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CODOMINANT DIMENSION OF RINGS AND MODULES

BY

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ABSTRACT. Expanding Nakayama's original concept of dominant dimension, Tachikawa, Müller and Kato have obtained a number of results pertaining to finite dimensional algebras and more generally, rings and their modules. The purpose of this paper is to introduce and examine a categorically dual notion, namely, codominant dimension. Special attention is given to the question of the relation between the codominant and dominant dimensions of a ring. In particular, we show that the two dimensions are equivalent for artinian rings. This follows from our main result that for a left perfect ring R the dominant dimension of each projective left R-module is greater than or equal to n if and only if the codominant dimension of each injective left R-module is greater than or equal to n. Finally, for computations, we consider generalized uniserial rings and show that the codominant dimension, or equivalently, dominant dimension, is a strict function of the ring's Kupisch sequence.

1. **Preliminaries.** Throughout this paper we shall assume that all rings are associative and have an identity and that all modules are unital. The category of all unital left R-modules of a ring R will be denoted by ${}_R\mathfrak{M}$ while J will be used to indicate the Jacobson radical of R. Further, if M is a module over R, $E_R(M)$ will denote the injective hull of M and $\mathrm{Soc}(M)$ the socle of M.

Let M be an R-module with minimal projective resolution

$$\cdots \longrightarrow P_n \xrightarrow{\phi_n} \cdots \longrightarrow P_1 \xrightarrow{\phi_1} M \longrightarrow 0,$$

i.e., $\phi_i\colon P_i \to \mathrm{Ker}\,\phi_{i-1}$ is a projective cover of $\mathrm{Ker}\,\phi_{i-1}$ for $i = 1, 2, \cdots$. The *codominant dimension* of M, denoted codom dim M, is defined to be $\min\{i\colon P_{i+1}$ is not injective$\}$ where we follow the convention of setting $\min \varnothing = \infty$.

So as to be assured of the existence of projective covers and therefore codominant dimension we will usually restrict our attention to perfect rings. In any case, whenever defined the codominant dimension of a module is well defined, as projective covers are unique to within an isomorphism.

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Example 1.1. *A ring with modules of arbitrary codominant dimension.* Let R be an algebra over a field F with basis $\{e_1, e_2, \cdots\} \cup \{x_1, x_2, \cdots\} \cup \{1\}$ where multiplication is defined by $e_i^2 = e_i$, $e_i x_i e_{i+1} = x_i$, 1 is a two-sided identity, all other products zero. Then the R-module Re_i/Je_i has codominant dimension $i-1$. (This ring is given in [19, p. 379] as an example of a ring for which $\mathfrak{M}_R$ possesses a noninjective cogenerator with no chain conditions on direct summands.)

We will need the following observation concerning codominant dimension's behavior over direct sums.

Proposition 1.2. *Let R be a left artinian ring. If ${}_RM \cong \bigoplus \Sigma_{\alpha\in\Omega} M_\alpha$, then* codom dim M = min {codom dim M_α: $\alpha \in \Omega$}. *Further, for finite Ω we can relax artinian to perfect.*

Proof. This is immediate, for if we take a direct sum of the minimal projective resolutions of the M_α's we obtain a minimal projective resolution of M.

Although we make no use of it, we also note a dual to a result of Tachikawa [21].

Proposition 1.3. *If* proj dim $M <$ codom dim M, *then M is injective and projective.*

Proof. In such a case any minimal projective resolution of M would necessarily be split exact.

Next we recall the definition of dominant dimension and note some results we need later.

Let M be an R-module with minimal injective resolution

$$0 \longrightarrow M \xrightarrow{\psi_1} E_1 \longrightarrow \cdots \longrightarrow E_n \xrightarrow{\psi_{n+1}} \cdots,$$

i.e., $E_i \cong E_R(E_{i-1}/\operatorname{Im}\psi_{i-1})$. The *dominant dimension* of M, denoted dom dim M, is defined to be $\min\{i : E_{i+1}$ is not projective$\}$. We define the *left dominant dimension* of R, denoted l-dom dim R, to be dom dim ${}_RR$.

