

STEENROD SQUARES IN SPECTRAL SEQUENCES. I

BY

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ABSTRACT. We define two kinds of Steenrod operations on the spectral sequence of a bisimplicial coalgebra. We show these operations compatible with the differentials of the spectral sequence, and with the Steenrod squares defined on the cohomology of the total complex. We give a general rule for computing the operations on E_2 .

The purpose of this paper is to define Steenrod squaring operations on a class of spectral sequences. In this paper's sequel (hereafter called "II") we describe these actions in some special cases.

Let \mathcal{O} be the category of finite ordered sets and nondecreasing maps. By a bisimplicial coalgebra we mean a purely contravariant functor from $\mathcal{O} \times \mathcal{O}$ to the category of commutative Z_2 coalgebras. We can regard the vector space dual of any bisimplicial coalgebra as a double cochain complex, and filter it in the usual way. Our main result is that on the E_r term of the resulting spectral sequence, for each $r \geq 2$, are defined Steenrod operations

$$(0.1) \quad \text{Sq}^k: E_r^{p,q} \rightarrow E_r^{p,q+k} \quad (0 \leq k \leq q),$$

$$(0.2) \quad \text{Sq}^k: E_r^{p,q} \rightarrow E_r^{p+k-q, 2q} \quad (k \geq q).$$

These operations commute with the differentials of the spectral sequence (Proposition 1.4). In particular they are defined on E_∞ , where they are compatible with the action of Steenrod squares on the cohomology of the total complex (Proposition 1.5).

We thus obtain a larger set of operations than Rector [12] and Smith [15], who consider the related problem of defining Steenrod squares on the spectral sequence of a "mixed" bisimplicial object . . . , i.e., a functor from $\mathcal{O} \times \mathcal{O}$ that is contravariant in one variable and covariant in the other. Rector and Smith obtain only operations of type (0.1). Whether operations of type (0.2) can be defined for mixed bisimplicial objects remain an open problem.

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In II we make a number of applications. The first is to the change of rings spectral sequence, relating the cohomology of an extension of Hopf algebras to the cohomologies of base and fiber. The extensions we consider are in general very bad . . . noncentral, algebra twisted, and even coalgebra twisted. Still we are able to define both types of operation (0.1), (0.2) and show how to compute them. Our second application is to the spectral sequence constructed by Moore in [11], passing from the homology of a topological group to the cohomology of its classifying space. Again we compute both kinds of Steenrod squares. Our final application is to the Serre spectral sequence: we use our general constructions to recover the results of [1], [7] and [18].

References are to the bibliography at the end of II. We owe a special debt to Kristensen's work on the Serre sequence [7]. Our Propositions 1.4 and 1.5 are formally identical with the results obtained by Kristensen in the special case.

1. Definitions and results. Our object in this section is to give enough definitions to state precisely the main results of the present paper. We then state the results. Proofs are given in later sections.

We take algebras, coalgebras and Hopf algebras over Z_2 in the sense of [10, Definitions 1.1, 2.1 and 4.1], but we add the convention that all comultiplications are commutative. We write comultiplications $\psi: \Lambda \rightarrow \Lambda \otimes \Lambda$, multiplications $\mu: \Lambda \otimes \Lambda \rightarrow \Lambda$, counits $\epsilon: \Lambda \rightarrow Z_2$, units $\eta: Z_2 \rightarrow \Lambda$. If Λ is a Hopf algebra then by a (left) Λ -coalgebra we mean a coalgebra M that is also a (left) Λ -module, in such a way that $\psi: M \rightarrow M \otimes M$ and $\epsilon: M \rightarrow Z_2$ are Λ -morphisms. Here Λ acts diagonally on $M \otimes M$ [10, Definition 4.2] and tensor products are always over Z_2 . Similarly we define (left) Λ -algebras and (left) Λ -Hopf algebras.

We write d_i^v, s_i^v for the "vertical" face and degeneracy operators on a bisimplicial object, and d_i^h, s_i^h for the horizontal operators. If X is a bisimplicial object over a category \mathcal{C} then by an augmentation for X we mean a simplicial \mathcal{C} object R , and a morphism of simplicial \mathcal{C} objects $\lambda: X_{0,-} \rightarrow R_-$ satisfying $\lambda d_1^h = \lambda d_0^h: X_{1,q} \rightarrow R_q$ for all $q \geq 0$.

