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ON THE MULTIPLICATIVE COMPLETION OF CERTAIN BASIC SEQUENCES IN L^p, $1 < p < \infty$(¹)

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ABSTRACT. Boas and Pollard proved that given any basis $\{f_n\}_{n=1}^{\infty}$ for $L^2(E)$ one can delete the first k basis elements and then find a bounded measurable function M such that $\{Mf_n\}_{n=k+1}^{\infty}$ is total in $L^2(E)$, that is, the closure of the linear span of the set $\{Mf_n: n \geq k+1\}$ is $L^2(E)$. We improve this result by weakening the hypothesis to accept bases of $L^p(E)$, $1 < p < \infty$, and strengthening the conclusion to read serially total, that is, given any $f \in L^2(E)$ one can find a sequence of reals $\{a_n\}_{n=k+1}^{\infty}$ such that $\sum_{n=k+1}^{\infty} a_n M f_n$ converges to f in the norm. We also show that certain infinite deletions are possible.

1. In this paper we strengthen and generalize the following result by Boas and Pollard [1].

Theorem 1.1. *If* $\{f_n\}_{n=1}^{\infty}$ *is an orthonormal set which is not complete, but can be completed by the addition of a finite number of functions to the set, then there is a bounded measurable function* M *such that* $\{Mf_n\}_{n=1}^{\infty}$ *is complete.*

The strengthened theorem can be viewed as a first step toward changing totality into serial totality in [2].

Theorem 1.2. *Let* $\{\phi_n\}_{n=1}^{\infty}$ *be a system of functions defined on the measurable set* $E \subset [0, 1]$, $|E| > 0$, *and forming a normal basis for* $L^p(E)$, $1 < p < \infty$. *Then, for any integer* N_0 *there exists a measurable function* M, $0 \leq M(x) \leq 1$, *such that for any given function* f *in* $L^p(E)$, *there is a series*

$$\sum_{k=N_0}^{\infty} \alpha_k (M\phi_k), \qquad \alpha_k \text{ a real number,} \tag{1.1}$$

with the properties

(a) *the series* (1.1) *converges in* L^p *to* f;

(b) $\alpha_k \to 0$ *as* $k \to \infty$.

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2. The main tool in the proof of Theorem 1.2 is the following

Lemma 2.1. *Let* $\Phi = \{\phi_n : n = 1, 2, \cdots\}$ *be a normalized basis for* $L^p(E)$, $1 < p < \infty$, f *a measurable function finite almost everywhere on* E, *and*

$$M(x) = \sum_{i=1}^{j} c_i \chi_{E_i} \quad \textit{with } E = \bigcup_{i=1}^{j} E_i \textit{ and } 0 < c_i \leq 1.$$

Then given $\epsilon > 0$ *and a positive integer* n, *there exists a measurable set* e_0 *and a* Φ*-polynomial* $P = \Sigma_{k=n}^{m} b_k \phi_k$ *such that*

(2.1) $e_0 \subset E$ *and* $|e_0| < \epsilon$, *where* $|e_0|$ *is the Lebesque measure of* e_0;

(2.2) $|b_k| < \epsilon$ *for* $n \leq k \leq m$;

(2.3) $\|MP - f\|_{(E\setminus e_0)} \equiv \|(MP - f)\chi_{(E\setminus e_0)}\| < \epsilon$;

(2.4) $\|M \Sigma_{k=n}^{s} b_k \phi_k\|_e^p \leq \epsilon + \|f\|_e^p$, *for all* $n \leq s \leq m$, *and every measurable subset* e *of* $E\setminus e_0$.

Proof. The lemma follows immediately from Lemma 3 of Talalyan [3].

