THE BRAUER GROUP OF GRADED AZUMAYA ALGEBRAS

BY

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ABSTRACT. We study G-graded Azumaya R-algebras for R a commutative ring and G a finite abelian group, and a Brauer group formed by such algebras. A short exact sequence is obtained which relates this Brauer group with the usual Brauer group of R and with a group of graded Galois extensions of R. In case G is cyclic a second short exact sequence describes this group of graded Galois extensions in terms of the usual group of Galois extensions of R with group G and a certain group of functions on Spec(R).

Introduction. In [20], C. T. C. Wall introduced the Brauer group of Z/2Z-graded central simple algebras over a field K. He called a graded K-algebra A = A_0 ⊕ A_1 graded simple if A had no homogeneous two-sided ideals, and graded central if A_0 ∩ center(A) = K. Wall’s definition resolved an inelegance in the theory of quadratic forms: the Clifford algebra A of a quadratic form q: V → K is always a graded central simple algebra, whereas when V is odd dimensional A is not central simple. Wall showed that a graded central simple K-algebra is described by three invariants: a central simple K-algebra, an element of Z/2Z, and a quadratic Galois extension of K (or more precisely, an element of K/φ(K) if char(K) = 2 and of K*/K^*2 if char(K) ≠ 2).

H. Bass [2, Chapter IV] and C. Small [16] extended Wall’s work to Z/2Z-graded algebras over a commutative ring R. They generalized Auslander and Goldman’s definition of separable R-algebras to Z/2Z-graded R-algebras [1] and defined a Brauer-Wall group of graded central separable R-algebras. This group, BW(R), was described by two exact sequences:

\[ 0 \to B(R) \to BW(R) \to Q_2(R) \to 0, \]

where B(R) is the usual Brauer group of R and Q_2(R) the group of Z/2Z-graded quadratic Galois extensions of R;

\[ 0 \to \text{Gal}(R, Z/2Z) \to Q_2(R) \to H(R) \to 0, \]

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which relates \( Q_2(R) \) to the group \( \text{Gal}(R, \mathbb{Z}/2\mathbb{Z}) \) of quadratic Galois extensions of \( R \) (defined by Harrison in \([10]\)), and to \( H(R) \), the group of those continuous maps from \( \text{Spec}(R) \) to \( \mathbb{Z}/2\mathbb{Z} \) which vanish on primes containing 2.

In §§1 and 2 of this paper we extend the definitions of Bass and Small to \( R \)-algebras graded by an abelian group \( G \). However, we do restrict to \( R \) being tri-vially graded. We adopt the device Knus introduced in \([14]\): graded concepts are defined relative to a fixed bilinear map \( \phi: G \times G \rightarrow \text{Units}(R) \). We do not assume that \( \phi \) is symmetric or nondegenerate. Surprisingly, to define graded central separable, the concept of graded separable is unnecessary (cf. Remark (a) following 2.1).

Thus our graded Azumaya \( R \)-algebras \( A \) are those separable \( R \)-algebras which are \( G \)-graded and whose graded center, relative to \( \phi \), is \( R \); the graded center of \( A \) is the set of homogeneous \( x \) satisfying \( xa = \phi(\deg x, \deg a)ax \) and \( ax = \phi(\deg a, \deg x)xa \) for all homogeneous \( a \) in \( A \). §2 develops the theory of \( G \)-graded Azumaya \( R \)-algebras. As in the ungraded and \( \mathbb{Z}/2\mathbb{Z} \)-graded cases the Morita theorems play an important role. However, we have tried to avoid duplicating the superstructure used in \([2]\) and \([1]\) to prove results about Azumaya \( R \)-algebras. It seems likely that one could develop machinery for graded objects which would generalize classical results about ungraded modules and algebras (cf. \([16]\)).

In §3 we define a map \( \pi \) from graded Azumaya \( R \)-algebras to graded Galois extensions, and obtain an exact sequence

\[
0 \rightarrow B(R) \rightarrow B(R, G) \rightarrow \text{Im}(\pi) \rightarrow 0.
\]

In contrast to the case \( G = \mathbb{Z}/2\mathbb{Z} \), \( \text{Im}(\pi) \) is not always the full set of graded Galois extensions of \( R \) with group \( G \). In fact, the latter set need not be a group under its natural multiplication, but \( \text{Im}(\pi) \) is a group.

In §4 we take \( G \) to be cyclic of order \( n \). We obtain an exact sequence

\[
0 \rightarrow \text{Gal}(R, G) \rightarrow \text{Im}(\pi) \rightarrow H(R) \rightarrow 0.
\]

\( \text{Gal}(R, G) \) denotes the group of Galois extensions of \( R \) with group \( G \) ([10], [15]). \( H(R) \) is a certain set of continuous functions from \( \text{Spec}(R) \) to a product of copies of \( \mathbb{Z}/2\mathbb{Z} \), as described in (4.1). This generalizes Bass's exact sequence. As a corollary of our exact sequences, we can deduce Knus's result that for \( G \) cyclic of prime order and \( R \) an algebraically closed field of characteristic prime to the order of \( G \), \( B(R, G) \) is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \).

1. Graded algebras and modules. This section contains basic definitions and results on graded algebras and modules. \( R \) will denote a commutative ring, \( G \) a finite abelian group. We shall assume the existence of a fixed bilinear form \( \phi: G \times G \rightarrow \text{Units}(R) \). We shall write \( G \) multiplicatively.

An \( R \)-algebra \( A \) will be called graded if \( A = \bigoplus_{g \in G} A_g \), where \( A_g \) is an \( R \)-submodule of \( A \) satisfying \( A_g A_r \subseteq A_{g_r} \) for all \( r \) in \( G \). It follows that \( 1 \in A_1 \).
If $A$ and $B$ are graded $R$-algebras, an $R$-algebra homomorphism $f: A \to B$ satisfying $f(A_{\sigma}) \subseteq B_{\sigma}$ for $\sigma$ in $G$ will be called a homomorphism of graded $R$-algebras.

The graded tensor product of graded $R$-algebras $A$ and $B$ is defined as follows: $A \otimes \phi B = A \otimes_R B$ as $R$-modules; multiplication is given by

$$(a \otimes b)(c \otimes d) = \phi(b, c)ac \otimes bd,$$

where the notation $\phi(b, c)$ is explained below.

Notational conventions. If $x$ and $y$ are homogeneous elements of the graded $R$-algebras $A$ and $B$, with $x$ in $A_{\sigma}$, $y$ in $B_{\tau}$, we shall write $\phi(x, y)$ instead of $\phi(\sigma, \tau)$. Moreover, the very fact of our writing $\phi(a, b)$ shall imply that $a$ and $b$ are homogeneous elements. Thus, in equations such as the one displayed above, it is to be understood that the formula extends from homogeneous elements to arbitrary ones by requiring linearity.

Since $\phi$ is a fixed pairing, we adopt the device of using boldface notation for constructions or concepts involving $\phi$. Thus $A \otimes B$ shall denote $A \otimes \phi B$, and the very use of $\otimes$ shall imply that we are dealing with graded objects. The grading on $A \otimes B$ is given by $(A \otimes B)_\sigma = \bigoplus_{\alpha \beta = \sigma} (A_\alpha \otimes B_\beta)$.

Let $A$ be a graded $R$-algebra. By a graded left $A$-module we shall mean a left $A$-module $M$ which decomposes as $M = \bigoplus_{\sigma \in G} M_{\sigma}$, with $A_{\alpha}M_{\beta} \subseteq M_{\sigma \tau}$ for $\sigma, \tau$ in $G$.

We define $A^\#$, the graded opposite algebra of $A$, by $(A^\#)_\sigma = \{a^\# | a \in A_{\sigma}\}$, with $a^\#b^\# = \phi(a, b)(ba)^\#$.

Let $M = \bigoplus M_\sigma$ be a graded $(A, B)$-bimodule, i.e. $A_\alpha M_\beta B_\gamma \subseteq M_{\sigma \tau \beta}$, $(am)b = a(mb)$, and $rm = mr$ for $\alpha, \beta, \sigma, \tau$ in $G$, $a$ in $A$, $m$ in $M$, $b$ in $B$ and $r$ in $R$. We make $M$ a graded $A \otimes B^\#$-module by setting

$$(a \otimes b^\#)m = \phi(b, m)amb$$

for $a$ in $A$, $b$ in $B$ and $m$ in $M$. Likewise, a graded left $A \otimes B^\#$-module is a graded $(A, B)$-bimodule.

Let $M$ and $N$ be graded left $A$-modules. Define a graded $R$-module

$${\text{Hom}}_A (M, N)$$

by

$${\text{Hom}}_A (M, N) = \{f: M \to N | f(M_{\sigma}) \subseteq N_{\sigma \tau} \text{ for } \sigma \in G \text{ and } f(ax) = \phi(\tau, a)f(x) \text{ for } a \in A, x \in M\}.$$ 

$${\text{Hom}}_A (M, N)$$

is the direct sum of the homogeneous pieces thus described. An element of $${\text{Hom}}_A (M, N)$$ will be called an $A$-homomorphism. We shall write $A$-$\text{Mod}$ for the category of graded left $A$-modules and $A$-homomorphisms. If $f$ is homogeneous of degree 1, the condition $f(ax) = \phi(f, a)f(x)$ is just the statement $f(ax) = af(x)$. 

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We define $A^e = A \otimes A^#$, $eA = A^# \otimes A$. We write $A^e = A \otimes_R A^0$ for the usual enveloping algebra of $A$; $A^e$ is graded via $(A^e)_\sigma = \bigoplus \alpha = \pi_\sigma (A^0 \otimes^R A^0)$. 

There is an action of $A^e$ (resp. $eA$) on $A$ which makes $A$ a graded left (resp. right) module. It is given by $(a \otimes b^#)x = \phi(b, x)axb$ (resp. $x(a^# \otimes b) = \phi(x, a)axb$), for $a, b$ and $x$ in $A$.

There are natural maps, both called $\pi_A$, from $A^e$ and $eA$ to $A$, given by deleting #'s and multiplying. These maps are homogeneous of degree 1. We would like to say that these maps are, respectively, an $A^e$ and $eA$-module homomorphism. For this, and for future reference, we must establish the relationship between left and right actions by graded rings.

Let $A$ be a graded $R$-algebra, $M$ and $N$ graded right $A$-modules, i.e. $M^\sigma A \subset M^\sigma R$. Define $\text{Hom}_A(M, N)$ to be $\text{Hom}_A(M, N)$ with the induced grading. The fact that the gradings do not affect the definition of homomorphism for right modules, in contrast with the situation for left modules, is consistent with (1.1) below.

Let $M, N$ be graded right modules over the respective graded algebras $A$ and $B$. It is easily verified that $M \otimes_R N$ is a graded right $A \otimes B$-module via $(m \otimes n)(a \otimes b) = \phi(n, a)ma \otimes nb$. If $P$ is a graded left module over the graded algebra $C$, then $P$ becomes a graded right $C^#$-module via $pc^# = \phi(p, c)cp$. With these conventions in mind we have

(1.1) Let $A$ and $B$ be graded $R$-algebras. Let $M$ and $N$ be graded right modules over $A$ and $B$ respectively. Let $P$ and $Q$ be graded left $A$-modules. Write $E$ for $\text{End}$. Let $C = A \otimes B$. Then

(a) $\text{Hom}_A(P, Q) = \text{Hom}_A^#(P, Q)$.
(b) There exists a homomorphism of graded $R$-algebras,

$$\theta: E_A(M) \otimes E_B(N) \rightarrow E_C(M \otimes_R N),$$

given by $\theta(a \otimes b)(m \otimes n) = \phi(b, m)a(m) \otimes b(n)$.

For $A$ a graded algebra we shall let $\text{Mod}^e_A$ denote the category of graded right $A$-modules, with morphisms set $\text{Hom}_A(M, N)$. Consider the following situations:

(a) $M$ is in $A^e$-Mod.
(b) $N$ is in $\text{Mod}^e_A$.

Let $S$ be a subset of $A$, $bS$ the set of homogeneous components of elements of $S$.

(1.2) Definition.

$$M^S = \{ x \in M | (s \otimes 1^#)x = (1 \otimes s^#)x \text{ for } s \in S \}$$

(a)

$$S^N = \{ x \in N | x(1^# \otimes s) = x(s^# \otimes 1) \text{ for } s \in S \}$$

(b)
(1.3) Proposition. The correspondences \( f \rightarrow f(1) \) establish natural isomorphisms:

(a) \( \text{Hom}_B(A, M) = MA \), \( B = A^e \).

(b) \( \text{Hom}_B(A, M) = AM \), \( B = eA \).

When \( A = M = S \) we obtain graded \( R \)-algebras \( A^A \) and \( A^A \), called respectively the left center and right center of \( A \).