As (1.2)'s dual, we have:

Proposition 1.4. *Let R be left noetherian and ${}_RM \cong \bigoplus \Sigma_{\alpha\in\Omega} M_\alpha$. Then,* dom dim M = min{dom dim M_α: $\alpha \in \Omega$}. *Also, R can be arbitrary if Ω is finite.*

Proof. By a result of Matlis [11, p. 514],

$$E\left(\bigoplus \sum_{\alpha\in\Omega} M_\alpha\right) \cong \bigoplus \sum_{\alpha\in\Omega} E(M_\alpha).$$

Thus, by taking a direct sum of the minimal injective resolutions of the M_α's we obtain a minimal injective resolution of M.

This means, in the case of a left perfect ring, that l-dom dim $R \geq n$ if and only if dom dim ${}_RP \geq n$ for each indecomposable projective ${}_RP$. For R left artinian, we can drop indecomposable.

Proposition 1.5. *Let R be left artinian. Then the left dominant dimension of R is greater than or equal to n if and only if the dominant dimension of each projective left R-module is greater than or equal to n.*

Proof. This follows from (1.4) and the characterization of projective left R-modules over left perfect rings as direct sums of indecomposable projectives, i.e., of Re's, e a primitive idempotent.

To this point we have not defined the codominant dimension of the ring itself. Further, we observe that imitating the procedure used for dominant dimension, using the dimension of the left regular module ${}_RR$, would be absurd. To motivate this promised but still unstated definition we make a few additional observations about its dual.

A projective generator ${}_RP$ is a *minimal projective generator* for ${}_R\mathfrak{M}$ provided that for every other projective generator ${}_RP'$, there exists an epimorphism

$$P' \longrightarrow P \longrightarrow 0,$$

i.e., P is isomorphic to a direct summand of every other projective generator.

Since ${}_RR$ is always a projective generator for ${}_R\mathfrak{M}$, it is immediate that if a minimal projective generator exists it must be isomorphic to Re for some idempotent $e \in R$.

Example 1.6. *A ring without minimal projective generators.* Let $R = \prod_{i=1}^{\infty} M_n(D)$, D a division ring, and $e_{ij}^n \in M_n(D)$ with all entries zero excepting the i, jth which is $1 \in D$. Then $R \cdot (e_{11}^1, \cdots, e_{11}^n, 1, \cdots, 1, \cdots) = P_n$ is a projective generator for $n = 1, 2, \cdots$. Suppose that ${}_RP$ is a minimal projective generator. In this case, for each n there exists an epimorphism $\phi_n: P_n \rightarrow P$. But ${}_RP$ is a generator and ${}_RR$ is cyclic, so there exists an epimorphism

$$\phi: P^m \longrightarrow R \longrightarrow 0$$

for some m, where $P^m = \bigoplus \Sigma_{i=1}^m P$. Hence, for each n there exists an epimorphism

$$P_n^m \longrightarrow R \longrightarrow 0.$$

This is impossible if $n > m$. Therefore, ${}_R\mathfrak{M}$ has no minimal projective generator.

In the case of semiperfect rings, however, minimal projective generators do exist. In particular, if $e_1, \cdots, e_n$ is a basic set of idempotents for such a ring, then $Re_1 \oplus \cdots \oplus Re_n$ is precisely such a module. Moreover, as we now show, minimal projective generators are unique to within an isomorphism for such rings.

That is, if ${}_RP$ and ${}_RP'$ are minimal projective generators for ${}_R\mathfrak{M}$, then

$P \cong P' \oplus P_1$, $P' \cong P \oplus P_2$ and $P \cong P \oplus P_1 \oplus P_2$. But then

$$P/JP \cong (P \oplus P_1 \oplus P_2)/(JP \oplus JP_1 \oplus JP_2) \cong P/JP \oplus P_1/JP_1 \oplus P_2/JP_2$$

where P/JP is artinian. Thus, $P_1/JP_1 \oplus P_2/JP_2 = 0$ from which we have $P_1 = P_2 = 0$ and $P \cong P'$.

We now note a categorical characterization of the left dominant dimension of R.

Proposition 1.7. *If ${}_RP$ is a minimal projective generator for ${}_R\mathfrak{M}$, then* l-dom dim R = dom dim ${}_RP$.

Proof. This follows directly from (1.4).