3. **Proof of Theorem 1.2.** The required function will be a certain infinite product. The individual factors of this product are inductively determined. Let $\epsilon_n = 2^{-n-2}$ and, for each n, choose a positive $\delta_n < \epsilon_n$ so that

$$\|\phi_i\|_G < \epsilon_n/3, \quad i = 1, 2, \cdots, n, \quad \text{whenever } |G| < \delta_n. \tag{3.1}$$

Applying Lemma 2.1 with $M(x) \equiv 1$ to ϕ_1, we choose a set D_1 whose complement E_1 has measure less than δ_1, and a Φ-polynomial

$$P_{11} = \sum_{j=\nu(1,0)+1}^{\nu(1,1)} b_j \phi_j, \quad \text{where } \nu(1, 0) = N_0,$$

satisfying the following conditions:

(3.2) $|b_j| < \epsilon_1$, if $\nu(1, 0) < j \leq \nu(1, 1)$;

(3.3) $\|\phi_1 - P_{11}\|_{D_1} < \epsilon_1/3$;

$$\sup_{\nu(1,0)<s\leq\nu(1,1)} \left\| \sum_{j=\nu(1,0)+1}^{s} b_j \phi_j \right\|_e^p < \frac{\epsilon_1}{2} + \|\phi_1\|_e^p, \tag{3.4}$$

for all measurable subsets e of D_1.

Define

$$M_1(x) = \begin{cases} 1, & \text{if } x \in D_1, \\ c_1, & \text{if } x \in E_1, \end{cases} \tag{3.5}$$

where

$$c_1 = \frac{\epsilon_1}{3}\left(1 + \sup_{\nu(1,0)<s\leq\nu(1,1)} \left\| \sum_{j=\nu(1,0)+1}^{s} b_j \phi_j \right\|\right)^{-1}$$

With the help of M_1 we will be able to extend approximations analogous to (3.3) and (3.4) to the whole set and obtain estimates appropriate to the first step. In fact, it follows from (3.3), (3.1) and the definition of M_1 that

$$\begin{aligned} \|\phi_1 - M_1P_{11}\| &\le \|\phi_1 - P_{11}\|_{D_1} + \|\phi_1 - M_1P_{11}\|_{E_1} \\ &\le \epsilon_1/3 + \|\phi_1\|_{E_1} + c_1\|P_{11}\|_{E_1} \le \epsilon_1/3 + \epsilon_1/3 + \epsilon_1/3 = \epsilon_1, \end{aligned} \tag{3.6}$$

and by virtue of (3.4) and the definition of M_1, we obtain for $\nu(1, 0) < s \le \nu(1, 1)$

$$\begin{aligned} \left\|M_1 \sum_{j=\nu(1,0)+1}^{s} b_j\phi_j\right\|^p &= \left\|\sum_{j=\nu(1,0)+1}^{s} b_j\phi_j\right\|_{D_1}^p + \left\|M_1 \sum_{j=\nu(1,0)+1}^{s} b_j\phi_j\right\|_{E_1}^p \\ &\le \frac{\epsilon_1}{2} + \|\phi_1\|^p + \left(\frac{\epsilon_1}{3}\right)^p \le 1 + \epsilon_1. \end{aligned} \tag{3.7}$$

Again, Lemma 2.1 allows us to choose, for $i = 1, 2$, sets D_{2i} with respective complements E_{2i} and Φ-polynomials $P_{2i} = \sum_{j=\nu(2,i-1)+1}^{\nu(2,i)} b_j\phi_j$, with $\nu(1, 1) < \nu(2, 0) < \nu(2, 1) < \nu(2, 2)$, satisfying the following conditions:

(3.8) $|E_{2i}| < \delta_2/2$, $i = 1, 2$;

(3.9) $|b_j| < \epsilon_2$, $\nu(2, 0) < j \le \nu(2, 2)$;

(3.10) $\|(\phi_1 - M_1P_{11}) - M_1P_{21}\|_{D_{21}} < \epsilon_2/3$;

(3.11) $\|\phi_2 - M_1P_{22}\|_{D_{22}} < \epsilon_2/3$;

$$\sup_{\nu(2,0)<s\le\nu(2,1)} \left\|M_1 \sum_{j=\nu(2,0)+1}^{s} b_j\phi_j\right\|_e^p < \frac{\epsilon_2}{2} + \|\phi_1 - M_1P_{11}\|_e^p \tag{3.12}$$

for all measurable subsets e of D_{21};

$$\sup_{\nu(2,1)<s\le\nu(2,2)} \left\|M_1 \sum_{j=\nu(2,1)+1}^{s} b_j\phi_j\right\|_e^p < \frac{\epsilon_2}{2} + \|\phi_2\|_e^p < 1 + \frac{\epsilon_2}{2} \tag{3.13}$$

for all measurable subsets e of D_{22}.