Suppose Λ a Hopf algebra and R a simplicial Λ -module. Then we write CR for the Λ -chain complex defined by $(CR)_n = R_n, d = \sum_{i=0}^n d_i$. If X is a bisimplicial Λ -module we write CX for the double Λ -chain complex given by $(CX)_{p,q} = X_{p,q}, d^v = \sum_{i=0}^q d_i^v; d^h = \sum_{j=0}^p d_j^h$. We will also regard CX as a Λ -chain complex with increasing filtration

$$(1.1) \quad F_p(CX)_n = \sum_{i=0}^p \bigoplus (CX)_{i,n-i}$$

Whenever in this paper a double chain complex appears with a single subscript, the subscript refers to total degree.

Suppose X is a bisimplicial Λ -module and N a Λ -module. Then we write

$\text{Hom}_\Lambda(CX, N)$ for the Z_2 -cochain complex with decreasing filtration defined by

$$(1.2) \quad F^p \text{Hom}_\Lambda^n(CX, N) = [f: (CX)_n \rightarrow N \mid f \text{ a } \Lambda\text{-morphism; } f(F_{p-1}CX) = 0].$$

We write $(E_r \text{Hom}_\Lambda(CX, N), d_r)$, or simply (E_r, d_r) , for the spectral sequence of this filtered cochain complex, and refer to it as the spectral sequence of the bisimplicial Λ -coalgebra X . If X has an augmentation $\lambda: X \rightarrow R$, then λ defines in an obvious way a Z_2 -homomorphism

$$(1.3) \quad \lambda^*: H^*(\text{Hom}_\Lambda(CR, N)) \rightarrow H^*(\text{Hom}_\Lambda(CX, N)).$$

In the cases of interest to us λ^* will be an isomorphism. Then $H^*(\text{Hom}_\Lambda(CR, N))$ is filtered, and is the target of the spectral sequence (E_r, d_r) . We write

$$(1.4) \quad \rho: F^p H^{p+q}(\text{Hom}_\Lambda(CR, N)) \rightarrow E_\infty^{p,q}$$

for the usual projection.

We can now state the results. Supposing Λ a Hopf algebra, R a simplicial Λ -coalgebra, X a bisimplicial Λ -coalgebra, and N a commutative Λ -algebra. We define products and Steenrod operations in $H^*(\text{Hom}_\Lambda(CR, N))$, as well as products and Steenrod operations in $H^*(\text{Hom}_\Lambda(CX, N))$ (see (3.8) and (3.9)).

Proposition 1.1. *Suppose $\lambda: X \rightarrow R$ is an augmentation. Then λ^* of (1.3) preserves products and Steenrod squares.*

Further, we define products in the spectral sequence $(E_r \text{Hom}_\Lambda(CX, N), d_r)$

$$(1.5) \quad E_r^{p,q} \otimes E_r^{p',q'} \rightarrow E_r^{p+p',q+q'}$$

(see (4.5)); and Steenrod squares

$$(1.6) \quad \text{Sq}^k: E_r^{p,q} \rightarrow E_r^{p,q+k} \quad (0 \leq k \leq q),$$

$$(1.7) \quad \text{Sq}^k: E_r^{p,q} \rightarrow E_r^{p+k-q,2q} \quad (k \geq q).$$

(See Proposition 4.2.)

Proposition 1.2. *Products and Steenrod squares on E_2 determine products and Steenrod squares on E_r for all $r \geq 2$. E.g. if $u \in E_2$ survives to E_r and represents $[u] \in E_r$, then $\text{Sq}^k u$ survives to E_r , and $[\text{Sq}^k u] = \text{Sq}^k [u]$.*

Proposition 1.3. *Under the pairing (1.5), (E_r, d_r) is a differential algebra.*

Proposition 1.4. *Suppose $u \in E_r^{p,q}$ ($r \geq 2$).*

Case 1. $k \leq q - r + 1$. *Then $d_r \text{Sq}^k u = \text{Sq}^k d_r u$.*

Case 2. $q - r + 1 \leq k \leq q$. *Then $\text{Sq}^k u$ survives to a class $[\text{Sq}^k u] \in E_t^{p,q+k}$, where $t = 2r + k - q - 1$. Also $\text{Sq}^k d_r u$ survives to a class $[\text{Sq}^k d_r u] \in E_t^{p+t,2q-2r+2}$, and $d_t [\text{Sq}^k u] = [\text{Sq}^k d_r u]$.*

Case 3. $q \leq k$. Then $Sq^k u$ survives to a class $[Sq^k u]$ in $E_t^{p+k-q, 2q}$ where $t = 2r - 1$. Also $Sq^k d_r u$ survives to a class $[Sq^k d_r u]$ in $E_t^{p+t+k-q, 2q-2r+2}$, and $d_t[Sq^k u] = [Sq^k d_r u]$.

