W)|$ is odd, F is solvable and 2-closed because $C_F(W)/\langle\zeta\rangle$, $\zeta \in W^{\#}$, is contained in some $E \in \mathcal{E}_{i+1}$ and E is solvable and 2-closed by induction. If $O_2(F) = 1$, then Theorem 2 of [7] implies that F has cyclic, quaternion, or semidihedral Sylow 2-subgroups, contradicting $m(F) \geq 3$. Thus, F is solvable, and the proposition is proven.

Lemma 10. *Let z be an involution of S. Then $C_G(z) \subseteq N_G(S)$.*

Proof. We know $C_G(z)$ is solvable of 2-length 1. Since $m(S) \geq 3$, Theorem C implies $O(C_G(z)) = 1$ as $O(G) = 1$. Thus $O_G(z)$ is 2-closed, i.e., $C_G(z) \subseteq N_G(S)$.

Lemma 11. *Theorem 2 holds.*

Proof. For each involution z of G, $C_G(z)$ is 2-closed. Thus, Suzuki's classification implies Theorem 2.

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