(3.14) Let $D_2 = D_{21} \cap D_{22}$ and $E_2 = E_{21} \cup E_{22}$.

Define

$$M_2(x) = \begin{cases} 1, & \text{if } x \in D_2; \\ c_2, & \text{if } x \in E_2; \end{cases} \tag{3.15}$$

where

$$c_2 = \frac{\epsilon_2}{3}\left(1 + \sum_{m=1}^{2}\sum_{k=1}^{m} \sup_{\nu(m,k-1)<s\le\nu(m,k)} \left\|\sum_{j=\nu(m,k-1)+1}^{s} b_j\phi_j\right\|\right)^{-1}.$$

With the help of M_2 we will be able to extend approximations analogous to (3.10) through (3.13) to the whole set and obtain the second-step estimates. In fact, it follows from (3.10), (3.14), (3.8) and (3.1) that

$$\begin{aligned}\|\phi_1 - M_1M_2(P_{11} + P_{21})\| &\le \|\phi_1 - M_1(P_{11} + P_{21})\|_{D_2} + \|\phi_1 - M_1M_2(P_{11} + P_{21})\|_{E_2}\\ &\le \epsilon_2/3 + \|\phi_1\|_{E_2} + c_2(\|P_{11}\| + \|P_{21}\|)\\ &\le \epsilon_2/3 + \epsilon_2/3 + \epsilon_2/3 = \epsilon_2,\end{aligned}$$

and by virtue of (3.12) and (3.15) we obtain for $\nu(2, 0) < s \le \nu(2, 1)$

$$\begin{aligned}\left\|M_1M_2 \sum_{j=\nu(2,0)+1}^{s} b_j\phi_j\right\|^p &= \left\|M_1 \sum_{j=\nu(2,0)+1}^{s} b_j\phi_j\right\|^p_{D_2.} + \left\|M_1M_2 \sum_{j=\nu(2,0)+1}^{s} b_j\phi_j\right\|^p_{E_2}\\ &\le \epsilon_2/2 + \|\phi_1 - M_1P_{11}\|^p + (\epsilon_2/3)^p\\ &\le \epsilon_2 + (\epsilon_1)^p \le \epsilon_1 + \epsilon_2.\end{aligned}$$

Similarly, $\|\phi_2 - M_1M_2P_{22}\| < \epsilon_2$, and for $\nu(2, 1) < s \le \nu(2, 2)$,

$$\left\|M_1M_2 \sum_{j=\nu(2,1)+1}^{s} b_j\phi_j\right\| < 1 + \epsilon_2.$$

Now we show that the estimates of the first step are weakened only slightly by the introduction of M_2. As a matter of fact, (3.6) and (3.7) are changed only by the introduction of an additional ϵ_2.

$$\begin{aligned}\|\phi_1 - M_1M_2P_{11}\| &< \|\phi_1 - M_1P_{11}\|_{D_2} + \|\phi_1 - M_1M_2P_{11}\|_{E_2}\\ &\le \epsilon_1 + \|\phi_1\|_{E_2} + c_2\|P_{11}\|_{E_2} \le \epsilon_1 + \epsilon_2/3 + \epsilon_2/3 < \epsilon_1 + \epsilon_2,\end{aligned}$$

and for $\nu(1, 0) < s \le \nu(1, 1)$,

$$\begin{aligned}\left\|M_1M_2 \sum_{j=\nu(1,0)+1}^{s} b_j\phi_j\right\|^p &= \left\|M_1 \sum_{j=\nu(1,0)+1}^{s} b_j\phi_j\right\|^p_{D_2} + \left\|M_1M_2 \sum_{j=\nu(1,0)+1}^{s} b_j\phi_j\right\|^p_{E_2}\\ &\le 1 + \epsilon_1 + (\epsilon_2/3)^p < 1 + \epsilon_2 + \epsilon_1.\end{aligned}$$