Finally we observe that (1.6), (1.7) imply in particular that we have defined squaring operations on E_∞ . These are related to operations on the target of the spectral sequence:

Proposition 1.5. *Suppose X a bisimplicial Λ -coalgebra with augmentation $\lambda: X \rightarrow R$, suppose N a commutative Λ -algebra, and suppose λ^* of (1.3) an isomorphism. Suppose $u \in F^p H^{p+q}(\text{Hom}_\Lambda(CR, N))$ and $v \in F^r H^{r+s}(\text{Hom}_\Lambda(CR, N))$. Then $u \cdot v \in F^{p+r} H^*(\text{Hom}_\Lambda(CR, N))$, and $\rho(u \cdot v) = \rho u \cdot \rho v$ (ρ as in (1.4)). Further:*

- Case 1. $0 \leq k \leq q$. Then $Sq^k u \in F^p H^*(\text{Hom}_\Lambda(CR, N))$ and $\rho Sq^k u = Sq^k \rho u$.
- Case 2. $q \leq k$. Then $Sq^k u \in F^{p+k-q} H^*(\text{Hom}_\Lambda(CR, N))$ and $\rho Sq^k u = Sq^k \rho u$.

The rest of this paper is devoted to proving these results, as well as some auxiliary ones. (See in particular Proposition 5.1, which gives a useful description of the products and Steenrod squares on E_2 .)

2. Simplicial Eilenberg-Zilber maps. Suppose for each integer $k \geq 0$ and for every pair of simplicial Z_2 -modules R, S we have defined a Z_2 -homomorphism $D_k = D_k(R, S): C(R \times S) \rightarrow CR \otimes CS$ homogeneous of degree k and natural in the variables R and S ; suppose that for all $k \geq 0$

$$(2.1) \quad dD_k + D_k d = D_{k-1} + TD_{k-1}T$$

where $T: C(R \times S) \rightarrow C(S \times R)$; $T: CR \otimes CS \rightarrow CS \otimes CR$ are the switching maps (read $D_{-1} = 0$). Suppose finally that $D_0: C(R \times S)_0 \rightarrow (CR \otimes CS)_0$ is the identity. Then we call the sequence (D_k) a simplicial Eilenberg-Zilber map. The existence of such a sequence is proved by Dold [3].

Suppose now Λ is a Hopf algebra. If R, S are simplicial Λ -modules we write $R \times S$ for the simplicial Λ -module $(R \times S)_p = R_p \otimes S_p$ with diagonal Λ -action. The tensor product $CR \otimes CS$ of Λ -chain complexes is again a Λ -chain complex, with diagonal Λ -action.

Proposition 2.1. *Suppose (D_k) a simplicial Eilenberg-Zilber map. Suppose Λ a Hopf algebra and R and S simplicial Λ -modules. Then $D_k: C(R \times S) \rightarrow CR \otimes CS$ is a Λ -homomorphism.*

This follows easily from the functorial properties of D_k . As a consequence we obtain for any Λ -module N and any simplicial Λ -modules R, S the Z_2 -homomorphisms $D_k^*: \text{Hom}_\Lambda(CR \otimes CS, N) \rightarrow \text{Hom}_\Lambda(C(R \times S), N)$ satisfying the dual of (2.1).