If \( M \) is a graded \( A \)-module, we shall write \( E_A(M) \) for \( \text{Hom}_A(M, M) \). If \( A = R \), then \( E_R(M) \) is simply \( \text{End}_R(M) \) as an \( R \)-module, but is endowed with a grading. We shall write \( E(M) \) for \( E_R(M) \), and \( E_A(M) \) for \( \text{End}_A(M) \).

(1.4) Proposition. Let \( M \) be a graded \( R \)-module and let \( E = E(M) \). Then \( E^E = E = \text{center}(E) = \text{center}(E^E) \).

The graded \( R \)-algebras of the type \( E(M) \) enjoy a commutativity property vis-a-vis graded tensor products:

(1.5) Proposition. Let \( M \) be a graded \( R \)-module and \( A \) a graded \( R \)-algebra. Then there exist isomorphisms of graded \( R \)-algebras.

\[ A \otimes E(M) \cong A \otimes_R E(M) \cong E(M) \otimes A. \]

Proof. Define maps by sending \( a \otimes a \) to \( a \otimes \phi(a( ), a)^{-1} a \), \( a \otimes a \) to \( \phi(a, )^{-1} a \otimes a \). These maps are \( R \)-algebra isomorphisms, homogeneous of degree 1. Their respective inverses take \( a \otimes a \) to \( a \otimes \phi(a( ), a^a \) and \( a \otimes a \) to \( a \otimes \phi(a, )^a \).

(1.6) Corollary. Let \( P \) and \( Q \) be graded projective \( R \)-modules of finite type. Then the map \( f \otimes g \rightarrow f \otimes \phi(g( ), f)^{-1} g \) establishes an isomorphism of graded \( R \)-algebras

\[ E(P) \otimes E(Q) \cong E(P \otimes Q). \]

The following technical results will be useful in §3.

(1.7) Let \( A \) and \( B \) be graded \( R \)-algebras. Then there exists isomorphisms of graded \( R \)-algebras

\[ (A \otimes B)^{\#} \cong B^{\#} \otimes A^{\#}, \quad A \cong A^{\#}. \]

Proof. Define a map \( A \rightarrow A^{\#} \) on homogeneous elements by sending \( x \) to \( \phi(x, x)^{\#} \). Define a map \( B^{\#} \otimes A^{\#} \rightarrow (A \otimes B)^{\#} \) be sending \( b^{\#} \otimes a^{\#} \) to \( \phi(b, a)(a \otimes b)^{\#} \). Both maps are easily seen to be \( R \)-algebra isomorphisms.

(1.8) Remark. Proposition (1.5) will be crucial in defining the Brauer group of graded Azumaya algebras. Except in the case \( G = \mathbb{Z}/2\mathbb{Z} \), the graded tensor product does not yield a commutative operation. Thus (1.5) will be needed to prove that the definition of equivalent Azumaya algebras, following (2.10), yields
an equivalence relation compatible with \( \otimes \). This point seems to have been overlooked in [14].

2. Separable algebras and Azumaya algebras. We begin by analyzing separability for graded algebras:

(2.1) Theorem. Let \( A \) be a graded \( R \)-algebra. The following conditions are equivalent:

(a) \( A \) is a projective left \( A^e \)-module.
(b) There exists an element \( e' \) in \( A^e \) satisfying \( \pi_A(e') = 1 \) and \( (1 \otimes a^h)e' = (a \otimes 1^h)e' \) for all \( a \) in \( A \).
(c) There exists an element \( e \) in \( (A^e)_1 \) satisfying \( \pi_A(e) = 1 \) and \( (1 \otimes a^h)e = (a \otimes 1^h)e \) for all \( a \) in \( A \).
(d) There exists an element \( e \) in \( A \otimes_R A^0 \) satisfying \( \pi_A(e) = 1 \) and \( (1 \otimes a^h)e = (a \otimes 1^h)e \) for all \( a \) in \( A \).
(e) There exists an element \( \epsilon' \) in \( eA \) satisfying \( \pi_A(\epsilon') = 1 \) and \( \epsilon'(a^h \otimes 1) = \epsilon(1 \otimes a) \) for all \( a \) in \( A \).
(f) \( A \) is a projective right \( eA \)-module.

Proof. (a) is equivalent to \( \pi_A: A^e \rightarrow A \) being a split epimorphism, and this is easily seen to be equivalent to (b); if \( f \) splits \( \pi_A \), set \( e' = f(1) \). Since \( \pi_A \) is homogeneous of degree 1, \( \pi_A \) is split by \( f \) if and only if \( \pi_A \) is split by \( f_1 \), the 1-homogeneous component of \( f \). It is thus clear that (b) is equivalent to (c).

Let \( e = \sum x_i \otimes y_i^h \) be in \( (A^e)_1 \); we may take \( x_i \) and \( y_i \) homogeneous, with \( x_i y_i \) in \( A_1 \). The condition \( (1 \otimes a^h) = (a \otimes 1^h) \) is then equivalent to the equality

\[
\sum x_i \otimes (y_i^h a)^h = \sum a x_i \otimes y_i^h.
\]

Noting that there is an \( R \)-module isomorphism \( A^e \cong A \otimes_R A^0 \), under which \( a \otimes b^h \) and \( a^0 \otimes b \) correspond, we may conclude that (c) is equivalent to (d). Similarly, using the \( R \)-module isomorphism \( A \otimes_R A^0 = A^0 \otimes_R A \) under which \( a \otimes b^h \) and \( a^0 \otimes b \) correspond, we conclude that (e) and (f) are equivalent to conditions (a)–(d).

Definition. We say that \( A \) is \( R \)-separable if conditions (a)–(e) above hold.

Remarks. (a) Condition (d) states that \( A \) is \( R \)-separable if and only if it is \( R \)-separable in the sense of ungraded algebras.
(b) It is easily computed that \( \epsilon' \), \( \epsilon \) and \( \epsilon' \epsilon \), referred to in (b)–(e) above, are idempotents.
(c) Let \( A \) be a separable \( R \)-algebra and choose \( \epsilon' \) and \( \epsilon' \epsilon \) as in (b) and (e) above. Let \( M \) be in \( A^e\text{-Mod} \) and \( N \) in \( \text{Mod}^eA \). Then \( M^A = \epsilon'M \) and \( AN = N'\epsilon \) (cf. (1.2), (1.3)). For if \( x \) is in \( M^A \) and \( \epsilon' = \sum x_i \otimes y_i^h \), then \( x = \sum (x_i y_i \otimes 1^h)x = \sum (x_i \otimes 1^h)(1 \otimes y_i^h)x = \epsilon'x \).
(2.2) **Proposition.** (a) Let \( f : A \to B \) be an onto map of graded \( R \)-algebras. If \( A \) is \( R \)-separable, so is \( B \). Moreover, in this case \( f(A^a) = B^b \) and \( f(A^a) = B^b \).

(b) Let \( S \) be a commutative \( R \)-algebra. If \( A \) is \( R \)-separable then \( B = S \otimes_R A \) is \( S \)-separable. Moreover, in this case \( B^B = S \otimes_R A^A \) and \( B^B = S \otimes_R A^A \).

**Proof.** Apply the preceding remarks.

(2.3) **Proposition.** (a) Let \( A \) be \( R \)-separable. Then \( A^A = B \) is \( R \)-separable. If in addition \( A^A = R \) then \( B^B = R \); if \( A^A = R \) then \( B^B = R \).

(b) Let \( A \) and \( B \) be \( R \)-separable. Then \( C = A \otimes B \) is \( R \)-separable. If \( A^A = R = B^B \) then \( C^C = R \). If \( A^A = R = B^B \) then \( C^C = R \).

**Proof.** (a) Let \( \epsilon = \Sigma x_i \otimes y_i^a \) be an element as described in (2.1)(c), with \( x_i \) and \( y_i \) homogeneous and \( x_i y_i \) in \( A_1 \). For each \( a \) in \( bA \) we have

\[
\sum ax_i \otimes y_i^a = \sum x_i \otimes (y_i a)^a.
\]

Let \( \epsilon^a = \Sigma \phi(x_i, x_i) y_i^a \otimes x_i^a \). Using the fact that \( x_i y_i \) is in \( A_1 \), we compute that, for \( a \) in \( bA \),

\[
(1) \quad (a^a \otimes 1^a) \epsilon^a = \sum \phi(a, y_i) x_i^a)^a \otimes x_i^a,
\]

\[
(2) \quad (1^a \otimes a^a) \epsilon^a = \sum \phi(x_i, ax_i) y_i^a \otimes (ax_i)^a.
\]

There is an \( R \)-module isomorphism \( A^a \otimes A^a = A \otimes A^a \), under which \( a^a \otimes b^a \) and \( \phi(a, b) b \otimes a^a \) correspond. Applying this map to the right-hand sides of (1) and (2), we obtain the two sides of (\( \ast \)). It follows that the left-hand sides of (1) and (2) are equal. It is clear that \( \pi_A^a(\epsilon^a) = 1 \). The rest of the proof of (a) is straightforward.

(b) This argument uses the same idea just employed. We note that there is an \( R \)-module isomorphism \( A \otimes B \otimes (A \otimes B)^a \cong A \otimes A^a \otimes B \otimes B^a \) under which \( a \otimes b \otimes (c \otimes d)^a \) corresponds to \( \phi(b, c) a \otimes c^a \otimes b \otimes d^a \). Let \( \epsilon_A = \Sigma x_i \otimes y_i^a \) and \( \epsilon_B = \Sigma a_j \otimes b_j^a \) be elements of \( A^e \) and \( B^e \) satisfying (2.1)(c). Let \( \epsilon = \Sigma \phi(a_j, x_i) x_i \otimes a_j \otimes (y_i \otimes b_j)^a \). A straightforward computation such as used above allows us to conclude that \( \epsilon \) satisfies the conditions required for \( A \otimes B \) to be \( R \)-separable.

Suppose that \( A^A = R = B^B \). By the remark preceding (2.2) we know that \( \epsilon_A^a A = R = \epsilon_B^B \). We wish to show that \( \epsilon(A \otimes B) = R \). Clearly, \( \epsilon(1 \otimes 1) = 1 \). Let \( a \) and \( b \) be homogeneous elements in \( A \) and \( B \) respectively. Keeping in mind that \( x_i y_i \) and \( a_j b_j \) are homogeneous of degree 1, we compute that

\[
\epsilon(a \otimes b) = \sum_i \phi(y_i, b) \phi(b, y_i) \epsilon_A(a) \otimes \epsilon_B(b).
\]
Now \( \epsilon_B(b) \) in \( R \) implies that \( \epsilon_B(b) = 0 \) for \( b \) not in \( B_1 \); if \( b \) is in \( B_1 \), \( \epsilon(a \otimes b) = \epsilon_A(a) \otimes \epsilon_B(b) \) is in \( R \).

We now introduce the objects of our principal interest. Let \( A \) be a graded \( R \)-algebra. We shall say that \( A \) is left central if \( A^A = R \), right central if \( A^A = R \), central if \( A^A = A = A^A \). We shall say that \( A \) is a (left, right) Azumaya \( R \)-algebra if \( A \) is \( R \)-separable and (left, right) central.

(2.4) Example. Let \( M \) be a graded \( R \)-module which is faithful, projective and of finite type. Then \( \text{End}_R(M) \) is a separable \( R \)-algebra with center \( R \) [1, Proposition 5.1]. It follows from (1.4) and Remark (a) after (2.1) that \( \text{End}_R(M) \) is an Azumaya \( R \)-algebra.

Remark. Let \( A \) be a left or right Azumaya \( R \)-algebra. Then the inclusion map embeds \( R \) as a direct summand of \( A \). For let \( \epsilon \) be as in (2.1) (c). Define \( t: A \rightarrow A \) by \( t(x) = \epsilon x \). For any \( y \) in \( bA \) we have

\[
y(tx) = (y \otimes 1^y_\epsilon)x = (1 \otimes y^\#_\epsilon)x = \phi(y, tx)(tx)y.
\]

Thus \( tx \) is in \( A^A = R \). The fact that \( \pi_A(\epsilon) = 1 \) implies that \( t(r) = r \) for \( r \) in \( R \).

A graded \( R \)-algebra \( A \) is said to be graded simple if \( A \) has no homogeneous two-sided ideals except (0) and \( A \).

The next result shows that our Azumaya algebras coincide with those studied by Wall and Knus ([20], [14]) for \( R \) a field.

(2.5) Proposition. Let \( A \) be a left or right Azumaya \( R \)-algebra. Then \( A \) is graded simple if and only if \( R \) is a field.

Proof. Suppose \( A \) is graded simple and \( l \) is a nonzero ideal of \( R \). Then \( lA = A \). Let \( t: A \rightarrow R \) be a splitting of the inclusion of \( R \) in \( A \). Then \( R = tA = ltA = l \).