Suppose we have completed the first n steps; that is, for each pair (m, k) with $m = 1, 2, \cdots, n$ and $k = 1, 2, \cdots, m$, we have

$$\left\|\phi_k - \left(\prod_{i=1}^{n} M_i\right) \sum_{j=k}^{m} P_{jk}\right\| < \sum_{j=m}^{n} \epsilon_j; \tag{3.16}$$

$$\sup_{(m,k-1)<s\le\nu(m,k)} \left\| \left(\prod_{i=1}^{n} M_i\right) \sum_{j=\nu(m,k-1)+1}^{s} b_j\phi_j \right\|^p$$

$$\le \begin{cases} \displaystyle\sum_{j=m-1}^{n} \epsilon_j, & \text{if } m > k; \\ 1 + \displaystyle\sum_{j=m}^{n} \epsilon_j, & \text{if } m = k; \end{cases} \tag{3.17}$$

and

$$|b_j| < \epsilon_m, \quad \text{for all } \nu(m,\ 0) < j \le \nu(m,\ m). \tag{3.18}$$

Now successively apply Lemma 2.1, with $M = \Pi_{i=1}^{n} M_i$, to the functions

$$\begin{aligned} \Psi_k &= \phi_k - \left(\prod_{i=1}^{n} M_i\right)\left(\sum_{j=k}^{n} P_{jk}\right), \quad k = 1, 2, \cdots, n, \\ \Psi_{n+1} &= \phi_{n+1}. \end{aligned} \tag{3.19}$$

By virtue of Lemma 2.1, we may choose, for $k = 1, 2, \cdots, n+1$, sets $D_{n+1\,k}$ with respective complements $E_{n+1\,k}$, and Φ-polynomials

$$P_{n+1\,k} = \sum_{j=\nu(n+1,k-1)+1}^{\nu(n+1,k)} b_j\phi_j,$$

where $\nu(n,\ n) < \nu(n+1,\ 0) < \nu(n+1,\ 1) < \cdots < \nu(n+1,\ n+1)$, satisfying the following conditions:

$$|E_{n+1\,k}| < \delta_{n+1}/(n+1); \tag{3.20}$$

$$|b_j| < \epsilon_{n+1}, \quad \nu(n+1,\ 0) < j \le \nu(n+1,\ n+1); \tag{3.21}$$

$$\left\| \Psi_k - \left(\prod_{i=1}^{n} M_i\right) P_{n+1\,k} \right\|_{D_{n+1\,k}} < \frac{\epsilon_{n+1}}{3}; \tag{3.22}$$

$$\sup_{\nu(n+1,k-1)<s\le\nu(n+1,k)} \left\| \left(\prod_{i=1}^{n} M_i\right) \sum_{j=\nu(n+1,k-1)+1}^{s} b_j\phi_j \right\|_e^p \le \frac{\epsilon_{n+1}}{2} + \|\Psi_k\|_e^p, \tag{3.23}$$

for all measurable subsets e of $D_{n+1\,k}$.

(3.24) Let $D_{n+1} = \bigcap_{k=1}^{n+1} D_{n+1\,k}$ and $E_{n+1} = \bigcup_{k=1}^{n+1} E_{n+1\,k}$. Next, define

$$M_{n+1}(x) = \begin{cases} 1, & \text{if } x \in D_{n+1}; \\ c_{n+1}, & \text{if } x \in E_{n+1}; \end{cases} \tag{3.25}$$

where

$$c_{n+1} = \frac{\epsilon_{n+1}}{3}\left(1 + \sum_{m=1}^{n+1}\sum_{k=1}^{m}\sup_{\nu(m,k-1)<s\le\nu(m,k)}\left\|\sum_{j=\nu(m,k-1)+1}^{s} b_j\phi_j\right\|\right)^{-1}$$

With the help of M_{n+1} we will be able to extend approximations analogous to (3.22) and (3.23) to the whole set. In fact, it follows from (3.22), (3.19), (3.24), (3.1) and (3.25) that

$$\begin{aligned}&\left\|\phi_k - \left(\prod_{i=1}^{n+1} M_i\right)\sum_{j=k}^{n+1} P_{jk}\right\|\\ &\quad\le \left\|\Psi_k - \left(\prod_{i=1}^{n} M_i\right)P_{n+1\,k}\right\|_{D_{n+1}} + \left\|\phi_k - \left(\prod_{i=1}^{n+1} M_i\right)\sum_{j=k}^{n+1} P_{jk}\right\|_{E_{n+1}}\\ &\quad\le \epsilon_{n+1}/3 + \epsilon_{n+1}/3 + \epsilon_{n+1}/3 = \epsilon_{n+1}.\end{aligned}$$