Now suppose N a commutative Λ -algebra, and R a simplicial Λ -coalgebra. We define a map of cochain complexes

$$(2.2) \quad \phi: \text{Hom}_\Lambda(CR, N) \otimes \text{Hom}_\Lambda(CR, N) \rightarrow \text{Hom}_\Lambda(CR \otimes CR, N)$$

by assigning to the pair $x: CR \rightarrow N, y: CR \rightarrow N$, the cochain $CR \otimes CR \xrightarrow{x \otimes y} N \otimes N \xrightarrow{\mu} N$. We then define cochain operations $\mu: \text{Hom}_\Lambda^p(CR, N) \otimes \text{Hom}_\Lambda^q(CR, N) \rightarrow \text{Hom}_\Lambda^{p+q}(CR, N)$ and $S^k: \text{Hom}_\Lambda^n(CR, N) \rightarrow \text{Hom}_\Lambda^{n+k}(CR, N)$ by the customary formulas $\mu(x \otimes y) = \psi^* D_0^* \phi(x \otimes y)$; $S^k x = \psi^* D_{n-k}^* \phi(x \otimes x) + \psi^* D_{n-k+1}^* \phi(x \otimes \delta x)$. Here $\psi: R \rightarrow R \times R$ is the coproduct. Then μ and S^k define products and Steenrod squares on $H^*(\text{Hom}_\Lambda(CR, N))$. These satisfy all the usual rules, although both Cartan formula and Adem relations must be interpreted with $\text{Sq}^0 \neq 1$.

The notion of a "special" Eilenberg-Zilber map will be useful to us in §4.

Definition 2.2. A simplicial Eilenberg-Zilber map (D_k) is called "special" if for all pairs of simplicial Z_2 -modules R, S ; and all integers n, k :

- (1) The image of $D_k: C(R \times S)_n \rightarrow (CR \otimes CS)_{n+k}$ is contained in the subspace $\sum_{i,j \leq n} \bigoplus ((CR)_i \otimes (CS)_j)$. In particular, $D_k(C(R \times S)_n) = 0$ if $k > n$.
- (2) The map $D_n: C(R \times S)_n \rightarrow (CR \otimes CS)_{2n}$ is given by $D_n(\sigma \otimes \tau) = \sigma \otimes \tau$ for all $\sigma \in R_n, \tau \in S_n$.

Proposition 2.3. *There exists a special Eilenberg-Zilber map.*

The proof is an easy modification of Spanier [16, p. 274].

3. Bisimplicial Eilenberg-Zilber maps. Suppose Λ a Hopf algebra and X, Y bisimplicial Λ -modules. We define a double Λ -complex called the "vertical tensor product": $(CX \otimes_V CY)_{p,q} = \sum_{k+l=q} \bigoplus (X_{p,k} \otimes Y_{p,l})$. Here the horizontal differential is $\sum_i d_i^h \otimes d_i^h$, and the vertical differential is $d^v \otimes \text{id} + \text{id} \otimes d^v$. We define the ordinary tensor product of the double complexes CX, CY by writing $(CX \otimes CY)_{p,q} = \sum_{i+j=p} \sum_{k+l=q} \bigoplus (X_{i,k} \otimes Y_{j,l})$ (action of Λ is always diagonal).

Fix a simplicial Eilenberg-Zilber map (D_k) . Define Λ -homomorphisms:

$$G_s: C(X \times Y) \rightarrow CX \otimes_V CY,$$

$$H_r: CX \otimes_V CY \rightarrow CX \otimes CY,$$

where G_s is homogeneous of bidegree $(0, s)$ and H_r is homogeneous of bidegree $(r, 0)$, as follows. The restriction of G_s to $C(X \times Y)_{p,q}$ is

$$(3.1) \quad D_s(X_{p,-}; Y_{p,-}): X_{p,q} \otimes Y_{p,q} \rightarrow \sum_{k+l=q+s} \bigoplus (X_{p,k} \otimes Y_{p,l})$$

and the restriction of H_r to the summand $X_{p,k} \otimes Y_{p,l}$ of $(CX \otimes_V CY)_{p,q}$ is $(k+l=q)$:

$$(3.2) \quad D_r(X_{-,k}; Y_{-,l}): X_{p,k} \otimes Y_{p,l} \rightarrow \sum_{i+j=p+r} \bigoplus (X_{i,k} \otimes Y_{j,l}).$$

Clearly

$$(3.3) \quad \begin{aligned} d^\nu G_s + G_s d^\nu &= G_{s-1} + TG_{s-1}T, \\ d^b H_r + H_r d^b &= H_{r-1} + TH_{r-1}T, \\ d^b G_s &= G_s d^b; \quad d^\nu H_r = H_r d^\nu. \end{aligned}$$

We now define for each integer $k \geq 0$ a Λ -homomorphism $K_k: C(X \times Y) \rightarrow CX \otimes CY$ homogeneous of degree k with respect to total degree by setting

$$(3.4) \quad K_k = \sum_{r+s=k} T^s H_r T^s G_s$$

where T^s is the s -fold iterate of the switching morphism. From the relations (3.3) we find after a short computation

$$(3.5) \quad dK_k + K_k d = K_{k-1} + TK_{k-1}T$$

where the d 's are total differentials. Thus (K_k) can be called a bisimplicial Eilenberg-Zilber map.

