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THE MODULUS OF THE BOUNDARY VALUES OF BOUNDED ANALYTIC FUNCTIONS OF SEVERAL VARIABLES

BY

CHESTER ALAN JACEWICZ

ABSTRACT. One necessary condition and one sufficient condition are given in order that a nonnegative function be the modulus of the boundary values of a bounded analytic function on the polydisc. As a consequence, a weak version of a theorem of F. Riesz is generalized to several variables. For special classes of functions several conditions are given which are equivalent to a function's being the modulus of the boundary values of a bounded analytic function. Finally, an algebraic structure is provided for these special classes of functions.

1. **Introduction.** Let U^n be the unit polydisc in complex n-space and let $H^\infty(U^n)$ be the linear space of all bounded analytic functions on U^n. By an extension of a well-known theorem of Fatou, every h in $H^\infty(U^n)$ has a radial limit which we denote by h^+. h^+ exists as a function on T^n, the distinguished boundary of U^n. T^n is an n-torus and h^+ exists almost everywhere in the sense of Haar measure on T^n. In this paper we study a converse problem of deciding when a function w on T^n is of the form $w = |h^+|^2$ for h in $H^\infty(U^n)$. When h exists we shall say the boundary modulus problem is solvable for w.

It is well known that for a nonnegative w in $L^\infty(T^1)$, which is not the zero function, the boundary modulus problem is solvable for w if and only if $\log w$ is in $L^1(T^1)$. For $n \geq 2$ it is still necessary that $\log w$ be integrable [2, p. 46], but not sufficient [2, p. 56]. Rudin (cf. §2, this paper) has given a sufficient condition when $n \geq 2$. We are able to give conditions which are necessary and sufficient for the solvability of the boundary modulus problem, but at the expense of restricting the problem to a smaller class of functions. These conditions should serve both as tools for future work and as models of the possible complexity of the complete solution.

In addition to the notations already used, we need to introduce a few more. $H(U^n)$ will be the space of all holomorphic functions on U^n. For $1 \leq p \leq \infty$, $H^p(U^n)$ is the usual Hardy space of holomorphic functions on U^n. "Almost everywhere" refers to Haar measure on T^n, and, for $1 \leq p \leq \infty$, $L^p(T^n)$ denotes the

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usual Lebesgue space on T^n with respect to normalized Haar measure dx and with the standard norm $\| \|_p$. A null function is one which is zero almost everywhere. We shall also have need of f^+ for f in $H^p(U^n)$ and the identification of $H^p(U^n)$ with a subspace $H^p(T^n)$ of $L^p(T^n)$ by means of the correspondence of f with f^+. The necessary results are all to be found in our basic reference [2].

Finally, let us recall that an extended real-valued function is *lower semicontinuous* if and only if

$$\liminf_{z \to z_0} f(z) \geq f(z_0) > -\infty.$$

The basic properties of lower semicontinuous functions which we shall use are either to be found in [1] or are straightforward exercises in real analysis. We will use the following easily proved lemma.

Lemma 1.1. *Suppose w is positive lower semicontinuous and f is nonnegative continuous. Set $w_1(z) = w(z)/f(z)$ (∞ when $f(z) = 0$). Then $w_1(z)$ is positive lower semicontinuous.*

2. **A sufficient condition.** We begin by stating the following result of Rudin [2, p. 55]:

Theorem. *Suppose w is positive, bounded, and lower semicontinuous on T^n. Then there is an h in $H^\infty(U^n)$ such that $|h^+|^2 = w$ almost everywhere.*

The condition of positivity cannot be weakened to mere nonnegativity for $\log |h^+|$ is in $L^1(T^n)$, provided h is not identically zero.

The obstruction to finding h in general seems to be related to the zeroes of w. Theorem 2.1 states that if the zeroes of w can be "removed" by a function in $H^1(U^n)$, then the boundary modulus problem can be solved. More precisely:

Theorem 2.1. *Given w in $L^\infty(T^n)$, suppose there is an h in $H^1(U^n)$ and a function w_1 on T^n such that:*

(i) *$w = w_1 |h^+|^2$ almost everywhere,*

(ii) *w_1 is positive and lower semicontinuous.*

Then there is an f in $H^\infty(U^n)$ with $w = |f^+|^2$, almost everywhere.