Let \( A \) be a separable \( R \)-algebra, with \( R \) a field. Since \( A \) is \( R \)-separable in the ungraded sense and \( R \)-projective, it is an \( R \)-module of finite type [19, Proposition 1.1]. Therefore \( A \) is a semisimple ring with\,d.c.c. [2, p. 100, Theorem 3.1].

Let \( \mathcal{U} \) be a two-sided homogeneous ideal of \( A \). There is a central idempotent \( \alpha \) in \( A \) with \( \mathcal{U} = \alpha A \). Suppose \( \alpha = \Sigma \alpha_\sigma \) is the decomposition of \( \alpha \) into homogeneous components; then \( \alpha_\sigma \) is in \( \mathcal{U} \) by homogeneity and \( \alpha_\sigma = \alpha_\sigma \alpha_\sigma \). Thus \( \alpha_\sigma = \Sigma_r \alpha_\sigma \alpha_\sigma \) and \( \alpha_\sigma \alpha_\sigma = 0 \) for \( r \neq 1 \). Similarly, \( \alpha \alpha_\sigma = \alpha_\sigma \alpha_\sigma \) implies \( \alpha_\sigma \alpha_\sigma = 0 \) for \( r \neq 1 \). But \( \alpha = \alpha_2^2 \) then yields \( \alpha = \alpha_2^2 \), thus \( \alpha \) is homogeneous of degree 1. However, \( \alpha \) being central is then equivalent to \( \alpha \) being in \( A^A \) and in \( A^A \). Thus \( \alpha \) is in \( R \) and \( \mathcal{U} = (0) \) or \( \mathcal{U} = A \).

(2.6) Lemma. Let \( A \) be a left or right Azumaya \( R \)-algebra. Then for any maximal homogeneous two-sided ideal \( \mathcal{U} \) of \( A \), we have \( \mathcal{U} = (\mathcal{U} \cap R)A \), and \( \mathcal{U} \cap R \) is a maximal ideal of \( R \).
Proof. Let $m = \mathfrak{m} \cap R$. By (2.2), $A/\mathfrak{m}$ is a left or right Azumaya $R/m$-algebra. Thus $R/m$ is a field by (2.5). But $A/mA$ is a left or right Azumaya $R/m$-algebra, and is therefore graded simple. $(R/m)$ is embedded in $A/mA$ by the remark following (2.4). Thus $mA \subset \mathfrak{m}$ implies that $mA = \mathfrak{m}$.

We show next that the Morita equivalences which underlie the theory of Azumaya algebras are valid for Azumaya algebras as well.

(2.7) Lemma. Let $A$ be a separable $R$-algebra. Let $\epsilon$ be a homogeneous element of $A^e$ (resp. $\epsilon A$) of degree 1, satisfying $\pi_A(\epsilon) = 1$ and $(a \otimes 1^e)\epsilon = (1 \otimes a^e)\epsilon$ (resp. $\epsilon(a^e \otimes 1) = (1 \otimes a^e)$) for all $a$ in $A$. Then the graded trace ideal, $\text{tr}_B(A)$, where $B = A^e$ (resp. $\epsilon A$), is $B \mathfrak{b} B$. If $A$ is an Azumaya $R$-algebra, then $\text{tr}_B(A) = B$.

Proof. By $\text{tr}_B(A)$ we of course mean the homogeneous ideal of $B$ generated by all $f(x)$, $f$ in $\text{Hom}_B(A, B)$, $x$ in $A$; as in the ungraded case, $\text{tr}_B(A)$ is a two-sided ideal of $B$. By (1.3), if $f$ is in $\text{Hom}_B(A, B)$, then $f(x) = \phi(w, x)(x \otimes 1^e)w$ (resp. $f(x) = w(1^e \otimes x)$) where $w = f(1)$ is in $B^A$ (resp. $A B$). Then $w = \epsilon w$ (resp. $w = \epsilon$) by the remark preceding (2.2), so that $\text{tr}_B(A) \subset B \mathfrak{b} B$. Since $\epsilon$ is clearly in $\text{tr}_B(A)$, and the latter is a two-sided ideal, the first result is clear.

Suppose $A$ is an Azumaya $R$-algebra. Then $B$ is an Azumaya $R$-algebra by (2.3). If $B \mathfrak{b} B$ is not all of $B$, (2.6) implies that $B \mathfrak{b} B$ is contained in some maximal ideal $\mathfrak{m} B$, $\mathfrak{m}$ a maximal ideal of $R$. Applying $\pi_A$, we see that $A = \pi_A(B \mathfrak{b} B) \subset mA$, so that $A = mA$. However, it follows from the remark preceding (2.5) that $mA \cap R = m$, so that $B \mathfrak{b} B = B$.

Let $A$ be a graded $R$-algebra. $E(A)$ denotes $\text{End}_R(A)$. Define homomorphisms of graded $R$-algebras:

$$\eta_A: A^e \rightarrow E(A), \quad \mu_A: \epsilon A \rightarrow E(A)^0,$$

by $\eta_A(w)(x) = wx$, $\mu_A(w) = f^0$, where $f(x) = x w$.

(2.8) Theorem. Let $A$ be a graded $R$-algebra. The following conditions are equivalent.

(a) $A$ is an Azumaya $R$-algebra.

(b) $A$ is a faithful projective $R$-module of finite type, and $\eta_A, \mu_A$ are isomorphisms.

(c) $\text{tr}_R(A) = R$, and $\eta_A, \mu_A$ are isomorphisms.

(d) Each of the following pairs of functors establishes an isomorphism of categories:

$$G: R\text{-Mod} \rightarrow A^e\text{-Mod}, \quad G(X) = A \otimes_R X,$$
$$H: A^e\text{-Mod} \rightarrow R\text{-Mod}, \quad H(X) = X^A.$$
\( K: \text{R-Mod} \to \text{Mod-}^e \text{A}, \quad K(X) = \bigotimes_R A. \)

(ii) \( L: \text{Mod-}^e \text{A} \to \text{R-Mod}, \quad L(X) = A^X. \)

\text{R-Mod} denotes the category of R-modules which have a G-grading.

**Proof.** (a) \( \Rightarrow \) (b). We know from (2.7) that \( \text{tr}_B(A) = B \), where \( B = A^e \) or \( ^eA. \)

Thus there exist homogeneous elements \( f_1, \ldots, f_n \) in \( \text{Hom}_B(A, B) \), \( x_1, \ldots, x_n \) in \( A \), satisfying \( \sum x_i f_i(1) = 1 \). Define \( g_i: A \to A \) by \( g_i(a) = f_i(1)a \). It is easily seen \( g_i(A) \subseteq A = R \). But then, for \( x \) in \( A \), we have \( x = \sum x_i g_i(x) \), proving that \( A \) is \( R \)-projective of finite type.

That \( \eta_A \) is monic follows from the equalities \( w = \sum w f_i(x_i) = \sum f_i(wx_i) \) for \( w \) in \( A^e \). Now let \( a \) be in \( E(A) \), and \( x \) in \( A \). Since \( (1 \otimes a^e) f_i(1)x = f_i(a)x = (a \otimes 1^e) f_i(1)x \), we see that \( f_i(1)x \) is in \( A^A = R \).

Let \( f_i(1)x = r_i \). Then \( \alpha(x) = \sum \alpha(f_i(x_i)x) = \sum \alpha(x_i r_i) = \sum (f_i(1)x) \alpha(x_i) \), which proves that \( \eta_A \) is onto. Likewise, \( \mu_A \) is on.

(b) \( \Rightarrow \) (c). This is well known.

(c) \( \Rightarrow \) (a). The condition \( \text{tr}_R(A) = R \) is easily seen to imply two facts: (1) \( A \) is projective as an \( E(A) \)-module and, (2) the natural map \( R \to \text{End}_{E(A)}(A) \) is an isomorphism. But (1) implies that \( A \) is a projective \( \text{A}^e \)-module, since \( \mu_A \) is an isomorphism, and (2) implies that \( A^A = R = \text{A}^A \), by (1.3).

(a)–(c) \( \iff \) (d). This can be proved using the computational techniques illustrated by the proof of (a) \( \Rightarrow \) (b). At this point enough facts have been established to permit duplication in our context of the arguments in [2, p. 104, Theorem 4.1].

(2.9) **Lemma.** Let \( A \) be any graded R-algebra, \( M \) and \( P \) graded projective faithful R-modules of finite type. Suppose there is an isomorphism of graded R-algebras, \( A \otimes E(M) \cong E(P) \). Then there exists a graded projective faithful R-module of finite type, \( H \), satisfying \( A \cong E(H) \) as graded R-algebras.

**Proof.** By (1.5), \( A \otimes E(M) \cong E(P) \). For ungraded algebras this implies that \( A \cong E(H) \), where \( H = \text{Hom}_{E(M)}(M, P) \) is a projective R-module of finite type [1, Proposition 5.3]. Since the isomorphism \( A \otimes E(M) \cong E(P) \) is homogeneous of degree 1, \( P \) is a graded \( E(M) \)-module and thus \( H \) and \( E(H) \) are graded. It is easy to see that the isomorphism \( A \cong E(H) \) is homogeneous of degree 1.

(2.10) **Proposition.** Let \( A \) and \( B \) be Azumaya R-algebras. Then the following conditions are equivalent:

(a) There exist graded projective faithful R-modules of finite type, \( P \) and \( Q \), for which \( A \otimes E(P) \cong B \otimes E(Q) \) as graded R-algebras.

(b) There exists a graded projective faithful R-module of finite type, \( M \), satisfying \( A \otimes B^M \cong E(M) \) as graded R-algebras.
Proof. This result follows easily by use of (1.5), (2.9) and the well-known formula $E(P \otimes Q) \cong E(P) \otimes E(Q)$.

Two algebras related as in (2.10) will be called equivalent. It is easily verified that this yields an equivalence relation compatible with $\otimes$; (1.5) plays a crucial role. The equivalence classes of Azumaya $R$-algebras form a group. Multiplication is given by $(A)(B) = (A \otimes B)$; If $P$ is a graded projective faithful $R$-module, $E(P)$ represents the identity element; the inverse of $(A)$ is $(A^g)$. This group is denoted by $B(R, G)$, and is called the Brauer group of graded Azumaya $R$-algebras.

A graded $R$-algebra $A$ will be called fully graded if, for each $\sigma$ in $G$, $A_{\sigma}A_{\sigma^{-1}} = A_1$. This is equivalent to having $A_{\sigma}A_{\tau} = A_{\sigma \tau}$ for $\sigma, \tau$ in $G$. If $B$ is fully graded, so is $A \otimes B$. Since $E(RG)$ is easily seen to be fully graded ($RG$ is the group ring) we see that every element in $B(R, G)$ has a fully graded representative Azumaya $R$-algebra.

(2.11) Let $A$ be a fully graded $R$-algebra. Then each $A_{\sigma}$ is a projective (left and right) $A_1$-module of finite type. If in addition $A$ is $R$-separable, then $A_1$ is $R$-separable.

Proof. Fix $\sigma$ in $G$, and choose $x_1, \ldots, x_n$ in $A_{\sigma^{-1}}$, $y_1, \ldots, y_n$ in $A_{\sigma}$ satisfying $\sum x_iy_i = 1$. Define $f_i : A_{\sigma} \to A_1$ by $f_i(x) = xx_i$; $f_i$ is a homomorphism of left $A_1$-modules. For $x$ in $A_{\sigma}$, $x = \sum f_i(x)y_i$. Thus $\{y_i, f_i\}$ is a projective coordinate system for $A_{\sigma}$ as a left $A_1$-module. Similarly, $\{x_i, y_i(f)\}$ is a projective coordinate system for $A_{\sigma^{-1}}$ as a right $A_1$-module.

Since $A$ is $A_1$-projective, $A \otimes_R A^0$ is $A_1 \otimes_R A_1^0$-projective. But $R$-separability of $A$ implies that $A$ is an $A \otimes_R A^0$-direct summand, and thus an $A_1 \otimes_R A_1^0$-direct summand, of $A \otimes_R A^0$. Since $A_1$ is an $A_1 \otimes_R A_1^0$-direct summand of $A$, it follows that $A_1$ is $A_1 \otimes_R A_1^0$-projective, and therefore $R$-separable.

Remark. Let $A$ be fully graded and $R$-separable. The separability of $A_1$ implies that there exists an element $e$ in $A_1 \otimes_R A_1^0$ satisfying $(1 \otimes a^0)e = (a \otimes 1^0)e$ for $a$ in $A_1$. Let $M$ be any $A^e$-module. It is clear that $eM$ is a subset of $M^{A^1}$.

(We are identifying $a \otimes b^0$ with $a \otimes b^1$ when $a, b$ are in $A_1$.) If $e$ also satisfies $\pi_{A_1}(e) = 1$, then $eM = M^{A^1}$ (cf. Remark (c) following (2.1)).