Similarly,

$$\begin{aligned}&\sup_{\nu(n+1,k-1)<s\le\nu(n+1,k)}\left\|\left(\prod_{i=1}^{n+1} M_i\right)\sum_{j=\nu(n+1,k-1)+1}^{s} b_j\phi_j\right\|^p\\ &\quad\le \epsilon_{n+1}/2 + \|\Psi_k\|^p + (\epsilon_{n+1}/3)^p \le \|\Psi_k\|^p + \epsilon_{n+1}\\ &\quad\le \begin{cases}\epsilon_n + \epsilon_{n+1}, & \text{if } k < n+1;\\ 1 + \epsilon_{n+1}, & \text{if } k = n+1.\end{cases}\end{aligned}$$

Now we show that the estimates of the first n steps are weakened only slightly by the introduction of M_{n+1}. Indeed, (3.16) and (3.17) are changed by no more than an additional ϵ_{n+1}. Let $m \le n+1$ and $k \le m$; then by (3.20), (3.1), and (3.25)

$$\begin{aligned}&\left\|\phi_k - \left(\prod_{i=1}^{n+1} M_i\right)\sum_{j=k}^{m} P_{jk}\right\|\\ &\quad\le \left\|\phi_k - \left(\prod_{i=1}^{n} M_i\right)\sum_{j=k}^{m} P_{jk}\right\|_{D_{n+1}} + \|\phi_k\|_{E_{n+1}} + \left\|\left(\prod_{i=1}^{n+1} M_i\right)\sum_{j=k}^{m} P_{jk}\right\|_{E_{n+1}}\\ &\quad\le \sum_{j=m}^{n}\epsilon_j + \frac{\epsilon_{n+1}}{3} + \frac{\epsilon_{n+1}}{3}.\end{aligned}$$

Similarly,

$$\sup_{\nu(m,k-1)<s\le\nu(m,k)} \left\| \left(\prod_{i=1}^{n+1} M_i\right) \sum_{j=\nu(m,k-1)+1}^{s} b_j\phi_j \right\|^p \le \begin{cases} \displaystyle\sum_{j=m-1}^{n+1} \epsilon_j, & \text{if } m > k; \\ \displaystyle 1 + \sum_{j=m}^{n+1} \epsilon_j, & \text{if } m = k. \end{cases}$$

Therefore inequalities (3.16) and (3.17) hold for $n+1$ and hence for all natural numbers.

We have constructed a sequence of measurable functions $\{M_i\}_{i=1}^{\infty}$ with $0 < M_i(x) \le 1$ and

$$\sum_{i=1}^{\infty} |\{x\colon M_i(x) \neq 1\}| \le \sum_{i=1}^{\infty} |E_i| \le \sum_{i=1}^{\infty} \epsilon_i = \sum_{i=1}^{\infty} 2^{-i-2} = \frac{1}{4}. \tag{3.26}$$

The sequence of partial products $\Pi_{i=1}^{n} M_i$ forms a nonincreasing sequence of positive functions. Hence

$$M(x) = \lim_{n} \prod_{i=1}^{n} M_i(x) \tag{3.27}$$

exists, is measurable and satisfies $0 \le M(x) \le 1$.

Fix a pair (m, k) with $m \ge k$. Since the norm is an absolutely continuous set function, we can find a $\delta > 0$ such that, whenever the measure of a set G is less than δ, we have

$$\|\phi_k\|_G + \left\| M \sum_{j=k}^{m} P_{jk} \right\|_G + \left(\sum_{n=\nu(m,k-1)+1}^{\nu(m,k)} \|Mb_j\phi_j\|_G \right)^p < 2^{-m-1}.$$

Now choose n large enough so that

$$B = \left\{ x\colon \prod_{i=n}^{\infty} M_i(x) \neq 1 \right\} \quad \text{has measure less than } \delta.$$