It is now apparent how to define cohomology operations on $H^*(\text{Hom}_\Lambda(CX, N))$ for any bisimplicial Λ -coalgebra X and any commutative Λ -algebra N . Each K_k defines

$$(3.6) \quad K_k^*: \text{Hom}_\Lambda(CX \otimes CY, N) \rightarrow \text{Hom}_\Lambda(C(X \times Y), N)$$

and we set up a map of cochain complexes

$$(3.7) \quad \phi: \text{Hom}_\Lambda(CX, N) \otimes \text{Hom}_\Lambda(CX, N) \rightarrow \text{Hom}_\Lambda(CX \otimes CX, N)$$

by analogy with (2.2). We define cochain operations $\mu: \text{Hom}_\Lambda^p(CX, N) \otimes \text{Hom}_\Lambda^q(CX, N) \rightarrow \text{Hom}_\Lambda^{p+q}(CX, N)$, $S^k: \text{Hom}_\Lambda^n(CX, N) \rightarrow \text{Hom}_\Lambda^{n+k}(CX, N)$ by

$$(3.8) \quad \mu(x \otimes y) = \psi^* K_0^* \phi(x \otimes y),$$

$$(3.9) \quad S^k x = \psi^* K_{n-k}^* \phi(x \otimes x) + \psi^* K_{n-k+1}^* \phi(x \otimes \delta x).$$

Then μ and S^k pass to products and Steenrod squares on $H^*(\text{Hom}_\Lambda(CX, N))$.

These operations are independent of the choice of (D_k) , and satisfy Cartan formulas and Adem relations. We do not prove these statements here, as we do not need them. The reader can construct his own proofs, by using an analogue of the Eilenberg-Zilber theorem which compares $C(X \times Y)$ to $CX \otimes CY$, where X, Y are bisimplicial objects.

Proof of Proposition 1.1. Suppose X a bisimplicial Λ -coalgebra, R a simplicial Λ -coalgebra, and $\lambda: X \rightarrow R$ an augmentation. Suppose $x: R_n \rightarrow N$ is a cocycle representing $u \in H^n(\text{Hom}_\Lambda(CR, N))$. Then $\text{Sq}^k \lambda^* u$ is represented by $\psi^* K_{n-k}^* \phi(\lambda^* x \otimes \lambda^* x): (CX)_{n+k} \rightarrow N$. Using (3.1), (3.2), (3.4) and the fact that

$H_0 = D_0(X_{-,n}; X_{-,n}): X_{0,n} \otimes X_{0,n} \rightarrow X_{0,n} \otimes X_{0,n}$ is the identity, we find that $\text{Sq}^k \lambda^* u$ is represented by

$$(3.10) \quad \begin{aligned} X_{0,n+k} &\xrightarrow{\psi} X_{0,n+k} \otimes X_{0,n+k} \xrightarrow{D_{n-k}^{n,n}(X_{0,-}; X_{0,-})} X_{0,n} \otimes X_{0,n} \\ &\xrightarrow{\lambda \otimes \lambda} R_n \otimes R_n \xrightarrow{x \otimes x} N \otimes N \xrightarrow{\mu} N \end{aligned}$$

where we have written $D_{n-k}^{n,n}$ for the map D_{n-k} followed by projection to the factor $X_{0,n} \otimes X_{0,n} \subset (CX \otimes_V CX)_{0,2n}$. But $D_{n-k}^{n,n}$ is natural with respect to the map of simplicial objects $\lambda: X_{0,-} \rightarrow R_-$, and since λ preserves the coproduct we can write (3.10) in the form $\mu(x \otimes x) D_{n-k}^{n,n}(R, R) \psi \lambda: X_{0,n+k} \rightarrow N$. But this cochain represents $\lambda^* \text{Sq}^k u$. A similar argument shows λ^* preserves products.