Now let $A$ and $B$ be graded $R$-algebras with $A_1$ and $B_1$ separable. Let $e_A$ in $A_1^e$, $e_B$ in $B_1^e$ be elements as above. Then $e_A \otimes e_B$ is identified in an obvious way with an element $e_A \otimes B$ of $(A_1 \otimes_R B_1)^e$ which guarantees separability of $A_1 \otimes_R B_1$. From the above discussion, $(A \otimes B)^{A_1 \otimes_R B_1} = e_A \otimes_B (A \otimes B) = e_A A \otimes e_B B$. Therefore

$$ (A \otimes B)^{A_1 \otimes_R B_1} = A^{A_1} \otimes B^{B_1}. $$

3. Azumaya algebras and Galois extensions. Given an Azumaya $R$-algebra $A$, we wish to associate to $A$ a Galois extension of $R$ with group $G$. We shall describe two equivalent ways of doing this.
For $S$ any ring, let $GS$ denote the collection of set maps from $G$ to $S$, a ring under pointwise operations. If $A$ is a graded $R$-algebra, so is $GA$; $(GA)_\sigma$ consists of those maps $\nu$ with $\nu(G) \subset A$.

Define a left $A^e$-module structure on $GA$ by setting

$$[(a \otimes b^n)d(o)](o) = \phi(b, d)ad(o)b,$$

for $a, b, d$ homogeneous elements (so that $ob$ makes sense). Define $\Gamma(A) = (GA)^A$.

There is a natural action of $G$ on $GA$: set $(od)(r) = d(o^{-1}r)$. Then $G$ acts as a group of $R$-algebra automorphisms of $GA$, and by restriction to $\Gamma(A)$, $G$ acts as a group of $R$-algebra automorphisms of $\Gamma(A)$, each homogeneous of degree 1.

The $R$-algebra $A$ is embedded in $GA$ as the set of constant functions, and in fact $A = (GA)^G$, the set of $G$-invariant elements. It is then easily checked that $\Gamma(A)^G = A$. The next result holds.

(3.1) If $A$ is central then $\Gamma(A)^G = R$.

Let $A$ be an Azumaya $R$-algebra. For $\Gamma = \Gamma(A)$, define $\theta: \Gamma \otimes R \Gamma \rightarrow G\Gamma$ by $\theta(x \otimes y)(o) = xo(y)$. Then $\theta$ is an isomorphism: By (2.8)(d), we have an isomorphism $A \otimes \Gamma \cong GA$ (unadorned tensor products are over $R$). We need to know this isomorphism explicitly: an examination of our proof that (a)$\rightarrow$(c)$\rightarrow$(d) in (2.8), and of [2, p.104, Theorem 4.1], shows that the above map is determined by sending $a \otimes d$ to $ad$. It is easily verified that $\eta: GA \otimes_A GA \rightarrow G(GA)$, defined analogously to $\theta$, is an isomorphism. We have a commutative diagram as below, from which it follows that $1 \otimes \theta$, and therefore $\theta$, is an isomorphism.

Using [4, Theorem 1.3], we obtain (3.2) below. It should be noted that the proof in [4, Theorem 1.3], (b)$\iff$(c)$\iff$(d)$\iff$(e) does not require $A$ to be commutative.

(3.2) $\Gamma(A)^G = (GA)^A$ is a Galois extension of $R$ with group $G$. Moreover, for $\sigma$ in $G$ and $\Gamma(A)_r$ the $r$-homogeneous component of $\Gamma(A)$, $\sigma(\Gamma(A)_r) \subset \Gamma(A)_r$.

We shall find it useful to have a different characterization of $\Gamma(A)$. Let $A$ be
a fully graded Azumaya $R$-algebra. Then $A_{\sigma^{-1}} A_{\sigma} = A_1$ implies that, for $\sigma$ in $G$, there exist elements $b_{\sigma, i}$ in $A_{\sigma^{-1}}$, $c_{\sigma, i}$ in $A_{\sigma}$, satisfying $\sum b_{\sigma, i} c_{\sigma, i} = 1$.

Let $\pi(A) = A^A_1$. Define an action of $G$ on $\pi(A)$ by

$$\sigma(a) = \phi(\sigma, a) \sum_i b_{\sigma, i} a c_{\sigma, i}.$$ 

This action is independent of the choice of $b_{\sigma, i}$ and $c_{\sigma, i}$. For let $\sum_i d_{\sigma, i} e_{\sigma, i} = 1$. Then $\sum_i b_{\sigma, i} a c_{\sigma, i} = \sum_i b_{\sigma, i} b_{\sigma, j} a c_{\sigma, i} d_{\sigma, j} d_{\sigma, i} e_{\sigma, j} = \sum_i b_{\sigma, i} b_{\sigma, i} c_{\sigma, i} d_{\sigma, j} e_{\sigma, j}$, since $a$ is in $\pi(A)$. A similar computation shows that $\sigma(a) \sigma(b) = \sigma(ab)$, and that $\sigma$ is an isomorphism with inverse $\sigma^{-1}$.

For $a$ in $\pi(A)$ and $x$ in $A_{\sigma}$, define $a^x = \sigma(a)$. We then have that $xa^x = \phi(x, a) \sum b_{\sigma, i} a c_{\sigma, i} = \phi(x, a) x \sum b_{\sigma, i} c_{\sigma, i} = \phi(x, a) ax$. It follows that

$$\pi(A) = \{a \text{ in } A \mid xa^x = \phi(x, a) ax\}.$$ 

It follows easily that $\pi(A)^G = A^A = R$. The $b_{\sigma, i}$ and $c_{\sigma, i}$ allow us to compute $\Gamma(A)$ more explicitly. It is straightforward to verify that $d$ is in $(GA)^A$ if and only if $a_\rho d(\rho^{-1}) = \phi(\rho, d(\rho)) d(\sigma) a_\rho$ for $\sigma, \rho$ in $G$, $a_\rho$ in $A_\rho$, and $d$ homogeneous. In particular, choose $a_\rho = c_{\rho, i}$, multiply on the left by $b_{\sigma, i}$ and obtain $d$ in $(GA)^A = d(\sigma^{-1}) = \rho d(\sigma)$.

It is an easy consequence of the above characterization of $(GA)^A$ that $d(1)$ is in $\pi(A)$, and therefore:

$$\Gamma(A) = \{d \mid d(\sigma) = \sigma^{-1}(c), c \in \pi(A)\}.$$ 

Define maps $\pi(A) \to \Gamma(A)$, $\Gamma(A) \to \pi(A)$ by $c \to \sum c_{\sigma^{-1}(c)} e_{\sigma}$, where $e_{\sigma}(r) = \delta_{\sigma, r}$ and $d \to d(1)$. These are inverse isomorphisms of graded $R$-algebras, and are $RG$-module maps as well. We summarize these facts:

(3.4) Proposition. Let $A$ be a fully graded Azumaya $R$-algebra. Then $(GA)^A$ and $A^A_1$ are isomorphic Galois extensions of $R$ with group $G$. The elements of $G$ act as isomorphisms of graded $R$-algebras.

(3.5) Remark. The set of isomorphism classes of Galois extensions of $R$ with group $G$, $\text{Gal}(R, G)$, is endowed with a group structure [15, §1]. The trivial element of this group is $GR$, and the multiplication is defined by $(S) \cdot (T) = (S \otimes R) (T)^{\Delta G}$, where $\Delta G$ consists of the elements $(\sigma, \sigma^{-1})$ in $G \times G$ [15, p. 486].

Let $\text{Gal}(R, G)$ denote the set of isomorphism classes of Galois extensions of $R$ with group $G$. We would like to show that the correspondence $A \to \Gamma(A)$ determines a map from $B(R, G)$ to $\text{Gal}(R, G)$. We shall first consider a special case of this problem.
Let $A$ and $B$ be fully graded Azumaya $R$-algebras. It is obvious that $\pi(A \otimes B) \subseteq (A \otimes B)^{A_1 \otimes B_1}$, and by (2.12), $\pi(A \otimes B) \subseteq \pi(A) \otimes \pi(B)$. The $G$-actions on $\pi(A)$ and $\pi(B)$ induce a $G \times G$-action on $\pi(A) \otimes \pi(B)$, since $G$ acts as homogeneous automorphisms in each case, $(\sigma, \tau)(a \otimes b) = \sigma a \otimes \tau b$. We shall write $[\pi(A) \otimes \pi(B)]^{AG}$ for the set of elements invariant under all automorphisms $(\sigma, \sigma^{-1})$, $\sigma$ in $G$. This $R$-algebra obtains a well-defined $G$-action via $\sigma x = (\sigma, 1)x$.

(3.6) Proposition. There is an equality of graded Galois extensions

$$\pi(A \otimes B) = [\pi(A) \otimes \pi(B)]^{AG}$$

provided that $\phi(\sigma, \alpha)\phi(\alpha, \sigma) = 1$ for all $\sigma$ in $G$ and for those $\alpha$ in $G$ satisfying $\pi(B)_{\alpha} \neq (0)$.

Proof. Let $w$ be an element of $\pi(A \otimes B)$. By the remarks above $w$ is in $\pi(A) \otimes \pi(B)$, and for ease of computation write $w = x \otimes y$, $x$ in $\pi(A)$, $y$ in $\pi(B)$, each homogeneous. Let $b_i$ in $A_{\sigma^{-1}}$, $c_i$ in $A_{\sigma}$, $\beta_j$ in $B_\sigma$, $\gamma_j$ in $B_{\sigma^{-1}}$ be elements satisfying

$$\phi(\sigma, x) \sum_i b_i uc_i, \quad \phi(\sigma, y) \sum_j \beta_j v \gamma_j,$$

for $u$ in $\pi(A)$, $v$ in $\pi(B)$. A straightforward computation yields

$$(\sigma, \sigma^{-1})(x \otimes y) = \alpha \sum_{i,j} (b_i \otimes \beta_j)(x \otimes y)(c_i \otimes \gamma_j),$$

where $\alpha = \phi(\sigma^{-1}, y)\phi(y, \sigma^{-1})\phi(\sigma, \sigma^{-1})$. But $b_i \otimes \beta_j$ is in $(A \otimes B)_1$, and therefore commutes with $x \otimes y$; recall moreover that $\sum_i b_i c_i = 1$, $\sum_j \beta_j \gamma_j = 1$. We therefore conclude that

$$(\sigma, \sigma^{-1})(x \otimes y) = \phi(\sigma^{-1}, y)\phi(y, \sigma^{-1})(x \otimes y),$$

and the desired equality of sets is clear. It is a straightforward computation that the $G$-action naturally obtained on $\pi(A \otimes B)$ agrees with the $G$-action induced on $[\pi(A) \otimes \pi(B)]^{AG}$; indeed one may choose $(b_i \otimes 1)$ in $(A \otimes B)_\sigma$, $(c_i \otimes 1)$ in $(A \otimes B)_{\sigma^{-1}}$ as the elements which determine the $G$-action on $\pi(A \otimes B)$. This concludes the proof of the Proposition.

Let $P = \bigoplus P_\sigma$ be a graded $R$-module, with each $P_\sigma$ a finitely generated faithful projective $R$-module. Let $E = \text{End}_R(P)$, and assume that $E$ is fully graded.

By (2.4), $E$ is an Azumaya $R$-algebra. We claim

(3.7) $E^{E_1} \cong GR$, with trivial grading.

Let $f$ be an element of $E^{E_1}$, homogeneous of degree $\sigma$. Let $b_\tau$ denote the element of $E_1$ which is the identity on $P_\tau$, zero elsewhere. Then $f b_\tau = b_\tau f$ implies that $f$ is homogeneous of degree 1. Write $f = \sum f^\sigma$, with $f^\sigma$ in $\text{End}_R(P_\sigma)$.
Clearly \( f \) is in the center of \( \text{End}_R(P_\sigma) \), which is \( R \) [1, Proposition 5.1].

Let \( P \) remain as above, and let \( A \) be a fully graded Azumaya \( R \)-algebra. By (3.6) and (3.7), \( \sigma(A \otimes E) = (\sigma(A) \otimes GR)^{\Delta G} = (\sigma(A) \otimes GR)^{\Delta G} \). The usual isomorphism from \( \sigma(A) \) to \( (\sigma(A) \otimes GR)^{\Delta G} \) given by \( a \mapsto \Sigma\sigma(a) \otimes v_{-1}^\sigma \) yields an isomorphism from \( \sigma(A) \otimes E \) to \( \sigma(A) \) as graded Galois extensions.