Dualizing, we say that an injective cogenerator is a *minimal injective cogenerator* for ${}_R\mathfrak{M}$ in case for every other injective cogenerator ${}_RE$ there exists a monomorphism

$$0 \longrightarrow {}_RU \longrightarrow {}_RE,$$

i.e., ${}_RU$ is isomorphic to a direct summand of every other injective cogenerator. We define the *left codominant dimension* of R, denoted l-codom dim R, to be codom dim ${}_RU$ where ${}_RU$ is a minimal injective cogenerator for ${}_R\mathfrak{M}$.

So as to settle questions concerning whether this definition is well defined, we give the following characterization of a minimal injective cogenerator.

Proposition 1.8. *Let ${}_RS = \bigoplus \sum_{\alpha\in\Omega} S_\alpha$ where $\{S_\alpha: \alpha \in \Omega\}$ is a complete set of pairwise nonisomorphic representatives of the simple modules in ${}_R\mathfrak{M}$. Then, ${}_RU \cong E_R(S)$ if and only if ${}_RU$ is a minimal injective cogenerator for ${}_R\mathfrak{M}$.*

Proof. It is well known (e.g., see Osofsky [19, p. 374]) that a module ${}_RM$ is a cogenerator if and only if ${}_RM$ contains an injective hull of every simple module in ${}_R\mathfrak{M}$. Consequently, $E_R(S)$ is necessarily a minimal injective cogenerator.

Conversely, if ${}_RU$ is a minimal injective cogenerator, then $E_R(S) \cong {}_RU \oplus N$ for some ${}_RN$. Let this isomorphism be given by ϕ. If $N \neq 0$, then Soc$(N) \neq 0$ since $\phi(S)$ is essential in $U \oplus N$. This is impossible as S is isomorphic to a submodule of U while $S \cong \text{Soc}(E_R(S))$ has simple homogeneous components.

Corollary 1.9. *Let R be left perfect with basic set of idempotents $e_1, \cdots, e_n$. Then the left codominant dimension of R is equal to the minimum of* codom dim $E(Re_i/Je_i)$ *for $i = 1, \cdots, n$.*

Next, we note the existence of rings of arbitrary codominant dimension.

Example 1.10. Let R be the ring of upper triangular $n \times n$ matrices over a division ring. The Jacobson radical J of R is the ideal of all strictly upper triangular matrices in R. Further, if e_{ij} denotes the matrix in R all of whose entries are zero excepting for the i, jth which is 1, then

$$Re_{22} \oplus \cdots \oplus Re_{n-1,n-1} \oplus Re_{nn} \oplus Re_{nn}/Je_{nn}$$

is seen to be a minimal injective cogenerator for the ring R/J^2. It follows that l-codom dim R/J^2 = codom dim $Re_{nn}/Je_{nn} = n - 1$.

Over a right perfect ring R every left module has a nonzero socle (see Bass [1, Theorem P]). It follows that every indecomposable injective left R-module is of the form $E_R(S)$ for ${}_RS$ is a simple R-module. Hence, the left codominant dimension of a right perfect ring is greater than or equal to n if and only if codom dim ${}_RE \geq n$ for each indecomposable injective left R-module ${}_RE$. For a left artinian ring one can drop the indecomposable restriction in this equivalence.

Proposition 1.11. *Let R be left artinian. Then the left codominant dimension of R is greater than or equal to m if and only if the codominant dimension of each injective left R-module is greater than or equal to m.*

Proof. Since R is left artinian, for an arbitrary injective R-module ${}_RE$ we have ${}_RE \cong \bigoplus \Sigma_{\alpha \in \Omega} E(S_\alpha)$ where each ${}_RS_\alpha$ is simple. By (1.2),

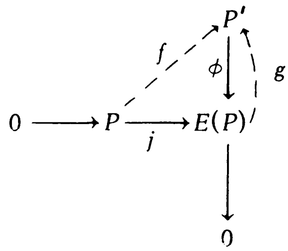
$$\begin{aligned} \text{codom dim } {}_RE &= \min\{\text{codom dim } E_R(S_\alpha) : \alpha \in \Omega\} \\ &\geq \min\{\text{codom dim } E(Re_i/Je_i) : i = 1, \cdots, n\} = l\text{-codom dim } R \end{aligned}$$

where $e_1, \cdots, e_n$ is a basic set of idempotents for R.