Proof. We may assume that h is not identically zero. Note that w_1 is measurable since it is lower semicontinuous. Since w_1 is positive and lower semicontinuous it achieves its greatest lower bound c on T^n. Thus, almost everywhere, we have

$$0 < c \leq w_1 \leq \|w\|_\infty |h^+|^{-2}.$$

Hence, almost everywhere

$$\log c \leq \log w_1 \leq \log \|w\|_\infty - 2(\log |h^+|).$$

Being bounded above and below by integrable functions, $\log w_1$ is also integrable. We may take c to be 1.

Now we use a modification of Rudin's argument [2, p. 56]. Let

$$P(k)(z) = \int p(z, x)k(x)dx$$

where $p(z, x)$ is the Poisson kernel for z in U^n and x on T^n, [2, p. 16]. By a theorem of Rudin [2, p. 34] applied to $\log (w_1)^{1/2}$ there is a singular measure s on T^n, $s \geq 0$, and a g in $H(U^n)$ with

$$\operatorname{Re}(g) = P(\log (w_1)^{1/2} - ds) \tag{2.1}$$

in U^n.

Let f be the function $h \cdot \exp(g)$ in $H(U^n)$. Then in U^n

$$\begin{aligned}
\log|f| &= \log|h| + \operatorname{Re}(g) = \log|h| + P(\log(w_1)^{1/2} - ds) \\
&\leq \log|h| + P(\log(w_1)^{1/2}) \qquad (\text{as } s \geq 0) \\
&\leq P(\log|h^+|) + P(\log(w_1)^{1/2}) \text{ (by [2, p. 47])} \\
&= P(\log(|h^+|(w_1)^{1/2})) = P(\log(w)^{1/2}) \\
&\leq P(\log\|(w)^{1/2}\|_\infty) = \log\|(w)^{1/2}\|_\infty < \infty.
\end{aligned}$$

Thus the function f is in $H^\infty(U^n)$.

By [2, p. 24] applied to (2.1), $(\operatorname{Re}(g))^+$ exists and is $\log(w_1)^{1/2}$. But then

$$|f^+| = |h^+|\exp((\operatorname{Re}(g)^+) = |h^+|\exp(\log(w_1)^{1/2}) = |h^+|(w_1)^{1/2} = (w)^{1/2}.$$

Thus f is a solution.

There is a well-known theorem of F. Riesz that if p is a nonnegative polynomial on T^1 there is a trigonometric polynomial q such that $p = |q|^2$. It is known [3, p. 209] that the theorem does *not* generalize directly to $n \geq 2$. By weakening the conclusion, we are able to obtain the following generalization of Riesz's theorem from Theorem 2.1.

Corollary. *If p is a trigonometric polynomial on T^n which is nonnegative, then there is an f in $H^\infty(U^n)$ such that $p(z) = |f^+(z)|^2$ almost everywhere on T^n.*

Proof. As a trigonometric polynomial, p is a finite linear combination of characters on T^n. Thus, there is a character c such that cp is in $H^\infty(T^n)$. Set

$$w_1(z) = p(z)/(|c(z)p(z)|^2) = 1/p(z).$$

w_1 is positive and lower semicontinuous by Lemma 1.1. We may assume $p(z) \neq 0$ almost everywhere so that with $h = cp$, $p = w_1|h^+|^2$ almost everywhere. Theorem 2.1 now yields the function f.

The proof of this corollary really shows the following: Suppose h is in $H^\infty(U^n)$, and $|h^+| = w$ almost everywhere with $-w$ lower semicontinuous (i.e. w is upper semicontinuous). Then the boundary modulus problem is solvable for w. That is, there is an f in $H^\infty(U^n)$ with $|h^+| = |f^+|^2$ almost everywhere.