4. **The spectral sequence of a bisimplicial Λ -coalgebra.** If X, Y are bisimplicial Λ -coalgebras, we give to $CX \otimes CY$ the usual (increasing) filtration associated with a tensor product of filtered complexes. Then $\text{Hom}_\Lambda(CX \otimes CY, N)$ has the dual, decreasing filtration (as in (1.2)). Then ϕ of (3.7) is filtration preserving. Further

Proposition 4.1. *Suppose the simplicial Eilenberg-Zilber map (D_k) is special. Then $K_k F_p C(X \times Y) \subset F_{p+k}(CX \otimes CY)$, and $K_k F_p C(X \times Y) \subset F_{2p}(CX \otimes CY)$.*

Proof. The first statement follows from the definition (3.4) of K_k , and the second follows from (1) of Definition 2.2.

Consequently the Z_2 -homomorphisms K_k^* of (3.6) satisfy

$$(4.1) \quad K_k^* F^p \subset F^{\text{lig}(p/2)},$$

$$(4.2) \quad K_k^* F^p \subset F^{p-k}$$

where $\text{lig}(x)$ denotes the least integer greater than or equal to x . From now on we assume that (D_k) is special.

The spectral sequence $E_r = E_r \text{Hom}_\Lambda(CX, N)$ (N a commutative Λ -algebra) is obtained by the usual formulas:

$$(4.3) \quad E_r^{p,q} = Z_r^{p,q} / [\delta Z_{r-1}^{p-r+1, q+r-2} + Z_{r-1}^{p+1, q-1}]$$

where

$$(4.4) \quad Z_r^{p,q} = [x \in F^p \text{Hom}_\Lambda^{p+q}(CX, N) \mid \delta x \in F^{p+r} \text{Hom}_\Lambda(CX, N)].$$

We write $Z_r^p = \sum_q \bigoplus Z_r^{p,q}$; $E_r^p = \sum_q \bigoplus E_r^{p,q}$. The differential δ on $\text{Hom}_\Lambda(CX, N)$ induces $d_r: E_r^{p,q} \rightarrow E_r^{p+r, q-r+1}$.

We define the product (1.5) by the composition

$$(4.5) \quad E_r \text{Hom}_\Lambda(CX, N) \otimes E_r \text{Hom}_\Lambda(CX, N) \rightarrow E_r(\text{Hom}_\Lambda(CX, N) \otimes \text{Hom}_\Lambda(CX, N)) \\ \xrightarrow{\mu_*} E_r(\text{Hom}_\Lambda(CX, N))$$

where μ_* is the map of spectral sequences induced by the map $\mu: \text{Hom}_\Lambda(CX, N) \otimes \text{Hom}_\Lambda(CX, N) \rightarrow \text{Hom}_\Lambda(CX, N)$ of filtered complexes given by (3.8). Steenrod squares in the spectral sequence are defined by virtue of

Proposition 4.2. *Suppose $r > 2$ and $x \in Z^{p,q}$.*

Case 1. $0 \leq k \leq q$. Then $S^k x \in Z_r^{p,q+k}$ (S^k as in (3.9)) and the correspondence $x \rightarrow S^k x$ passes to a homomorphism $\text{Sq}^k: E_r^{p,q} \rightarrow E_r^{p,q+k}$.

Case 2. $q \leq k$. Then $S^k x \in Z_r^{p+k-q,2q}$, and the correspondence $x \rightarrow S^k x$ passes to a homomorphism $\text{Sq}^k: E_r^{p,q} \rightarrow E_r^{p+k-q,2q}$.

Proof. The proofs of these statements depend only on the definition (3.9) and the properties (3.5), (4.1), (4.2) of K_k . Case 1 requires (4.1) and Case 2 requires (4.2). We write out the proof for Case 2. If $x \in Z_r^{p,q}$ then $\phi(x \otimes x) \in F^{2p} \text{Hom}_\Lambda(CX \otimes CX, N)$, so from (4.2) we have $\psi^* K_{n-k}^* \phi(x \otimes x) \in F^{p-q+k} \text{Hom}_\Lambda(CX, N)$; here we have set $n = p + q$. Also $\delta x \in F^{p+r} \text{Hom}_\Lambda(CX, N)$ so

$$\psi^* K_{n-k+1}^* \phi(x \otimes \delta x) \in F^{p-q+k+r-1} \text{Hom}_\Lambda(CX, N) \subset F^{p-q+k} \text{Hom}_\Lambda(CX, N) \quad (r \geq 1).$$

Hence $S^k x \in F^{p-q+k} \text{Hom}_\Lambda(CX, N)$. Further $\delta S^k x = S^k \delta x = \psi^* K_{n-k+1}^* \phi(\delta x \otimes \delta x)$ which by (4.2) lies in $F^{p+k-q+r} \text{Hom}_\Lambda(CX, N)$. This proves $S^k x \in Z_r^{p+k-q}$.