2. **Relating codominant dimension to dominant dimension.** We now give our main result, which reflects the dual nature of codominant dimension to dominant dimension. It might be noted that this generalizes a theorem of Fuller [4] which states that a left artinian ring is QF-3 if and only if each of its injective left modules has an injective projective cover.

Theorem 2.1. *Let R be left perfect. Then,* codom dim ${}_RE \geq n$ *for each injective ${}_RE$ if and only if* dom dim ${}_RP \geq n$ *for each projective ${}_RP$.*

Proof. ($\Rightarrow$). We shall induct on n. Assuming that codom dim ${}_RE \geq 1$ for every injective ${}_RE$ we let ${}_RP$ be projective and consider



where P' is a projective cover of $E_R(P)$. Since P is projective, the diagram can be completed with a homomorphism f such that $\phi f = j$. But now, using the injectivity of P', there exists a homomorphism $g: E_R(P) \to P'$ with $gj = f$. Furthermore, g is monic since f is and Im j is essential in $E_R(P)$. Hence, $E_R(P)$ is isomorphic to a direct summand of P' and is projective.

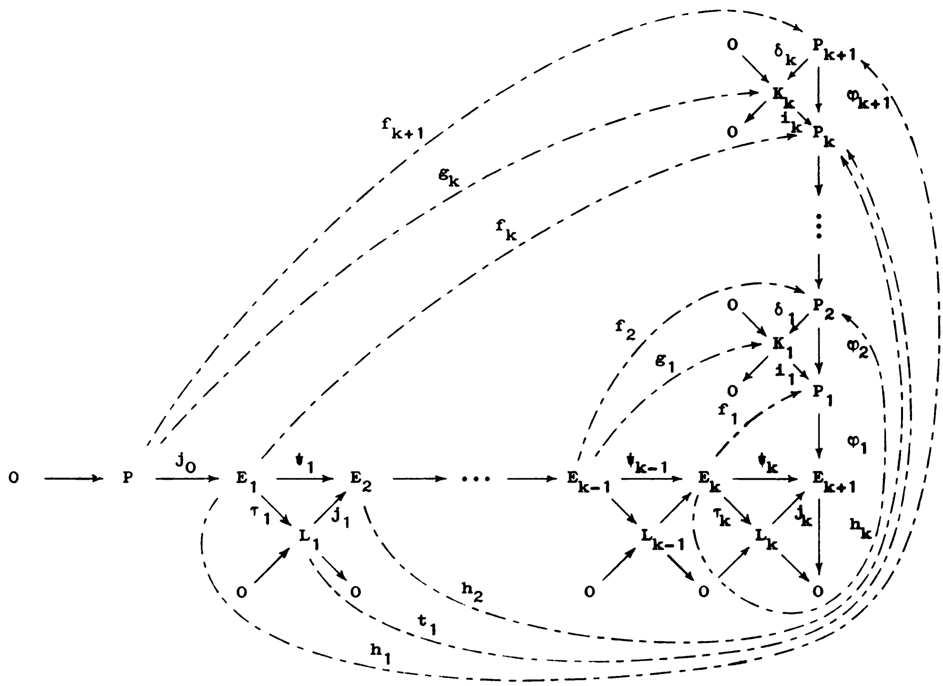


Figure 1

As the induction hypothesis we assume that codom dim ${}_RE \geq k+1$ for each injective ${}_RE$ and that dom dim ${}_RP' \geq k$ for all projective ${}_RP'$. We then let ${}_RP$ be projective and consider Figure 1. In this diagram the row is a minimal injective resolution of ${}_RP$ and the column is a minimal projective resolution of E_{k+1}, with the L_m's and K_m's being the naturally associated cokernels and kernels respectively. That is, $L_m = E_m/\operatorname{Im} j_{m-1}$ and $K_m = \operatorname{Ker} \phi_m$ while the maps τ_m, j_m, δ_m, and i_m are the obvious ones.