3. **A necessary condition.** We begin by introducing some notation in order to state and prove the necessary condition. Z^n will be the Cartesian product of n copies of the integers and will be identified with the Pontryagin dual of T^n. Z^n_+ will be those points of Z^n with all coordinates nonnegative. If H is a subset of Z^n, $L^p(T^n, H)$ will be the subspace of $L^p(T^n)$ consisting of those functions whose Fourier coefficients vanish off H. $H^p(T^n, H)$ will be the intersection of $L^p(T^n, H)$ and $H^p(T^n)$. It is a well-known fact that $H^p(T^n) = H^p(T^n, Z^n_+)$, [2, p. 53]. If m is in Z^n, H_m will be the translate of H by m. If f is in $L^1(T^n)$, $(f)_H$ will denote the formal (multiple) Fourier series formed by defining the Fourier coefficients of $(f)_H$ to be those of f on H and zero off H. $(f)_H \sim k$ means that the formal series $(f)_H$ is the Fourier series of the $L^1(T^n)$-function k. Finally, if H is also a subgroup of Z^n, Σ' will denote summation with respect to m over one representative m from each coset H_m.

We then have the following compound necessary condition on a function w in order that it have a solution to the boundary modulus problem.

Theorem 3.1. *Let $w = |h^+|^2$ for h in $H^2(U^n)$. Choose any subgroup H of Z^n. Then*

(i) *$(w)_H$ is the Fourier series of an $L^1(T^n)$-function of the form $\Sigma'|p_m|^2$, where each p_m is in $H^2(T^n, H_m)$.*

(ii) *$\log(w)_H$ is integrable, provided w is not the null function.*

(iii) *Suppose H and Z^n_+ have only the origin in common. The p_m may then be taken as polynomials in $H^2(T^n, H_m)$.*

Proof. (i) Since the H_m are cosets of the subgroup H, they decompose Z^n into disjoint subsets. This induces a decomposition of Z^n_+. Thus $H^2(T^n) = H^2(T^n, Z^n_+)$ may be written as the Hilbert space direct sum $\bigoplus H^2(T^n, H_m)$, where the sum runs over disjoint H_m (*not* over m). Now write

$$h^+ = \sum{}' g_m, \tag{3.1}$$

where g_m lies in $H^2(T^n, H_m)$, and the sum converges in $H^2(T^n)$.

Using the Hölder inequality on partial sums of the Fourier series of g_m and $\bar{g}_{m'}$, one easily sees that $g_m\bar{g}_{m'}$ lies in $L^1(T^n, H_{m-m'})$. But then

$$(g_m\bar{g}_{m'})_H \sim 0 \quad \text{if } m - m' \text{ is not in } H,$$

$$(g_m\bar{g}_{m'})_H \sim g_m\bar{g}_{m'} \quad \text{if } m - m' \text{ is in } H.$$

However, since the m's in (3.1) run over disjoint H_m, $m - m'$ is in H if and only if $m = m'$. Thus,

$$(3.2)\qquad \begin{aligned} (g_m \bar{g}_{m'})_H &\sim 0 \quad \text{if } m \neq m', \\ (g_m \bar{g}_{m'})_H &\sim |g_m|^2 \quad \text{if } m = m'. \end{aligned}$$

Now $G = \Sigma' |g_m|^2$ is in $L^1(T^n)$, because

$$\int \left(\sum{}' |g_m|^2 \right) dx = \sum{}' \left(\int |g_m|^2 dx \right)$$

$$= \sum{}' \|g_m\|_2^2 = \|h\|_2^2 \qquad \text{from (3.1).}$$

We may also conclude that $G_N = \Sigma'_{|m| \leq N} |g_m|^2$ converges to G in the norm of $L^1(T^n)$, where $|m| = |m_1| + \cdots + |m_n|$ for $m = (m_1, \cdots, m_n)$.

We now let $h_N = \Sigma'_{|m| \leq N} g_m$. Then h_N converges to h in $L^2(T^n)$ so $|h_N|^2$ converges to $|h|^2$ in $L^1(T^n)$ by the Hölder inequality. But $(|h_N|^2)_H \sim \Sigma'_{|m| \leq N} |g_m|^2$ by (3.2). Because the Fourier coefficients are continuous in the $L^1(T^n)$ norm, the formal Fourier series $(h)_H$ is the Fourier series of the $L^1(T^n)$-function G.