We obtain

$$\begin{aligned} \left\| \phi_k - M \sum_{j=k}^{m} P_{jk} \right\| &\le \left\| \phi_k - \left(\prod_{i=1}^{n} M_i\right) \sum_{j=k}^{m} P_{jk} \right\|_{E\setminus B} + \left\| \phi_k - M \sum_{j=k}^{m} P_{jk} \right\|_B \\ &\le 2^{-m-1} + 2^{-m-1} = 2^{-m}. \end{aligned} \tag{3.28}$$

Similarly,

$$(3.29)\qquad \sup_{\nu(m,k-1)<s\leq\nu(m,k)} \left\| M \sum_{j=\nu(m,k-1)+1}^{s} b_j\phi_j \right\|^p = \begin{cases} 2^{-m+2}, & \text{if } m > k; \\ 2, & \text{if } m = k. \end{cases}$$

Now we are ready to show that given any function f in $L^p(E)$, we can find a series $\Sigma a_j M\phi_j$ with $\nu(m, 0) \leq j \leq \nu(m, m)$, $m = 1, 2, \cdots$, which will converge to f in the norm.

In fact, if $\Sigma_{k=1}^{\infty} a_k\phi_k$ is the Schauder basis expansion of f, then $\Sigma_{m=1}^{\infty} \Sigma_{k=1}^{m} a_k P_{mk}$ converges to f in the norm.

Let $\epsilon > 0$ be given. Choose N_1 so that

$$(3.30)\qquad \left\| \sum_{k=1}^{n} a_k\phi_k - f \right\| < \frac{\epsilon}{3}, \quad \text{for all } n > N_1.$$

Setting $a = \sup_k |a_k|$, choose $N_2 > N_1$ so that, $a \cdot n \cdot 2^{-n} < \epsilon/3$, for all $n > N_2$. By virtue of (3.28) we obtain

$$(3.31)\qquad \begin{aligned} \left\| \sum_{j=1}^{n} \sum_{k=1}^{j} a_k MP_{jk} - \sum_{k=1}^{n} a_k\phi_k \right\| &= \left\| \sum_{k=1}^{n} a_k \left(M \sum_{j=k}^{n} P_{jk} - \phi_k \right) \right\| \\ &\leq \sum_{k=1}^{n} a \left\| M \sum_{j=k}^{n} P_{jk} - \phi_k \right\| \leq n \cdot a \cdot 2^{-n} < \frac{\epsilon}{3}. \end{aligned}$$

Last, choose $N_3 > N_2$ so that

$$(3.32)\qquad |2^{1/p} \cdot a_n| < \epsilon/3, \quad \text{whenever } n > N_3.$$

By virtue of (3.30) and (3.31) we obtain

$$(3.33)\qquad \left\| \sum_{j=1}^{n} \sum_{k=1}^{j} a_k MP_{jk} - f \right\| < \frac{2\epsilon}{3}, \quad \text{for all } n > N_3.$$

Obviously,

$$\left\| \sum_{j=1}^{n} \sum_{k=1}^{j} a_k MP_{jk} + \sum_{k=1}^{n+1} a_k MP_{j+1\,k} - f \right\| < \frac{2\epsilon}{3}, \quad \text{for all } n > N_3.$$

If we add in only part of the second sum, that is, $\Sigma_{k=1}^{m} a_k MP_{n+1\,k}$ with $m < n + 1$, then it is easy to see from (3.28) that the basis elements ϕ_i, $i = 1, 2, \cdots, m$, will be approximated better than before, by 2^{-n-1} instead of by 2^{-n}. Hence via the calculations in (3.31), we find that

$$(3.34)\qquad \left\| \sum_{j=1}^{n} \sum_{k=1}^{j} a_k MP_{jk} + \sum_{k=1}^{m} a_k MP_{n+1\,k} - f \right\| < \frac{2\epsilon}{3}.$$

Lastly, if we add to the summations in (3.34) only part of the Φ-polynomial $a_{m+1} MP_{n+1\,m+1}$, let us say $\Sigma_{j=\nu(n+1,m)+1}^{s} a_{m+1} b_j M\phi_j$, where $\nu(n + 1, m) < s < \nu(n + 1, m + 1)$, then (3.29) and (3.32) in addition to (3.34) give us

$$\left\|\sum_{j=1}^{n}\sum_{k=1}^{j} a_k MP_{jk} + \sum_{k=1}^{m} a_k MP_{n+1\,k} + \sum_{j=\nu(n+1,m)+1}^{s} a_{n+1} b_j M\phi_j - f\right\| < \epsilon, \tag{3.35}$$

whenever $n > N_3$, $m < n + 1$, and $\nu(n + 1, m) < s < \nu(n + 1, m + 1)$. Thus, as a consequence of (3.33), (3.34) and (3.35), we obtain the desired series convergence. Furthermore, the coefficients of $M\phi_j$ go to zero, since the a_n are bounded by a and the b_j go to zero.