To complete the proof we must show that the left R-module E_{k+1} is projective. This will be done by reducing the situation to the setting where $n = 1$. That is, we will establish the existence of a commutative diagram

$$\begin{array}{ccccc} & & & & P_1 \\ & & & \overset{f}{\nearrow} & \downarrow \phi_1 \\ 0 & \longrightarrow & E_k/\operatorname{Im} \psi_{k-1} & \xrightarrow{j_k} & E_{k+1} \cong E_R(E_k/\operatorname{Im} \psi_{k-1}) \end{array}$$

from which it will follow that $E_{k+1} \cong P_1$ is projective. In short, the remainder of the proof consists of constructing the map f indicated above. This will be done in two steps.

Claim 1. There exist maps $f_r\colon E_{k+1-r} \to P_r$, $1 \le r \le k+1$, $g_r\colon E_{k-r} \to K_r$, $1 \le r \le k$, such that $\delta_{r-1} f_r = g_{r-1}$, $i_r g_r = f_r \psi_{k-r}$ where we set $E_0 = P$, $\psi_0 = j_0$, $\delta_0 = \phi_1$ and $g_0 = \psi_k$.

Proof. Since E_k is projective, we can complete

$$\begin{array}{ccccc} & & E_k & & \\ & \overset{f_1}{\swarrow} & \downarrow \psi_k & & \\ P_1 & \xrightarrow[\phi_1]{} & E_{k+1} & \longrightarrow & 0 \end{array}$$

with a homomorphism f_1. Hitting the exact sequence

$$0 \longrightarrow K_1 \xrightarrow{i_1} P_1 \xrightarrow{\phi_1} E_{k+1} \longrightarrow 0$$

with the functor $\operatorname{Hom}_R(E_{k-1},\ )$, which is exact because by the induction hypothesis E_{k-1} is projective, gives the exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(E_{k-1}, K_1) \xrightarrow{\operatorname{Hom}(1,i_1)} \operatorname{Hom}_R(E_{k-1}, P_1)$$
$$\xrightarrow{\operatorname{Hom}(1,\phi_1)} \operatorname{Hom}_R(E_{k-1}, E_{k+1}) \longrightarrow 0.$$

Noting that $f_1\psi_{k-1} \in \operatorname{Hom}_R(E_{k-1}, P_1)$ while $\phi_1 f_1 \psi_{k-1} = \psi_k \psi_{k-1} = 0$, we have that there exists $g_1 \in \operatorname{Hom}_R(E_{k-1}, K_1)$ such that $i_1 g_1 = f_1 \psi_{k-1}$. Now assume that f_r, g_r have been defined for $1 \le r \le s \le k$ such that $f_r\colon E_{k+1-r} \to P_r$, $g_r\colon E_{k-r} \to K_r$ and $\delta_{r-1} f_r = g_{r-1}$, $i_r g_r = f_r \psi_{k-r}$ where we set $E_0 = P$, $\psi_0 = j_0$,

$\delta_0 = \phi_1$ and $g_0 = \psi_k$. Since E_{k-s} is projective we can complete

$$\begin{array}{ccccc} & & E_{k-s} & & \\ & {}^{f_{s+1}}\swarrow & \downarrow g_s & & \\ P_{s+1} & \xrightarrow{\delta_s} & K_s & \longrightarrow & 0. \end{array}$$

And for $s < k$, by applying $\mathrm{Hom}_R(E_{k-s-1}, \ )$ to

$$0 \longrightarrow K_{s+1} \xrightarrow{i_{s+1}} P_{s+1} \xrightarrow{\delta_s} K_s \longrightarrow 0$$

we have the exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(E_{k-s-1}, K_{s+1}) \xrightarrow{\mathrm{Hom}(1, i_{s+1})} \mathrm{Hom}_R(E_{k-s-1}, P_{s+1})$$
$$\xrightarrow{\mathrm{Hom}(1, \delta_s)} \mathrm{Hom}_R(E_{k-s-1}, K_s) \longrightarrow 0.$$

Since $f_{s+1}\psi_{k-s-1} \in \mathrm{Hom}_R(E_{k-s-1}, P_{s+1})$ while

$$i_s\delta_s f_{s+1}\psi_{k-s-1} = i_s g_s \psi_{k-s-1} = f_s\psi_{k-s}\psi_{k-s-1} = 0,$$

where i_s is monic, we have that $\delta_s f_{s+1}\psi_{k-s-1} = 0$. But then there exists $g_{s+1} \in \mathrm{Hom}_R(E_{k-s-1}, K_{s+1})$ such that $i_{s+1}g_{s+1} = f_{s+1}\psi_{k-s-1}$. This completes the verification of our first claim. Observe that by combining the two relations involving the f_r's and g_r's we get $\phi_r f_r = f_{r-1}\psi_{k-r+1}$.