(ii) Since w is not the null function, the Fourier coefficient at the origin (which is in H) is $\int w dx = \int |w| dx \neq 0$. Thus $(w)_H$ is not the zero Fourier series, so that there is a p_{m_0} which is not the null function. The inequalities

$$(w)_H \geq \log (w)_H = \log \left(\sum{}' |p_m|^2 \right) \geq \log |p_{m_0}|^2$$

show that $\log (w)_H$ is integrable, since it is bounded above and below by integrable functions.

(iii) We need only show the p_m of part (i) must be polynomials under the assumption of part (iii). For this we need only show that H_m and Z^n_+ have only a finite number of points in common, so that $H^2(T^n, H_m)$ consists of polynomials. We assume the contrary and derive a contradiction to the hypothesis of (iii).

Suppose now that $h_1 + m, h_2 + m, \cdots$ are distinct elements of H_m and lie in Z^n_+. By the first assumption $h_1, h_2, \cdots$ are distinct elements of H, while by the second assumption the coordinates of the h_i are all bounded below. That is, if $(u)_k$ denotes the kth coordinate of u in Z^n, then $(h_i)_k \geq (-m)_k$ for $1 \leq k \leq n$. After extracting subsequences, for each $1 \leq k \leq n$, either (1) $\lim_{i \to \infty} (h_i)_k = +\infty$ or (2) $(h_i)_k = (h_j)_k$ for all i and j.

Now since the h_i are distinct (2) cannot hold for all k and we assume (for simplicity of notation) that the alternative (1) holds for $k = 1$. We then choose i_0 such that $(h_{i_0})_k \geq (h_1)_k + 1$ for $k = 1$ and all other k where (1) holds. Set

$f = h_{i_0} - h_1$. Then f lies in H as H is a subgroup of Z^n. Choose k with $1 \leq k \leq n$. If (1) holds for k,

$$(f)_k = (h_{i_0})_k - (h_1)_k \geq (h_1)_k + 1 - (h_1)_k.$$

Thus,

$$(f)_k \geq 1 \geq 0. \tag{3.3}$$

If (2) holds for k, $(f)_k = (h_{i_0})_k - (h_1)_k = 0$. Thus f also lies in Z^n_+, hence in the intersection of H and Z^n_+. Finally, with $k = 1$, (3.3) shows f is not the origin. This contradiction of the hypothesis in (iii) proves the last part of Theorem 3.1.

Example. We give an example for $n \geq 2$ of a continuous, logarithmically integrable function w for which the necessary condition fails. For this w the boundary modulus problem has no solution, even though both the function and its zeroes are well behaved.

Let w_1 be any continuous, nonnegative, logarithmically integrable function on T^1 with infinitely many zeroes. It is easy to construct such a function. Define

$$w(e^{ix_1}, \cdots, e^{ix_n}) = w_1(e^{ix_1} e^{-ix_2}).$$

w is obviously continuous, and easily checked to be logarithmically integrable.

Choose the subgroup H to be all $(n, -n, 0, \cdots, 0)$ where n runs over the integers. Thus H and Z^n_+ have only the origin in common. Note that w lies in $L^\infty(T^n, H)$ so that $(w)_H \sim w$. Hence, if the boundary modulus problem is assumed to be solvable for w, part (iii) of Theorem 3.1 applies to yield polynomials p_m with

$$w(z) = \sum{}' |p_m(z)|^2 \quad \text{almost everywhere.} \tag{3.4}$$

It is clear from continuity and (3.4) that

$$w(z) \geq \sum_{|m| \leq N}{}' |p_m(z)|^2.$$

Thus, $w(z) \geq \Sigma' |p_m(z)|^2$ for every z in T^n. Fix $x_2 = a_2, \cdots, x_n = a_n$. Then

$$w(e^{ix_1}, e^{ia_2}, \cdots, e^{ia_n}) \geq \sum{}' |p_m(e^{ix_1}, e^{ia_2}, \cdots, e^{ia_n})|^2. \tag{3.5}$$

Since w_1 has infinitely many zeroes on T^1 so does the left-hand side of (3.5). Consequently, every polynomial $p_m(e^{ix_1}, e^{ia_2}, \cdots, e^{ia_n})$ has infinitely many zeroes on T^1 and so is identically zero. Since $a_2, \cdots, a_n$ are arbitrary real numbers p_m is the zero polynomial. But then w is zero almost everywhere, hence not logarithmically integrable. This contradiction completes the example.