We claim next that if $x, y \in Z_r^{p,q}$ then $S^k(x + y) - S^k x - S^k y$ represents zero in E_r^{p+k-q} . In fact (3.5) and (3.9) imply

$$(4.6) \quad S^k(x + y) - S^k x - S^k y = \psi^* [\delta K_{n-k+1}^* \phi(x \otimes y) + K_{n-k+1}^* \phi(1 + T)(\delta x \otimes y)].$$

But use of (4.2) shows that the first term on the right of (4.6) lies in $\delta Z_{r-1}^{p+k-q-r+1}$, and that the second term lies in $Z_{r-1}^{p+k-q+1}$. Hence both terms represent zero in E_r^{p+k-q} .

We claim finally that S^k passes to a well-defined map $E_r^{p,k} \rightarrow E_r^{p+k-q,2q}$. In view of our analysis of (4.6) we need only show that if $x \in Z_{r-1}^{p+1,q-1}$ then $S^k x \in Z_{r-1}^{p+k-q+1}$, and that if $x \in \delta Z_{r-1}^{p-r+1,q+r-2}$ then $S^k x \in \delta Z_{r-1}^{p+k-q-r+1}$. But both these statements follow easily from (4.2) and the relation $\delta S^k = S^k \delta$.

Proof of Proposition 1.2. Immediate from the definitions.

Proof of Proposition 1.3. Immediate.

Proof of Proposition 1.4. All the statements are consequences of the relation $\delta S^k = S^k \delta$. We write out the proof for Case 2. Suppose $x \in Z_r^{p,q}$ represents $u \in E_r^{p,q}$. Then $\delta x \in F^{p+r} \text{Hom}_\Lambda(CX, N)$, and by repeating an argument in the proof of Proposition 4.2 we find $S^k \delta x \in F^{p+t} \text{Hom}_\Lambda(CX, N)$, where $t = 2r - q + k - 1$.

Since $\delta S^k x = S^k \delta x$ we find $S^k x \in Z_t^p$. Hence $Sq^k u$ survives to E_t . Next we observe that $Sq^k d_r u$ is represented in Z_r^{p+t} by $S^k \delta x$. But this element surely lies in Z_t^{p+t} , since $\delta S^k \delta x = S^k \delta \delta x = 0$. Hence $Sq^k d_r u$ survives to E_t . Finally, the relation $\delta S^k = S^k \delta$ implies $d_t [Sq^k u] = [Sq^k d_r u]$.

Proof of Proposition 1.5. Let $\underline{\rho}: F^p H^{p+q}(\text{Hom}_\Lambda(CX, N)) \rightarrow E_\infty^{p,q}$ be the standard projection. Then $\rho = \underline{\rho} \lambda^*$. To show that ρ commutes with products and Steenrod squares it suffices, in view of Proposition 1.1, to show that $\underline{\rho}$ does. But this is immediate from the definitions (Proposition 4.2).

5. Operations on E_2 . One representation of E_2 is given by (4.3), but it is sometimes more convenient to work with

$$(5.1) \quad E_2^{p,q} = H_b^p H_v^q.$$

Here the right-hand side is the p th homology group of the complex

$$\dots \rightarrow H^q(\text{Hom}_\Lambda((CX)_{p,-}; N)) \xrightarrow{\delta^b} H^q(\text{Hom}_\Lambda((CX)_{p+1,-}; N)) \rightarrow \dots$$

Using (5.1) we can represent an element of E_2 by a Λ -homomorphism $x: X_{p,q} \rightarrow N$ satisfying $\delta^v x = 0$ and $\delta^b x = \delta^v y$ for some $y: X_{p+1,q-1} \rightarrow N$. The next result tells how to compute Steenrod squares on E_2 , using this representation