Claim 2. There exist maps $h_r: E_r \to P_{k+2-r}$, $1 \leq r \leq k$, and $t_r: L_r \to P_{k+1-r}$, $1 \leq r \leq k-1$, such that $h_r j_{r-1} = t_{r-1}$, $t_r \tau_r = f_{k+1-r} - \phi_{k+2-r}h_r$ where we set $t_0 = f_{k+1}$.

Proof. Since P_{k+1} is injective, we can complete

$$\begin{array}{ccccc} 0 & \longrightarrow & P & \xrightarrow{j_0} & E_1 \\ & & f_{k+1}\downarrow & \swarrow h_1 & \\ & & P_{k+1} & & \end{array}$$

Also, since by hypothesis P_k is injective, by applying $\mathrm{Hom}_R(\ , P_k)$ to

$$0 \longrightarrow P \xrightarrow{j_0} E_1 \xrightarrow{\tau_1} L_1 \longrightarrow 0,$$

we get the exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(L_1, P_k) \xrightarrow{\mathrm{Hom}(\tau_1, 1)} \mathrm{Hom}_R(E_1, P_k) \xrightarrow{\mathrm{Hom}(j_0, 1)} \mathrm{Hom}_R(P, P_k) \longrightarrow 0.$$

Further, by observing that $f_k - \phi_{k+1}h_1 \in \mathrm{Hom}_R(E_1, P_k)$ and that

$$(f_k - \phi_{k+1}h_1)j_0 = f_k j_0 - \phi_{k+1}f_{k+1} = f_k j_0 - f_k j_0 = 0$$

we have a homomorphism $t_1 \in \mathrm{Hom}_R(L_1, P_k)$ such that $t_1 r_1 = f_k - \phi_{k+1}h_1$. Now by assuming that h_r, t_r have been defined for $1 \le r \le s \le k-1$ such that h_r: $E_r \to P_{k+2-r}$, t_r: $L_r \to P_{k+1-r}$, $h_r j_{r-1} = t_{r-1}$, and $t_r r_r = f_{k+1-r} - \phi_{k+2-r}h_r$ (where we set $t_0 = f_{k+1}$), we define h_{s+1} by completing

$$\begin{array}{ccccc} 0 & \longrightarrow & L_s & \xrightarrow{\;j_s\;} & E_{s+1} \\ & & {\scriptstyle t_s}\Big\downarrow & \swarrow {\scriptstyle h_{s+1}} & \\ & & P_{k+1-s} & & \end{array}$$

On the other hand, if $s \le k-2$ we define t_{s+1} by first hitting

$$0 \longrightarrow L_s \xrightarrow{\;j_s\;} E_{s+1} \xrightarrow{\;r_{s+1}\;} L_{s+1} \longrightarrow 0$$

with the exact functor $\mathrm{Hom}_R(\ , P_{k-s})$, to get

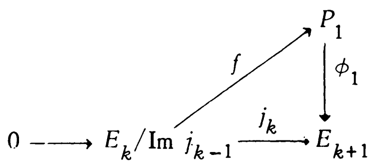
$$0 \longrightarrow \mathrm{Hom}(L_{s+1}, P_{k-s}) \xrightarrow{\;\mathrm{Hom}(r_{s+1},1)\;} \mathrm{Hom}(E_{s+1}, P_{k-s})$$
$$\xrightarrow{\;\mathrm{Hom}(j_s,1)\;} \mathrm{Hom}(L_s, P_{k-s}) \longrightarrow 0,$$

and then observing that $f_{k-s} - \phi_{k+1-s}h_{s+1} \in \mathrm{Hom}_R(E_{s+1}, P_{k-s})$ and that