Rudin [2, p. 114] has given an example of a positive continuous w which is not the boundary modulus of a function h in $H^\infty(U^n)$ such that h has a continuous extension to the closure of U^n.

4. Necessary and sufficient conditions for a special class of functions. By restricting ourselves to functions w in $L^2(T^n, H)$ for nice subgroups H we are able to clarify and in some sense solve the boundary modulus problem by presenting three conditions equivalent to its solution. This is obtained by combining the results of §§2 and 3.

Theorem 4.1. *Let H be a subgroup of Z^n with only the origin in common with Z^n_+. Let w be a function in $L^2(T^n, H)$. Then the following statements are equivalent*:

(i) $w = |h^+|^2$ *almost everywhere with h in $H^\infty(U^n)$.*

(ii) $w = \Sigma_{i=1}^\infty |p_i|^2$ *almost everywhere with the p_i trigonometric polynomials in $L^2(T^n, H)$.*

(iii) *There is a polynomial p in $L^2(T^n, H)$ such that $w/(|p|^2) = w_1$ almost everywhere with w_1 both positive and lower semicontinuous.*

(iv) *There is a function h in $H^1(U^n)$ such that $w/(|h^+|^2) = w_1$ almost everywhere with w_1 both positive and lower semicontinuous.*

Proof. We will prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii). Note that $(w)_H \sim w$. By Theorem 3.1 (iii), we may assume $w = \Sigma' |q_m|^2$ with q_m in $H^2(T^n, H_m)$. If $m = (m_1, \cdots, m_n)$, we set $p_m = e^{-im_1x_1} \cdots e^{-im_nx_n} q_m$. Then p_m is in $L^2(T^n, H)$ and $w = \Sigma' |p_m|^2$. Since there are only a countable number of cosets H_m, (ii) is proved.

(ii) $\Rightarrow$ (iii). We may assume that p_1 is not the zero polynomial. Let

$$\begin{aligned} w_1 &= \left(\sum_{i=1}^{\infty} |p_i|^2\right) \Big/ (|p_1|^2) \qquad \text{where } p_1 \neq 0, \\ &= 1 + \left(\sum_{i=2}^{\infty} |p_i|^2\right) \Big/ (|p_1|^2) \\ w_1 &= 1 \quad \text{where } p_1 = 0. \end{aligned}$$

Since $p_1 \neq 0$ almost everywhere, $w/(|p_1|^2) = w_1$ almost everywhere, and, clearly, $w_1 \geq 1$. It remains to show that w_1 is lower semicontinuous.

At a point z_0 where p_1 vanishes,

$$\liminf_{z \to z_0} w_1(z) \geq 1 = w_1(z_0).$$

If $p_1(z_0) \neq 0$, take a neighborhood about z_0 where p_1 is not zero. In this neighborhood, w_1 equals $w/(|p_1|^2)$ which is a quotient of a lower semicontinuous function by a positive continuous function. As in Lemma 1.1, w_1 is lower semicontinuous in this neighborhood. In particular, w_1 is lower semicontinuous at z_0.

Since z_0 may be arbitrary, w_1 is lower semicontinuous and (iii) is proved.

(iii) $\Longrightarrow$ (iv). Since p is a polynomial we can find a character c such that $c \cdot p = h$ lies in $H^\infty(U^n)$. But then

$$\begin{aligned} w/(|h^+|^2) &= w/(|c \cdot p|^2) \\ &= w/(|p|^2) \quad \text{since } c \text{ is a character} \\ &= w_1 \quad \text{almost everywhere.} \end{aligned}$$

(iv) $\Longrightarrow$ (i). This is an immediate application of Theorem 2.1.

5. **Remarks.**

Remark 1. In the theorem of §4, (ii) is, of course, better as a necessary condition, while (iv) is better as a sufficient condition.

Remark 2. For this section, fix H, a subgroup of Z^n with only the origin in common with Z^n_+. Let W_H consist of those functions w in $L^\infty(T^n, H)$ for which $w = |h^+|^2$ a.e., with h in $H^\infty(T^n)$. By (ii) of Theorem 4, W_H is composed of functions almost everywhere equal to a lower semicontinuous function.