The following remark is an immediate consequence of the above proof.

Remark 3. 1. *One can always delete certain infinite collections of the basis elements without affecting the conclusion of Theorem* 1.2.

Proposition 3.2. *The function M defined in Theorem* 1.2 *is positive almost everywhere.*

Proof. By virtue of (3.26), (3.27) and the definitions of M_n, $n = 1, 2, \cdots$, we have $M(x) = \Pi_{n=1}^{\infty} M_n(x)$, where

(a) $M_n(x)$ is measurable and $0 < M_n(x) \leq 1$;

(b) if $E_n = \{x: M_n(x) \neq 1\}$, then $\Sigma_{n=1}^{\infty} |E_n| < \infty$.

$M(x)$ is positive almost everywhere since the product defining $M(x)$ has infinitely many factors different from unity only on the set $\limsup E_n$. But (b) insures that $|\limsup E_n| = 0$.

Definition 3.3. $\{\phi_n\}_{n=1}^{\infty}$ *is serially total in* L^p *if and only if for any function* $f \in L^p$ *we can find a series* $\Sigma_{k=1}^{\infty} a_k\phi_k$ *which converges to f in the norm.*

Theorem 3.4. *Let* $\{\phi_n\}_{n=1}^{\infty}$ *be a normalized basis for* $L^p(E)$, $E \subset [0, 1]$, $1 < p < \infty$. *Then, given any natural number m and $\epsilon > 0$, there exists a set* $G = G(m, \epsilon)$, *G contained in E and satisfying* $|G| > |E| - \epsilon$, *such that* $\{\phi_n\}_{n=m}^{\infty}$ *is serially total in* $L^p(G)$.

Proof. By virtue of Theorem 1.2 we can find a bounded measurable function M such that $\{M\phi_n\}_{n=m}^{\infty}$ is serially total in $L^p(E)$.

Choose $\alpha > 0$ so that $|[x: M(x) > \alpha]| > |E| - \epsilon$; denote this set by G. We assert that $\{\phi_n\}_{n=m}^{\infty}$ is serially total in $L^p(G)$. To see this, take any f in $L^p(G)$ and any positive δ. If ϕ is the element of $L^p(E)$ that agrees with f on G and vanishes outside of G, then $M\phi \in L^p(E)$. Hence, by Theorem 1.2, there is a series $\Sigma_{k=m}^{\infty} a_k M\phi_k$, with $a_k \to 0$, with converges to $M\phi$ in the norm; that is,

$$\left\|M\phi - \sum_{k=m}^{n} a_k M\phi_k\right\| < \alpha\delta, \quad \text{for all } n > N(\alpha, \delta).$$

Now

$$\alpha^p \left\| f - \sum_{k=m}^{n} a_k\phi_k \right\|_G^p = \int_G \alpha^p \left(f(x) - \sum_{k=m}^{n} a_k\phi_k(x) \right)^p dx$$

$$\leq \int_G M^p(x) \left(f(x) - \sum_{k=m}^{n} a_k\phi_k(x) \right)^p dx$$

$$\leq \int_E \left(M(x)\phi(x) - \sum_{k=m}^{n} a_k M(x)\phi_k(x) \right)^p dx$$

$$= \left\| M\phi - \sum_{k=m}^{n} a_k M\phi_k \right\|^p < \alpha^p\delta^p.$$

Hence,

$$\left\| f - \sum_{k=m}^{n} a_k\phi_k \right\|_G < \delta, \quad \text{for all } n > N(\alpha, \delta).$$

Since δ may be taken to be arbitrarily small we have

$$\{\phi_k\}_{k=m}^{\infty} \quad \text{is serially total in } L^p(G).$$

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