Now suppose \( A \) and \( B \) are fully graded Azumaya \( R \)-algebras in the same equivalence class in \( B(R, G) \). By (2.10), \( A \otimes E(M) \cong B \otimes E(N) \), with \( M, N \) faithful projective \( R \)-modules of finite type. Tensor both sides with \( E(RG) \). Using (1.6), we can conclude that \( A \otimes E(P) \cong B \otimes E(Q) \) where \( P \) and \( Q \) have the nice properties that our module \( P \) discussed above was assumed to possess. It then follows that \( \sigma(A) \) and \( \sigma(B) \) are isomorphic as graded Galois extensions.

Let \( \text{Gal}(R, G) \) denote the set of isomorphism classes of graded Galois extensions, i.e. of graded \( R \)-algebras which are Galois extensions of \( R \), with group \( G \) and on which \( G \) acts as a group of \( R \)-algebra automorphisms, each homogeneous of degree 1. In taking isomorphism classes, only maps of degree one are considered. Then

(3.8) \( \pi \) determines a map from \( B(R, G) \) to \( \text{Gal}(R, G) \). \( \sigma(A) \) is the trivial element of \( \text{Gal}(R, G) \), viz. \( (GR) \), if and only if the equivalence class of \( A \) contains a trivially graded central separable \( R \)-algebra.

Proof. Only the second statement remains to be proved. Let \( A \) be an ungraded central separable \( R \)-algebra; \( D = A \otimes E(RG) \) is a fully graded Azumaya \( R \)-algebra in the equivalence class of \( A \). Then \( \sigma(A) = \sigma(D) = (R \otimes GR) \) by (2.12) and (3.7).

Conversely, suppose \( \sigma(A) = (GR) \). Then there exist pairwise orthogonal idempotents \( e_\sigma \) in \( \sigma(A) \), \( \sigma \) in \( G \), with \( \Sigma\sigma e_\sigma = 1 \). Note that since \( \sigma(A) \) is trivially graded, \( \sigma(A) \) is a subset of \( A_1 \). Our \( G \)-action was defined so that \( \sigma(e_\sigma) = e_{\sigma^G} \).

From (3.3) we conclude that \( xe_\sigma = e_1x \) for \( x \) in \( A_\sigma \). It follows that \( B = e_1Ae_1 = \bigoplus_{\sigma} A_\sigma e_\sigma e_1 = A_1e_1 \), and is an ungraded \( R \)-algebra. In fact, we shall show that \( B \) is an Azumaya \( R \)-algebra, with \( (B) = (A)^{-1} \) in \( B(R, G) \).

Consider the graded left \( A \)-module \( Ae_1 \). It is a projective \( A \)-module of finite type. Define a map \( \text{Hom}_A(Ae_1, A) \rightarrow e_1A \) by sending \( f \) to \( f(e_1) \); \( \text{Hom} \) is as defined preceding (1.1). Since \( e_1 \) is in \( \pi(A) \), and thus in \( A_1 \), this map is well defined. It is easily verified that this map is an isomorphism, whose inverse sends \( x \) to \( \phi(x)(x) \). It follows that the graded trace ideal, \( \text{tr}_A(Ae_1) \) is \( Ae_1A \) (cf. Proof of (2.7)). But \( Ae_1A = A \), as is easily verified from the fact, mentioned above, that \( A_\sigma e_\sigma = e_1A_\sigma \). Thus \( \text{tr}_A(Ae_1) = A \). A computation similar to the one carried out above shows that \( \text{End}_A(Ae_1) = (e_1Ae_1)^0 \).

Write \( P = Ae_1 \), \( Q = A(1 - e_1) \). These are graded projective left \( A \)-modules, and as explained preceding (1.2), may be viewed as graded right \( A^\sigma \)-modules.
Also, viewing $A$ as a graded right $A$-module yields that $A = \text{End}_A(A)$. Write $C$ for $A^\# \otimes A = eA$. From (1.2) we have a map of graded $R$-algebras:

$$\theta: E_{A^\#}(P) \otimes E_A(A) \to E_C(P \otimes_R A).$$

This map is in fact an isomorphism: for $A = P \oplus Q$, and $\theta$ would be an isomorphism were $A$ substituted for $P$. Using (1.2), we have that there exists an isomorphism of graded $R$-algebras

$$E_A(P) \otimes A \cong E_C(P \otimes_R A).$$

Because of the category isomorphism noted in (2.8)(d)(ii), there is an isomorphism of graded $R$-algebras,

$$E_C(P \otimes_R A) \cong E(M) \quad \text{where} \quad M = A(P \otimes_R A).$$

Because $P$ is $A$-projective and $\text{tr}_A(P) = A$, it follows that $P \otimes_R A$ is $C$-projective and $\text{tr}_C(P \otimes_R A) = C$. But then $M$ is a graded projective $R$-module of finite type, and is faithful. Thus $E(M)$ represents the trivial element of $\mathcal{B}(R, G)$. It remains only to show that $E_A(P)$ is an Azumaya $R$-algebra. As noted earlier, $E_A(P)$ is concentrated in degree 1; thus we have an isomorphism of ungraded $R$-algebras

$$E_A(P) \otimes A \cong E(M).$$

It follows from [2, Theorem 4.1, Condition (6), p. 105] that $E_A(P)$ is an Azumaya $R$-algebra. This completes the proof of (3.8).

We now turn our attention to $\text{Gal}(R, G)$ and to the image of $\pi$. Let $S$ be a graded Galois extension of $R$ with group $G$. Define a $G$-action on $S^\#$ by $\sigma(s^\#) = (\sigma^{-1}s)^\#$. It is easily verified that $S^\#$ is a Galois extension, using criterion (b) of [4, Theorem 1.3]: If $x_1, \ldots, x_n, y_1, \ldots, y_n$ are homogeneous elements of $S$ satisfying $\sum_i x_i \sigma(y_i) = \delta_1, \sigma$, then $\phi(x_i, y_i) = y_i^\#$, $x_i^\#$, $i = 1, \ldots, n$, are corresponding elements of $S^\#$; the fact that $(S^\#)^G = R$ follows from $S^G = R$.

If $S$ and $T$ are graded Galois extensions with group $G$, then $S \otimes T$ is a graded Galois extension with group $G \times G$: The existence of elements $u_j, v_j$ satisfying $\sum_i u_i(\sigma, \tau)v_j = \delta_{i, \sigma} \delta_{j, \tau}$ is a consequence of the existence of similar elements in $S$ and $T$; that $(S \otimes T)^{G \times G} = R$ is derivable from the fact that $\text{trace}_{G \times G}(S \otimes T) = R$ [4, Lemma 1.6].

Let $S$ be a graded Galois extension with group $G$, and let $H$ be a subgroup of $G$. Then $S^H$ is a graded Galois extension of $R$ with group $G/H$. This is easily derived using, e.g. [8, Lemma 1], (cf. [4, Theorem 2.2], [17, Proposition 1]).

Let $\Delta G = \{(\sigma, \sigma^{-1}) | \sigma \in G\}$. Putting together the preceding three paragraphs, we conclude that
We wish to show that if \( \pi(A) = \pi(B) \) for \( A \) and \( B \) fully graded Azumaya \( R \)-algebras, then \( \pi(A \otimes B^\#) = GR \), with trivial grading. First, we investigate the structure of \( \pi(B^\#) \) as a graded Galois extension. It is clear that as a graded \( R \)-algebra, \( \pi(B^\#) = \pi(B)^\# \). To determine the \( G \)-action on \( \pi(B^\#) \), one first chooses elements \( b_{\sigma,i} \) in \( B_{\sigma^{-1}} \), \( c_{\sigma,i} \) in \( B_{\sigma} \) for which \( \sigma(b) = \phi(\sigma, b) \sum \sigma c_{\sigma,i} b_{\sigma,i} c_{\sigma,i} \) for \( b \) in \( \pi(B) \).

It is then a straightforward matter to compute that

\[
\sigma(x^\#) = \phi(\sigma, x)\phi(x, \sigma)(\sigma^{-1}x)^\#
\]

for \( x^\# \) a homogeneous element of \( \pi(B^\#) = \pi(B)^\# \).

As noted prior to (3.6), \( \pi(A \otimes B^\#) \subset \pi(A) \otimes \pi(B^\#) \). The formula \((*)\), derived in the proof of (3.6), implies that \( \sum a_i \otimes b_i^\# \) is in \( \pi(A \otimes B^\#) \) if and only if

\[
(\sigma, \sigma^{-1}) \sum a_i \otimes b_i^\# = \sum \phi(\sigma^{-1}, b_i) \phi(b_i, \sigma^{-1}) a_i \otimes b_i^\#.
\]

(3.10) The action of \( G \) on \( \pi(B^\#) \) is given by \( \sigma(x^\#) = \phi(\sigma, x)\phi(x, \sigma)(\sigma^{-1}x)^\# \)

Considering the \( G \times G \)-homogeneous components of \( \pi(A) \otimes \pi(B^\#) \), and using

(3.10) to compute \( \sigma^{-1}(b^\#) \), we obtain

(3.11) An element \( \sum a_i \otimes b_i^\# \), \( a_i \) in \( \pi(A) \), \( b_i \) in \( \pi(B) \), is in \( \pi(A \otimes B^\#) \) if and only if \( \sum_i a_i \otimes b_i^\# = \sum_i \sigma(a_i) \otimes (\sigma b_i)^\# \) for all \( \sigma \) in \( G \).

Now suppose \( \pi(A) = \pi(B) = S \). It is clear from (3.11), and from the discussion preceding (3.9), that we have an equality of graded Galois extensions, \( \pi(A \otimes B^\#) = (S \otimes S^\#)^{\Delta G} \).

(3.12) Let \( S \) be a graded Galois extension of \( R \) with group \( G \). There is an isomorphism of graded Galois extensions,

\[
(S \otimes S^\#)^{\Delta G} = GR,
\]

where \( GR \) is trivially graded. If \( A \) and \( B \) are fully graded Azumaya \( R \)-algebras with \( \pi(A) = \pi(B) \), then \( A \otimes B^\# \) is equivalent to a trivially graded central separable \( R \)-algebra.

Proof. The last statement follows from the first statement, the paragraph preceding (3.12), and (3.8). For the first statement, let \( \sum_i s_i \otimes t_i^\# \) be in \( (S \otimes S^\#)^{\Delta G} \); this is equivalent to having \( \sum_i s_i \otimes t_i^\# = \sum_i \sigma s_i \otimes (\sigma t_i)^\# \) for all \( \sigma \) in \( G \). Note also that because the \( G \)-action on \( (S \otimes S^\#)^{\Delta G} \) is via either factor of \( G \times G \), it follows that the map \( j \) defined below is indeed a map into \( GR \):

\[
j : (S \otimes S^\#)^{\Delta G} \rightarrow GR
\]

by \( j(\sum_i s_i \otimes t_i^\#)(t) = \sum_i s_i t_i^\#(t_i) \). For \( w \) an element of \( (S \otimes S^\#)^{\Delta G} \), let \( w_1 \) be the 1-homogeneous component of \( w \). It is clear that \( j(w) = j(w_1) \), since \( \sum_i s_i t_i^\#(t_i) \) is in \( R \subset S_1 \). It is then a straightforward matter to compute that \( j \) is an \( R \)-algebra homomorphism, and a \( G \)-module map as well; recall that \( G \) acts on \( GR \) by

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\( \sigma u(r) = u(\sigma^{-1} r) \). As a consequence of [4, Theorem 3.4] or [8, Proposition 1], \( j \) is an isomorphism. As we remarked above, the restriction of \( j \) to the 1-homogeneous component of \((S \otimes S^h)^A \) is already onto \( GR \) since \( j \) is onto \( GR \). It follows that \((S \otimes S^h)^A \) is trivially graded. This concludes the proof.