Remark 3. Let W be composed of all functions in $L^\infty(T^n)$ for which the boundary modulus problem is solvable. Suppose H' is another subgroup of Z^n with only the origin in common with Z^n_+. Let w be in W_H and w' be in $W_{H'}$. Then the following functions belong to W:

(i) $w + w'$,

(ii) $w \cdot w'$,

(iii) $\min(w, w')$,

(iv) $\max(w, w')$,

(v) w^a with $a > 0$.

Proof. We may assume that neither w nor w' is a null function. Each of these five functions is certainly in $L^\infty(T^n)$. Let p, w_1, and p', w_1' be a polynomial and positive lower semicontinuous function corresponding to w and w' respectively from Theorem 4.1 (iii). Then pp' is a polynomial so there is a character c such that cpp' is in $H^1(T^n)$, and of course $|cpp'| = |pp'|$.

To prove (i) note that

$$(w + w')/(|pp'|^2) = w_1/(|p'|^2) + w_1'/(|p|^2)$$

almost everywhere. By Lemma 1.1, $w_1/(|p'|^2)$ and $w_1'/(|p|^2)$ are both positive and lower semicontinuous. Since the sum of two positive lower semicontinuous functions is still positive lower semicontinuous, Theorem 2.1 applies so that $w + w'$ lies in W.

(ii) This is clear as the product of two bounded analytic functions is again a bounded analytic function.

For (iii) note that

$$[\min(w, w')]/(|pp'|^2) = \min\{w_1/(|p'|^2), w_1'/(|p|^2)\}$$

almost everywhere. In the proof of (ii) we have seen that $w_1/(|p'|^2)$ and $w_1'/(|p|^2)$ are both positive and lower semicontinuous. Since the minimum of two positive lower semicontinuous functions is again a positive lower semicontinuous function, (iii) follows from Theorem 2.1.

(iv) is proved as was (iii), replacing "minimum" by "maximum".

To prove (v), take an integer $m \geq a$. Then

$$w^a/(|p^m|^2) = w^a/(|p^2|^a|p^2|^{(m-a)}) \tag{5.1}$$
$$= (w_1)^a/(|p^2|^{(m-a)}) \quad \text{almost everywhere.}$$

But $(w_1)^a$ is positive and lower semicontinuous and so by Lemma 1.1 the right-hand side of (5.1) is also. Theorem 2.1 then applies to give (v).

Remark 4. Let w be lower semicontinuous. Suppose there is a nonnull w' in W_H such that $w' \leq w$ almost everywhere. Then w is in W.

Proof. Take a polynomial p' in $H^2(T^n, H)$ and a positive lower semicontinuous function $(w')_1$ corresponding to w' as in Theorem 4.1 (iii). We have

$$w/(|p'|^2) \geq w'/(|p'|^2) \quad \text{almost everywhere,}$$
$$= (w')_1 \quad \text{almost everywhere.}$$

But there is a constant c such that $(w')_1 \geq c > 0$, everywhere. Let $w_1 = \max(w/(|p'|^2), c)$. Then w_1 is lower semicontinuous as it is the maximum of two lower semicontinuous functions. Also $w_1 = w/(|p'|^2)$ almost everywhere as the latter was shown to be $\geq c$ almost everywhere. By Theorem 2.1 we are finished.

Remark 5. W_H can be provided with a certain amount of structure. This structure is somewhat "logarithmic". For $n = 1$ the same properties are easily derived for the class W of all solutions to the boundary modulus problem by their characterization in terms of logarithmic integrability. This structure is contained in the following:

Suppose w and w' are both in the same W_H. Then Remarks 3 and 4 may be strengthened by replacing W by W_H.

Proof. For Remark 4 there is nothing to prove.

It is easy to check that $L^\infty(T^n, H)$ is a norm closed subalgebra of $L^\infty(T^n)$. Also it is a standard argument using the Weierstrass approximation theorem to show that $L^\infty(T^n, H)$ is closed under (iii), (iv), and (v).

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DEPARTMENT OF MATHEMATICS, JOHNS HOPKINS UNIVERSITY, BALTIMORE, MARYLAND 21218