Notation. If (D_k) is a (special) Eilenberg-Zilber map, and if R, S are simplicial Λ -modules, then for any pair of integers (i, j) for which $i + j = p + k$ write $D_k^{i,j}(R, S): R_p \otimes S_p \rightarrow R_i \otimes S_j$ for the composition

$$R_p \otimes S_p \xrightarrow{D_k(R,S)} (CR \otimes CS)_{p+k} \xrightarrow{\text{project}} R_i \otimes S_j.$$

Proposition 5.1. *Suppose $x: X_{p,q} \rightarrow N, y: X_{r,s} \rightarrow N$ are Λ -homomorphisms representing $u \in E_2^{p,q}, v \in E_2^{r,s}$. Then $u \cdot v \in E_2^{p+r,q+s}$ is represented by the composition*

$$(5.1) \quad X_{p+r,q+s} \xrightarrow{\psi} X_{p+r,q+s} \otimes X_{p+r,q+s} \xrightarrow{D_0^{q,s}(X_{p+r,-}; X_{p+r,-})} X_{p+r,q} \otimes X_{p+r,s}$$

$$\xrightarrow{D_0^{p,r}(X_{-,q}; X_{-,s})} X_{p,q} \otimes X_{r,s} \xrightarrow{x \otimes y} N \otimes N \xrightarrow{\mu} N.$$

If $k \leq q$ then $Sq^k u \in E_2^{p,q+k}$ is represented by the composition

$$(5.2) \quad X_{p,q+k} \xrightarrow{\psi} X_{p,q+k} \otimes X_{p,q+k} \xrightarrow{D_{q-k}^{q,q}(X_{p,-}; X_{p,-})} X_{p,q} \otimes X_{p,q} \xrightarrow{x \otimes x} N \otimes N \xrightarrow{\mu} N,$$

and if $k \geq q$, $Sq^k u \in E_2^{p+k-q,2q}$ is represented by the composition

$$\begin{aligned}
 & X_{p+k-q, 2q} \xrightarrow{\psi} X_{p+k-q, 2q} \otimes X_{p+k-q, 2q} \\
 (5.3) \quad & \frac{D_0^{q,q}(X_{p+k-q,-}; X_{p+k-q,-})}{D_{p+q-k}^{p,p}(X_{-,q}; X_{-,q})} \rightarrow X_{p+k-q,q} \otimes X_{p+k-q,q} \\
 & \xrightarrow{X_{p,q} \otimes X_{p,q} \xrightarrow{x \otimes x} N \otimes N \xrightarrow{\mu} N}
 \end{aligned}$$

Proof. We give the proof for (5.2); the others are similar. The Steenrod squares on E_r were originally defined in terms of the representation (4.3). In that language u is represented by a cochain of the form $x + y$, for some $y \in F^{p+1} \text{Hom}_\Lambda(CX, N)$, and $\text{Sq}^k u$ is represented by $S^k(x + y) \in \sum_{i \geq p} \bigoplus \text{Hom}_\Lambda(X_{i,j}; N)$. To translate back to the language of (5.1) we need only pick out from $S^k(x + y)$ the component that lies in $\text{Hom}_\Lambda(X_{p,q+k}; N)$. But use of (4.1) shows that terms in $S^k(x + y)$ involving either y or δx (such as $K_{n-k}^* \phi(x \otimes y)$ or $K_{n-k+1}^* \phi(x \otimes \delta x)$) make no contribution to the crucial component. Thus it suffices to find the appropriate component of $\psi^* K_{n-k}^* \phi(x \otimes x)$ ($n = p + q$). Use of (3.1), (3.2), (3.4) shows that this is the composition

$$(5.4) \quad \mu(x \otimes x) D_p^{p,p}(X_{-,q}; X_{-,q}) D_{q-k}^{q,q}(X_{p,-}; X_{p,-}) \psi: X_{p,q+k} \rightarrow N.$$

But by virtue of (2) of Definition 2.2, $D_p^{p,p}(X_{-,q}; X_{-,q})$ is the identity map on $X_{p,q} \otimes X_{p,q}$. So (5.4) reduces to (5.2).

Proposition 5.1 can be used to show that the Steenrod squares on E_2 (and hence on E_r for all $r \geq 2$) are independent of the choice of special Eilenberg-Zilber map. It can also be used to prove Adem relations and Cartan formulas. But in II we will obtain these results with no extra work, for each of the special cases we consider. For this reason we omit the general proofs.