We now state the main result of this section:

\[ \text{(3.13) Theorem. There exists an exact sequence of groups} \]
\[ 1 \to B(R) \xrightarrow{\iota} B(R, G) \xrightarrow{\pi} \text{Im}(\pi) \to 1, \]

where \( \iota(A) = A \) and \( \pi(B) = B^{B_1} \) for \( B \) a fully graded Azumaya \( R \)-algebra.

\[ \text{Proof. It is easily seen that} \ \iota \ \text{is a homomorphism into the center of} \ B(R, G) \ \text{and is monic by (2.9). From (3.12) it follows that if} \ \pi(A) = \pi(B) \ \text{then there exist} \ \text{trivially graded Azumaya} \ R \text{-algebras} \ A_0 \ \text{and} \ B_0 \ \text{with} \ A \otimes A_0 \ \text{and} \ B \otimes B_0 \ \text{in the same equivalence class in} \ B(R, G) \ \text{. The converse follows from (3.6) and (3.8). Thus} \ \pi \ \text{induces a bijection between} \ B(R, G)/\iota B(R) \ \text{and the image of} \ \pi \ \text{in} \ \text{Gal}(R, G) \ \text{. Giving} \ \text{Im}(\pi) \ \text{the induced multiplication clearly makes} \ \pi \ \text{into a homomorphism whose kernel is the image of} \ \iota.} \]

\[ \text{(3.14) Remarks. Although the multiplication on} \ \text{Im}(\pi) \ \text{is defined formally,} \ \text{Proposition (3.6) describes the multiplication explicitly in certain situations, and} \ \text{provides enough information to enable us to obtain, in \S 4, a precise description of} \ \text{Im}(\pi) \ \text{when} \ G \ \text{is cyclic.} \]

\[ \text{We now indicate, without giving detailed proofs, that the image of} \ \pi \ \text{and the} \ \text{multiplication on it can be made explicit when the following conditions hold: (i) The only idempotents in} \ R \ \text{are} \ 0 \ \text{and} \ 1, \ \text{(ii) there exists a primitive} \ m \ \text{th root of} \ 1 \ \text{in} \ R, \ \text{where} \ m \ \text{is the exponent of} \ G \ \text{and (iii) the order of} \ G, \ \text{call it} \ n, \ \text{is a unit in} \ R. \ \text{Let} \ G^* \ \text{denote} \ \text{Hom}(G, U(R)). \ \text{The above assumptions imply that} \ G \cong G^* \ \text{and that the following relations hold for} \ \sigma \ \text{in} \ G \ \text{and} \ \chi \ \text{in} \ G^* \ [(18, \ p. 178), \ (11, \ Corollary 2.5)]:} \]

\[ \sum_{\chi \in G^*} \chi(\sigma) = \delta_{1, \sigma}, \ \sum_{\sigma \in G} \chi(\sigma) = \delta_{1, \chi}. \]

\[ \text{For} \ A \ \text{a fully graded Azumaya} \ R \text{-algebra there exist} \ x_{\sigma, i} \ \text{in} \ A_{\sigma^{-1}}, \ y_{\sigma, i} \ \text{in} \ A_\sigma \ \text{satisfying} \ \sum_i x_{\sigma, i} y_{\sigma, i} = 1/n. \ \text{Define an action of} \ G^* \ \text{on} \ A \ \text{by} \ \chi(a) = \chi(\sigma)a \ \text{for} \ a \ \text{in} \ A_{\sigma}. \ \text{Using the ideas of [4, Theorem 1.3], it can be shown that the existence of} \ x_{\sigma, i} \ \text{and} \ y_{\sigma, i} \ \text{imply that the map} \ j: \ D(A, G^*) \to \text{End}_{A_1}(A) \ \text{is an isomorphism;} \ D(A, G^*) \ \text{is the trivial crossed product,} \ A \ \text{is viewed as a right} \ A_1 \text{-module, and} \ j(au_x)(b) = a\chi(b). \ \text{Thus} \ A \ \text{is a Galois extension of} \ A_1 \ \text{with group} \ G^*, \ \text{according to Kanzaki's definition [12]. Define} \]

\[ A_{\sigma, x} = \{a \in A_{\sigma} \mid ba = \phi(r, \sigma)\chi(r)ab \ \text{for} \ b \ \text{in} \ A_{\tau} \}. \]
It is clear that $A_{\sigma_x} \subset \pi(A)$. In fact one can obtain that $\pi(A) = \bigoplus_{\sigma_x} A_{\sigma_x}$. This is accomplished by noting that $b: \pi(A) \to \text{End}_{A_1}(A_1)A$ via $b(x)(y) = \phi(y, x)^{-1}yx$ is an $R$-module isomorphism, and thus $\pi(A) = \text{End}_{A_1}(A_1)A \cong D(A, G^*)A$.

Each $A_{\chi}$ is a projective $R$-module of rank 1: for, the argument that $\pi(A) \cong D(A, G^*)^A$ shows implicitly that $A_{\chi} \cong (A_{\chi})^A$ and one may then apply (2.8)(d). It follows, as in [5, Theorem 1], that $A_{\chi}^A = A_{\chi}$. Define an action of $G$ on $\pi(A)$ by $\sigma(a) = \chi(\sigma)a$ for $a$ in $A_{\chi}$. This agrees with our usual action on $\pi(A)$. The fact that $n$ is a unit can be used to obtain a direct proof that $\pi(A)$ is a Galois extension of $R$ with group $G$.

Since $A_{\chi}$ is projective of rank 1, there exists a unique $\sigma$ for which $A_{\sigma, x} \neq 0$, and $A_{\sigma, x} = A_{\sigma}$. Thus determines a homomorphism $f_A: G^* \to G$. If $A$ and $B$ are fully graded Azumaya $R$-algebras, the product $\pi(A) \cdot \pi(B)$ is determined by the following observation: for $\theta$ in $G^*$, $(A \otimes B)_{\theta} = A_{\theta} \otimes B_{\theta}$ where

$$\theta'(\sigma) = \phi(f_B(\theta), \sigma)\phi(\sigma, f_A(\theta))\phi(\sigma).$$

The above decomposition of our Galois extensions relates to the ones of [13] and [5] (cf. (4.2)).

4. The image of $\pi$. In §3 we showed that there is an exact sequence of groups

$$0 \to B(R) \to B(R, G) \to \text{Im}(\pi) \to 1,$$

where, for $A$ a fully graded Azumaya $R$-algebra, $\pi(A) = A_{A^1}$. In this section we describe $\text{Im}(\pi)$ for $G$ a finite cyclic group. Note that for $G$ a cyclic group the bilinear map $\phi$ is necessarily symmetric, being determined by $\phi(\sigma, \tau) = 1$ for all $\tau$ in $G$; otherwise $\phi$ will be said to be nondegenerate on $G$.

Let $\phi$ be a bilinear map on a group $G$. We shall say that $\phi$ is degenerate on $G$ if there exists $\sigma \neq 1$ in $G$ satisfying $\phi(\sigma, \tau) = 1$ for all $\tau$ in $G$; otherwise $\phi$ will be said to be nondegenerate on $G$.

Let $p$ be in $\text{Spec}(R)$ and $f$ in $\text{Cont}(\text{Spec}(R), (\mathbb{Z}/2\mathbb{Z})^r)$, the set of continuous functions from $\text{Spec}(R)$ to the $r$-fold product of $\mathbb{Z}/2\mathbb{Z}$. Write $f(p)_i$ for the projection of $f(p)$ on the $i$th copy of $\mathbb{Z}/2\mathbb{Z}$.

(4.1) Theorem. Let $G = \prod_{i=1}^r G_i$ be a cyclic group of order $n$, with $G_i$ of order $p_i^e_i$ and $p_1, \ldots, p_r$ distinct primes. Then there is an exact sequence of groups:

$$0 \to \text{Gal}(R, G) \to \text{Im}(\pi) \to \text{Cont}(\text{Spec}(R), (\mathbb{Z}/2\mathbb{Z})^r).$$

The image of $\beta$ is the set of functions $f$ satisfying $f(p)_i = 0$ whenever $p_i \in p$ or $\phi$ is degenerate on $G_i$. 

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The rest of this section is devoted to proving the result just stated. The preliminary results will, however, not use the fact that $G$ is cyclic; rather, the commutativity of $\pi(A)$ will be a sufficient assumption. We shall focus our attention on two types of gradings for commutative Galois extensions in $\text{Im}(\pi)$. One is the trivial grading. The other is called the identity grading and is described as follows:

(4.2) Let $R$ have no idempotents but 0 and 1. Assume that $G$ is a finite abelian group, that its order is a unit in $R$, and that $\phi$ is symmetric and nondegenerate. Let $S$ be a commutative Galois extension of $R$ with group $G$. The nondegeneracy of $\phi$ implies that $R$ contains a primitive $m$th root of unity, where $m$ is the exponent of $G$. Since the order of $G$ is also a unit in $R$, $S$ decomposes as $S = \bigoplus_{\sigma \in G} S_\sigma$, with

$$S_\sigma = \{ s \in S \mid \tau(s) = \phi(\sigma, \tau)s \text{ for } \tau \in G \},$$

a rank one projective $R$-module; this is proved as Theorem 1 of [5], using the orthogonality relations (3.15). The $S_\sigma$'s satisfy $S_\sigma S_\tau = S_{\sigma \tau}$, and the nondegeneracy of $\phi$ then implies that $S$ is a fully graded Azumaya $R$-algebra (separability is a consequence of $S$ being a Galois extension). Moreover, $\pi(S) = S$ since $S$ is commutative, and the action of $G$ on $\pi(S)$ (defined in (3.2)) agrees with the original action of $G$ on $S$.

A commutative graded Galois extension $S$ will be said to be identity graded if $S = \bigoplus_{\sigma \in G} S_\sigma$, with $S_\sigma$ homogeneous of degree $\sigma$ and $\sigma(s) = \phi(\sigma, \tau)s$ for $s$ in $S_\tau$ and all $\sigma, \tau$ in $G$.

If $A$ is an Azumaya $R$-algebra, it is $R$-separable, and therefore separable over its ungraded center $Z$ [1, Theorem 2.3]. Thus $A$ is an Azumaya $Z$-algebra and $Z$ is therefore a $Z$-direct summand of $A$ [1, Proposition 1.2]. Moreover, $Z = \bigoplus_{\sigma \in G} Z_\sigma$, with $Z_\sigma = A_\sigma \cap Z$. Since $A$ is a projective $R$-module, so is each $Z_\sigma$. We shall assume throughout the next proof that $G$ is a finite abelian group. However, the full force of this assumption is not needed for each of the desired conclusions. This is mentioned in the comments following (4.3).

(4.3) Theorem. Let $R$ have no idempotents but 0 and 1. Let $A$ be an Azumaya $R$-algebra with ungraded center $Z$. Let $H = \{ \sigma \in G \mid Z_\sigma \neq 0 \}; H^\perp = \{ \sigma \in G \mid \phi(\sigma, r) = 1 \text{ for all } r \text{ in } H \}$.

(a) $H$ is a subgroup of $G$ and the order of $H$ is a unit in $R$. If $R$ is local then $Z$ is a crossed product $RH$, for $f : H \times H \to U(R)$ a suitable cocycle,

(b) Suppose $A$ is fully graded and $\pi(A)$ is commutative. Let $Z(A_1)$ denote the center of $A_1$. The $G$-action on $\pi(A)$ induces a $H$-action (resp. $H^\perp$-action) making $Z$ (resp. $Z(A_1)$) an identity graded (resp. trivially graded) Galois extension of $R$. $\phi$ is nondegenerate on $H$, $G = H \times H^\perp$, and the multiplication map $Z \otimes_R Z(A_1) \to \pi(A)$ is an isomorphism of graded $R$-algebras and Galois extensions.
(c) Under the assumptions in (b), $Z_A$, the graded commutator of $Z$ in $A$, is an Azumaya $R$-algebra, graded by $H^1$. The multiplication map yields an isomorphism of graded $R$-algebras, $\pi(A) \cong Z \otimes_R Z_A$. If $\phi$ is symmetric, $Z_A$ is an $H^1$-graded Azumaya $R$-algebra.

Proof. Since $R$ has no nontrivial idempotents, each $Z_\sigma$ is a faithful projective $R$-module of finite type. Thus, for $p$ a maximal ideal of $R$, $pZ_\sigma \neq Z_\sigma$, and the condition $Z_\sigma \neq 0$ is equivalent to $(Z/pZ)_\sigma \neq 0$. Thus, it suffices to prove (a) under the assumption that $R$ is a field, since if $\text{card}(H)$ is a unit in $R/p$ for each $p$, it is a unit in $R$; that $A/pA$ is an Azumaya $R$-algebra with ungraded center $Z/pZ$ is a consequence of (2.2) and of [1, Proposition 1.4].

Let $R$ be a field. By (2.5), $A$ is an Azumaya $R$-algebra if and only if it has center $R$, and no homogeneous two-sided ideals. Following Knus [14, p. 126], we have that if $x, y$ are nonzero elements of $hZ$ and $xy = 0$, the homogeneous ideal generated by $\{a \in hA\mid ay = 0\}$ is a nonzero, proper, homogeneous two-sided ideal. Thus $H$ is closed under multiplication.

Continue to assume that $R$ is a field. Note that $Z_1 = R$. Choose a nonzero element $x_\sigma$ in $Z_\sigma$ and assume $\sigma^m = 1$; then $x_\sigma^m$ is in $R$, hence $Z_\sigma = Z_\sigma x_\sigma^m \subseteq Z_1 x_\sigma \subseteq Z_\sigma$. This implies each $Z_\sigma$ is of rank 1 over $R$. Hence, for $\sigma, \tau$ in $H$, $x_\sigma^m x_\tau = f(\sigma, \tau) x_{\sigma \tau}$ for a suitable cocycle $f: H \times H \to U(R)$. This shows that $Z \cong RH_f$, a crossed product; this argument is taken from [14, Theorem 3.1]. Now $Z$ is $R$-separable by the tower lemma [1, Theorem 2.3]; but a crossed product $RH_f$ is $R$-separable if and only if the order of $H$ is a unit in $R$ [9, Lemma 4].

Let $R$ now be any commutative ring without nontrivial idempotents. We claim that $Z_\sigma Z_\tau = Z_{\sigma \tau}$ for $\sigma, \tau$ in $H$. Each $Z_\sigma$ is a projective $R$-module having a well-defined rank. This rank is in fact one for $\sigma$ in $H$, since for each maximal ideal $p$ of $R$, $Z_\sigma/pZ_\sigma$ is a one-dimensional $R/p$-space, as shown above. The multiplication map $m: Z_\sigma \otimes Z_\tau \to Z_{\sigma \tau}$ induces an onto map $(Z_\sigma/pZ_\sigma) \otimes (Z_\tau/pZ_\tau) \to Z_{\sigma \tau}/pZ_{\sigma \tau}$ for each maximal ideal $p$ of $R$; this implies that $p(Z_{\sigma \tau}/\text{Im}(m)) = Z_{\sigma \tau}/\text{Im}(m)$ and that, for some $r_p$ in $R - p$, $r_p(Z_{\sigma \tau}/\text{Im}(m)) = 0$ [2, Lemma 5.8, p. 69]. The annihilator of $Z_{\sigma \tau}/\text{Im}(m)$ is therefore not contained in any maximal ideal, and $m$ is thus onto.

If $R$ is local each $Z_\sigma$ is a free $R$-module with generator $x_\sigma$, $\sigma$ in $H$. Since $Z_\sigma Z_\tau = Z_{\sigma \tau}$, it follows that $x_\sigma x_\tau = f(\sigma, \tau) x_{\sigma \tau}$ with $f(\sigma, \tau)$ a unit in $R$, and $f$ a cocycle. This proves (a).

Let $B = \bigoplus_{\sigma \in H} A_\sigma$. Since $Z_\sigma Z_\tau = Z_{\sigma \tau}$, it is easily verified that the multiplication map $Z_\sigma \otimes_R A_1 \to A_\sigma$ is onto for $\sigma$ in $H$. This map is in fact an isomorphism: to check this, it suffices to assume $R$ is a field. But then $Z_\sigma = Rx_\sigma$, $A_\sigma = A_1 x_\sigma$ and no element of $A_1$ annihilates $x_\sigma$, since $A$ is graded simple. It follows that $A_1$ and $A_\sigma$ have the same dimension, and $Z_\sigma \otimes_R A_1 \to A_\sigma$ is an
isomorphism. Hence we may identify \( B = ZA_1 = Z \otimes_R A_1 \).

Now assume \( A \) is fully graded. Then \( A_1 \) is \( R \)-separable by (2.11), hence \( B \) is \( R \)-separable. Since \( B = ZA_1 \), and since \( \pi(A) \) is the ungraded commutator of \( A_1 \) in \( A \), \( \pi(A) \) is the ungraded commutator of \( B \) in \( A \). By [12, Theorem 2], \( B \) is the ungraded commutator of \( \pi(A) \) in \( A \). If \( \pi(A) \) is commutative, \( \pi(A) \) must be contained in \( B \). It is easily seen that in fact \( \pi(A) \) is the center of \( B \). Since \( Z \otimes_R A_1 \to B \) is an isomorphism, it follows from [1, Proposition 1.4] that the multiplication map determines an isomorphism \( Z \otimes_R Z(A_1) \to \pi(A) \).

We can now show that \( \phi \) is nondegenerate on \( H \), i.e. \( H \cap H^\perp = 1 \). The \( G \)-action on \( \pi(A) \), defined in (3.2), is such that \( H \) acts trivially on \( Z(A_1) \), since \( Z_{\sigma^{-1}}Z_{\sigma} = Z_1 \). By definition of \( H^\perp \), this group acts trivially on \( Z \). Thus, an element of \( H \cap H^\perp \) acts trivially on \( \pi(A) \), and must be 1. It follows that the homomorphism \( \phi' : G \to H^* \), defined by \( \phi'(a)(r) = \phi(a, r) \) is onto, and an isomorphism onto \( H^* \) when restricted to \( H \). Since the kernel of \( \phi \) is \( H \), we may conclude that \( G = H \times H^\perp \). The discussion above, and that of (3.14), may be applied to yield that \( Z \) is a Galois extension of \( R \) with group \( H \). Moreover, \( H^\perp \) acts as a group of \( R \)-algebra automorphisms of \( Z(A_1) \), and the multiplication map preserves the action of \( G = H \times H^\perp \). To prove that \( Z(A_1) \) is a Galois extension of \( R \) with group \( H^\perp \), it will suffice to prove that \( Z(A_1) = \pi(A)^H \) [4, Theorem 2.2].

That \( Z(A_1) \) is included in \( \pi(A)^H \) follows from the definition of the action (in (3.2)) and from the fact that \( Z_{\sigma^{-1}}Z_{\sigma} = R \). Suppose that \( x \) is a homogeneous element in \( \pi(A)^H \). Since the multiplication map \( Z \otimes_R Z(A_1) \to \pi(A) \) is an isomorphism, \( x \) is in \( \pi(A)_r \) for some \( r \) in \( H \). The equations \( \sigma x = \phi(\sigma, r)x = x \) may be summed over \( \sigma \) in \( H \), yielding \( (\Sigma_\sigma \phi(\sigma, r))x = nx \), where \( n \) is the order of \( H \). The above comments concerning \( H \) and \( H^* \), and the orthogonality relations (3.15), allow us to conclude that \( nx = 0 \) unless \( r = 1 \). Since \( n \) is a unit in \( R \), \( r \) must be 1. This concludes the proof of (b).

Let \( C = \bigoplus_{\sigma \in H^\perp} A_\sigma \). Then \( C \subseteq ZA \). The multiplication map \( Z \otimes_R C \to A \) is onto, because \( G = H \times H^\perp \) and \( Z_{\sigma}Z_{\tau} = Z_{\sigma \tau} \); in fact this map is an isomorphism. To prove this, it suffices to show that the multiplication map \( Z \otimes_R ZA \to A \) is an isomorphism. This is true modulo each maximal ideal of \( R \) by Knus's argument in [14, Theorem 3.1]; we have shown in (b) that the hypothesis needed in Knus's proof, viz. that \( \phi \) is nondegenerate on \( H \), holds. It follows that \( Z \otimes_R ZA \to A \) is an isomorphism, and that \( ZA = \bigoplus_{\sigma \in H^\perp} A_\sigma \).

Since \( Z \otimes_R ZA \) is an Azumaya \( Z \)-algebra, \( ZA \) is an Azumaya \( R \)-algebra [2, Corollary 2.9 (b), p. 93]. It is easily computed that \( C^C = R \). If \( \phi \) is symmetric, \( C \) is a central \( R \)-algebra.

Remarks. (a) Some of the assumptions used above can be weakened. By a localization argument, it can be shown that connectedness of \( R \) is not necessary
in order for the multiplication maps $Z \otimes_R Z(A) \rightarrow \pi(A)$ and $Z \otimes_R ZA \rightarrow A$ to be isomorphisms. If one wishes to treat Azumaya algebras for which the grading group $G$ is not necessarily finite, or abelian (as Knus does in [14]), one can still obtain some of the results of (4.3). For example, the condition $Z_\sigma Z_r = Z_\sigma r$ implies that $H$ is a central subgroup of $G$, and since $H$ is finite because $Z$ is an $R$-module of finite type the arguments used before apply more generally.

(b) DeMeyer has shown that if $G$ is a finite cyclic group, any Galois extension of $R$ with group $G$ is commutative [7, Theorem 11]. We shall use this result freely below.

Proof of (4.1). First define $\alpha: \text{Gal}(R, G) \rightarrow \text{Im}(\pi)$: Let $S$ be a Galois extension of $R$ with group $G$; $S$ is commutative since $G$ is cyclic. Let $D = D(S, G)$ be the trivial crossed product of $S$ with $G$; i.e. $D = \bigoplus_{\sigma \in G} S u_\sigma$, with multiplication given by $(s u_\sigma) (t u_\tau) = s \sigma(t) u_{\sigma \tau}$. Theorem 1.3(c) of [4] implies that $S$ is a projective $R$-module of finite type, and that there is an $R$-algebra isomorphism $D \cong \text{End}_R(S)$. Consequently $D$ is an Azumaya $R$-algebra [1, Proposition 5.1], and in particular separable. We may grade $D$ by $D_\sigma = S u_\sigma$. It is easily seen that $s u_\sigma$ is in $D^D$ if and only if $s(t - \sigma(t)) = 0$ for all $t$ in $S$. It follows, because $S$ is a Galois extension [4, Theorem 1.3(f)], that $s = 0$ or $\sigma = 1$. Hence $\tau(s) = s$ for all $\tau$ in $G$, so that $D^D = R$. Then $D$ is an Azumaya $R$-algebra. It is easily computed that $\pi(D) = S u_1 = S$. Thus $\alpha(S) = \pi(D) = S$ gives a well-defined map which is clearly one-one.

Let $\sigma_i$ be a generator of $G_i$. We now define $\beta$. Let $A$ be a fully graded Azumaya $R$-algebra, with ungraded center $Z$. For $\mathfrak{p}$ in $\text{Spec}(R)$, $A_\mathfrak{p}$ is a fully graded Azumaya $R_\mathfrak{p}$-algebra with ungraded center $Z_\mathfrak{p}$ ([2, Corollary 2.9, p. 93]). Let $H_\mathfrak{p}$ be the subgroup of $G$ consisting of those $\sigma$ for which $(Z_\mathfrak{p})_{\sigma} \neq 0$. Define $\beta(\pi A)(\mathfrak{p})_i$ to be 1 or 0 depending upon whether $\sigma_i$ is in $H_\mathfrak{p}$ or not; equivalently, $\beta(\pi A)(\mathfrak{p})_i = \text{rank}_{R_\mathfrak{p}} [(Z_{\sigma_i})_{\mathfrak{p}}]$, and $\beta(A)$ is therefore a continuous map from $\text{Spec}(R)$ to $\Pi \mathbb{Z}/2\mathbb{Z}$. That $\beta$ is well defined follows from the observation that two equivalent algebras have the same ungraded center ([1.5], [2, Corollary 2.6, p. 91], [1, Proposition 5.1]). Suppose $\beta(\pi A)(\mathfrak{p})_i = 1$; then $G_i$ is included in $H_\mathfrak{p}$. Since by (4.2) the order of $H_\mathfrak{p}$ is a unit in $R_\mathfrak{p}$, so is the order of $G_i$, and consequently $p_i$ is not in $\mathfrak{p}$. Again by (4.2), $\phi$ is nondegenerate on $H_\mathfrak{p}$; since $H_\mathfrak{p} = G_i \times L_i$, with the orders of $G_i$ and $L_i$ relatively prime, $\phi$ is nondegenerate on $G_i$. Thus, the image of $\beta$ is contained in the indicated set.

It is easily seen that $\beta \alpha = 0$, since for $S$ a commutative Galois extension of $R$ with group $G$, $D(S, G)$ has ungraded center $R$. Conversely, if $\beta(\pi A) = 0$, then the ungraded center $Z$ of $A$ is $R$, since $(Z_{\sigma})_{\mathfrak{p}} = 0$ for $\mathfrak{p}$ in $\text{Spec}(R)$ and $\sigma \neq 1$ in $G$. By (4.2), $\pi(A) = Z(A_1)$, a trivially graded Galois extension. Consequently, $\pi(A) = \alpha(\pi A)$. The sequence is thus exact.
We now show that the image of $\beta$ is as claimed. Fix an index $i$ and suppose $\phi$ is nondegenerate on $G_i$. Let $f$ be a continuous map from $\text{Spec}(R)$ to $\prod_{i \neq i} Z/2Z$ with $f(p_{ij}) = 0$ for $j \neq i$ and $f(p_{ij}) = 0$ or 1 depending on whether $p_i$ is in $p_j$ or not. Then ([3, p. 130])

$$\text{Spec}(R) = \{ p | f(p_{ij}) = 1 \} \cup \{ p | f(p_{ij}) = 0 \},$$

a disjoint union of open and closed sets. This leads to a decomposition $R = S \times T$, with $\text{Spec}(S) = \{ p | f(p_{ij}) = 1 \}$. Consider the group ring $SG_i$; first of all $p_i$ is not in any element of $\text{Spec}(S)$ and is therefore a unit in $S$. Coupled with the fact that $\phi$ is nondegenerate on $G_i$, this implies that the orthogonality relations hold between $G_i$ and $G_i^*$ (cf. (3.15)) and that the center of $SG_i$ is $S$ (the grading is by $G_i$). Because the order of $G_i$ is a unit in $S$, $SG_i$ is $S$-separable [21, Theorem 1.1], and thus an Azumaya $S$-algebra.

Let $H_i = \prod_{j \neq i} G_j$, so that $G = G_i \times H_i$. Let $T_i$ be any commutative Galois extension of $T$ with group $H_i$; e.g. $T_i = H_i T$, the algebra of set functions from $H_i$ to $T$ (cf. [10, §1, p. 68]). By the argument used previously in this proof, $D_i = D(T_i, H_i)$ is a $H_i$-graded Azumaya $T$-algebra, with ungraded center $T$.

Let $C = SG_i \times D_i$, an $R$-algebra since $R = S \times T$. The formula $C_{\sigma,T} = (SG_i)^{\sigma} \times (D_i)^T$ defines a $G_i \times H_i$-grading, i.e. a $G$-grading on $C$. That $C$ is $R$-separable is a result about ungraded algebras [2, Proposition 2.20, p. 99]. A computation shows that $C$ is $R$-central, and thus an Azumaya $R$-algebra. The ungraded center of $C$ is $SG_i \times T$. Now, $C$ is equivalent to a fully graded Azumaya $R$-algebra $E$ whose ungraded center is that of $C$ (this was shown previously in this proof). Then $\beta(\pi E) = f$, and the image of $\beta$ is as claimed in the statement of the theorem.

To complete the proof it suffices to prove that $\alpha$ and $\beta$ are homomorphisms. Let $S$ and $T$ be Galois extensions of $R$ with group $G$. The product of $S$ and $T$ in $\text{Gal}(R, G)$ is $S \cdot T = (S \otimes_R T)^{AG}$, the set of elements of $S \otimes_R T$ fixed by $\{(\sigma, \sigma^{-1}) | \sigma \in G\}$. Write $A$ for $D(S, G)$, $B$ for $D(T, G)$. The multiplication in $\text{Im}(\pi)$ is such that $\pi(A) \cdot \pi(B) = \pi(A \otimes B)$ (cf. (3.13)). But $\pi(A)$ and $\pi(B)$, i.e. $S$ and $T$, are trivially graded. By (3.6), we conclude that $\alpha$ is a homomorphism.

Since $\beta$ being a homomorphism depends on the behavior of our algebras at localizations, we shall assume that $R$ is local. Let $E$ be a fully graded Azumaya $R$-algebra with ungraded center $Z(E)$, $H$ the set of $\sigma$ in $G$ with $Z(E)^{\sigma} \neq 0$. We obtain from (4.3) that $Z(E) = \pi(E)^H$ and $Z(E^1) = \pi(E)^{H^\perp}$, with $G = H \times H^\perp$. Fix an index $i$ and let $G_i = \langle \sigma \rangle$. Now $Z(E)^{\sigma} = 0$ is equivalent to $\langle \sigma \rangle$ not being a subgroup of $H$, which is in turn equivalent to $\langle \sigma \rangle$ being a subgroup of $H^\perp$ since $G$ is cyclic. We therefore conclude:

(A) $Z(E)^{\sigma} = 0$ (resp. $\neq 0$) if and only if $\pi(E)^{\sigma}$ is included in $Z(E^1)$ (resp. $Z(E)$).
Let $A$ and $B$ be fully graded Azumaya $R$-algebras (with $R$ still local and $G_i = \langle \sigma \rangle$), and let $C = A \otimes B$. We must relate $\pi(A)_\sigma$ and $\pi(B)_\sigma$ being nonzero to $\pi(C)_\sigma$ being nonzero, and to this end we shall examine $\pi(C)$ more closely. Formula $(\ast)$ of (3.6) allows us to identify $\pi(C)$ as a subset of $\pi(A) \otimes \pi(B)$. We may rephrase this formula by saying that $\pi(C) = [\pi(A) \otimes \pi(B)]^{DG}$, where $G \times G$ acts on $\pi(A) \otimes \pi(B)$ via $(\theta, r)(a \otimes b) = \phi(r^{-2}, b)\theta(a) \otimes r(b)$, and $DG$ is the kernel of the multiplication map from $G \times G$ to $G$. The $G$-action on $[\pi(A) \otimes \pi(B)]^{DG}$ compatible with that on $\pi(C)$ is $(a \otimes b) = (a \otimes r(b) = \phi(r^2, b)r(a) \otimes b$. We therefore obtain that $\pi(C)^G = [\pi(A)^G \otimes \pi(B)^G]^{DG}$, where $G_+ \times G_+$ acts on $[\pi(A)^G \otimes \pi(B)^G]$ via $(a \otimes b, A) = a \otimes (A \otimes A) = \phi(r, A)r(a) \otimes (A \otimes A)$. We therefore obtain that $\pi(C)$ is included in $[\pi(A)^G \otimes \pi(B)^G]$ under an inclusion $[\pi(A)^G \otimes \pi(B)^G] \subseteq \pi(C)$, since both algebras are Galois extensions of $R$ with group $K$, this inclusion is an equality [4, Theorem 3.4]. We therefore have

$$\pi(C) = [\pi(A)^G \otimes \pi(B)^G]^{DK}$$

To complete the proof that $\beta$ is a homomorphism, i.e. that $\beta(\pi(A) \cdot \pi(B))(\pi)_i = \beta(\pi(A) \cdot \pi(B))(\pi)_i$, we must examine four cases.

Case (0-0). If $Z(A)_\sigma = 0 = Z(B)_\sigma$ then $\pi(A)^G \subseteq Z(A)_1$ and $\pi(B)^G \subseteq Z(B)_1$ by (A), whence $Z(C)^G \subseteq Z(C)$ by (B).

Case (0-1). If $Z(A)_\sigma = 0$ and $Z(B)_\sigma \neq 0$ then the grading on $\pi(C)^G$ comes from that on $\pi(B)^G$. For $r \in G$, $a \in \pi(A)^G$ and $b \in \pi(B)^G$ we have $r(a \otimes b) = a \otimes r(b) = \phi(r, b)a \otimes b$ (since $Z(B)$ is identity graded). Hence $\pi(C)^G$ is identity graded and contained in $Z(C)$.

Case (1-0). If $Z(A)_\sigma \neq 0$ and $Z(B)_\sigma = 0$, a computation similar to the above shows that $Z(C)_\sigma = 0$.

Case (1-1). If $Z(A)_\sigma \neq 0$ and $Z(B)_\sigma \neq 0$, then an element $a_\theta \otimes b_\tau$ of $\pi(C)^G$ satisfies

$$a_\theta \otimes b_\tau = \phi(\sigma^2, \tau)\sigma(a_\theta) \otimes \sigma^{-1}(b_\tau) = \phi(\sigma^2, \tau)\phi(\sigma, \theta)\phi(\sigma^{-1}, r)(a_\theta \otimes b_\tau) = \phi(\sigma, \theta)\phi(\sigma, \tau)(a_\theta \otimes b_\tau).$$

Since $\phi$ is nondegenerate on $G_i$, $\theta = r^{-1}$ and $\pi(C)^G$ is trivially graded. This completes the proof of (4.1).

(4.5) Example. The group $B(R, G)$ need not be abelian. We shall in fact see that even $\text{Im}(\pi)$ need not be abelian.

Let $G = \mathbb{Z}/3\mathbb{Z}$ and let $\sigma$ be a generator of $G$. Let $R$ contain $1/3$, a primitive cube root of unity $\xi$, and a unit $b$ which is not a cube in $R$. Let $a$ be a unit in $R$. Choose a bilinear map $\phi$ satisfying $\phi(\sigma, \sigma) = \xi$. Let $S = R \oplus R \oplus R \oplus R$ with $u^3 = a$, $T = R \oplus R \oplus R \oplus R$ with $v^3 = b$. View $S$ as fully graded by $G$, $T$ as
trivially graded. Each of $S$ and $T$ is in $\text{Im}(\pi)$, since $S = \pi(S)$ and $T = \pi(D(T, G))$.

We have that $S \cdot T = (S \otimes T)^{AG}$ in $\text{Im}(\pi)$, where $G \times G$ acts on $S \otimes T$ by $(\sigma, r)(a \otimes b) = \phi(r, b)s(a) \otimes r(b)$. One verifies that

$$S \cdot T = R \oplus R(u \otimes v) \oplus R(u \otimes v)^2,$$

with $s(u \otimes v) = \xi(u \otimes v)$ and $(u \otimes v)^3 = ab$, whereas

$$T \cdot S = R \oplus R(v^2 \otimes u) \oplus R(v^2 \otimes u)^2,$$

with $s(v^2 \otimes u) = \xi(v^2 \otimes u)$ and $(v^2 \otimes u)^3 = ab^2$. Since $S \cdot T$ and $T \cdot S$ are both Galois extensions with normal bases, and $R$ possesses a cube root of unity and $1/3$, $S \cdot T$ and $T \cdot S$ are isomorphic if and only if $ab$ and $ab^2$ represent the same element in $H^2(G, U(R)) \cong U(R)/U(R)^3$ [5, Remark 3]. Since $b$ is not a cube in $R$, we have that $S \cdot T 
 T \cdot S$ in $\text{Im}(\pi)$.

We are able to describe $\text{Im}(\pi)$ intrinsically for $G$ any finite abelian group and $\phi$ trivial. In this case $\text{Im}(\pi)$ is $\text{Gal}(R, G)$, the group of ungraded Galois extensions of $R$ with group $G$. As the next example shows, however, we cannot expect to describe $\text{Im}(\pi)$ as easily for general $G$ as for finite cyclic $G$.

(4.6) Example. We shall show that the technique of splitting off the ungraded center (cf. (4.3)) does not necessarily work if $\pi(A)$ is not commutative. We shall also show that the conclusion of [4, Theorem 3.1], viz. that an Azumaya $R$-algebra decomposes as $RH \otimes_R M$, with $M$ a matrix ring, may hold even if the hypotheses of that theorem are not satisfied.

Let $Q$ be the field of rationals, $R = Q(i)$, $c$ a nonsquare in $R$ and $Z = R(a)$, with $a^2 = c$. Let $A$ (resp. $B$) be the generalized quaternion algebra over $Z$ generated by $u$ and $v$, with $u^2 = \alpha$, $uv = -vu$ and $v^2 = \alpha$ (resp. $v^2 = c\alpha$). Let $G = Z/4Z \times Z/2Z$ with $\sigma$ and $\tau$ generators of $Z/4Z$ and $Z/2Z$ respectively. Let $C$ stand for either $A$ or $B$; grade $C$ by setting $C_{\sigma} = Ru$, $C_{\sigma \tau} = Rv$. Define $\phi$ by $\phi(\sigma, \sigma) = i$, $\phi(\sigma, \tau) = \phi(\tau, \sigma) = \phi(\tau, \tau) = -1$. A computation shows that $C$ is central over $R$. $C$ is a central simple $Z$-algebra (since it is a generalized quaternion algebra), and is therefore separable over $Z$. In turn $Z$ is separable over $R$ [11, Corollary 2.4] and, by the tower lemma, $C$ is separable over $R$. Hence, $C$ is an Azumaya $R$-algebra. Now $\pi(C) = C$ since $C_1 = R$; thus $C$ is a graded noncommutative Galois extension of $R$ with group $G$. The action of $G$ is determined by the formulas $s(u) = iu$, $s(v) = iv$, $r(u) = u$, $r(v) = -v$. Now $Z$ is graded by $Z = C_1 \oplus C_{\sigma^2}$, hence $H = \{1, \sigma^2\}$ (cf. (4.3)). It is clear that $H$ is contained in $H^1$, so that the hypothesis of Knus's theorem, viz. that $\phi$ is nondegenerate on $H$, does not hold. The multiplication map $Z \otimes_R Z \rightarrow C$ is neither one-one nor onto (a computation shows that $Z_C = C_1 \oplus C_{\sigma 2} \oplus C_{\sigma 2} \oplus C_{\tau}$).

However, $A$ is isomorphic to the algebra of $2 \times 2$ matrices over $Z$, with $u$
and $v$ corresponding to the respective matrices $\begin{pmatrix} 0 & i \sigma \\ -i & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & c \\ 1 & 0 \end{pmatrix}$. It is not difficult to show that $B$ is not isomorphic to this matrix algebra over $\mathbb{Z}$; one uses the fact that a $2 \times 2$ matrix is similar to one with a zero diagonal entry and pursues a straightforward computation based on the equations $u^2 = \alpha$, $v^2 = c\alpha$, $uv = -vu$.

For $R$ a field let $\mathcal{O}_R$ be the abelianized Galois group of $R$. For $G$ cyclic, $\text{Gal}(R, G) \cong \text{Hom}(\mathcal{O}_R, G)$ [10, Theorem 8]. Thus, our exact sequences provide a connection between the Brauer group of $R$ and the fundamental group of $R$.

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