$$\begin{aligned}(f_{k-s} - \phi_{k+1-s}h_{s+1})j_s r_s &= f_{k-s}\psi_s - \phi_{k+1-s}t_s r_s \\ &= f_{k-s}\psi_s - \phi_{k+1-s}(f_{k+1-s} - \phi_{k+2-s}h_s) = f_{k-s}\psi_s - \phi_{k+1-s}f_{k+1-s} \\ &= f_{k-s}\psi_s - f_{k-s}\psi_s = 0\end{aligned}$$

where r_s is epic. Thus, $(f_{k-s} - \phi_{k+1-s}h_{s+1})j_s = 0$ and so there exists $t_{s+1} \in \mathrm{Hom}_R(L_{s+1}, P_{k-s})$ such that $t_{s+1}r_{s+1} = f_{k-s} - \phi_{k+1-s}h_{s+1}$. This completes the verification of our second claim.

We are now ready to construct the map f indicated in the first part of the proof. We first consider $f_1 - \phi_2 h_k$: $E_k \to P_1$. We note that $\mathrm{Im}\, j_{k-1} \le \mathrm{Ker}(f_1 - \phi_2 h_k)$ since, as seen in the proof of the second claim, $(f_1 - \phi_2 h_k)j_{k-1} = 0$. On the other hand, if $(f_1 - \phi_2 h_k)(x) = 0$ for some $x \in E_k$, then $\psi_k(x) = \phi_1\phi_2 h_k(x) = 0$, i.e., $x \in \mathrm{Ker}\,\psi_k = \mathrm{Im}\, j_{k-1}$. Therefore, $\mathrm{Ker}(f_1 - \phi_2 h_2) = \mathrm{Im}\, j_{k-1}$. But then $f_1 - \phi_2 h_k$ induces a map $E_k/\mathrm{Im}\, j_{k-1} \to P_1$. This is our map f. Now consider



It is easy to check that this diagram is commutative. This is exactly the situation we had when $n = 1$. It follows that E_{k+1} is isomorphic to P_1 and so, is projective. This completes the proof of the sufficiency part of (2.1). Observe that since E_{k+1} is projective the column in Figure 1 is trivial for all but E_{k+1} and P_1.

($\Leftarrow$). This half of the proof is the dual of the first.

Corollary 2.2. *Let R be a left artinian ring. Then, l-dom dim $R \geq n$ if and only if l-codom dim $R \geq n$.*

Proof. This follows immediately from (1.5) and (1.11).

Corollary 2.3. *If R is artinian, then the left codominant dimension of R is equal to the right codominant dimension of R.*

Proof. Müller [14, Theorem 10] has proved that for R perfect, r-dom dim $R > 0$ and l-dom dim $R > 0$ implies l-dom dim $R = r$-dom dim R. On the other hand, Harada [5] has shown that if R is artinian, then l-dom dim $R > 0$ if and only if r-dom dim $R > 0$.

Theorem 2.4. *Let R be semiperfect and left noetherian. Then the following are equivalent*:

(a) l-dom dim $R \geq n$;

(b) dom dim ${}_RP \geq n$ *for each finitely generated projective* ${}_RP$;

(c) dom dim ${}_RP \geq n$ *for each finitely generated indecomposable projective* ${}_RP$;

(d) codom dim ${}_RQ \geq n$ *for each finitely generated injective* ${}_RQ$ *and each finitely generated projective* ${}_RP$ *has a minimum injective resolution whose first* n *terms are finitely generated*;

(e) codom dim ${}_RQ \geq n$ *for each finitely generated indecomposable injective* ${}_RQ$ *and each finitely generated indecomposable projective* ${}_RP$ *has a minimum injective resolution whose first* n *terms are finitely generated.*

Proof. By (1.2) and the characterization of finitely generated projective modules over semiperfect rings, we have that (a), (b) and (c) are equivalent. The equivalence of (d) and (e) is immediate from (1.4).

To show that (b) is sufficient for (d) it will suffice to indicate how the sufficiency proof of (2.1) can be applied. Let ${}_RQ$ be finitely generated injective with minimum projective resolution

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow Q \longrightarrow 0.$$

The module P_1 is finitely generated since ${}_RQ$ is. Further, the kernel of $P_1 \to Q$ is finitely generated since R is left noetherian. But then P_2 is finitely generated. By induction, $P_1, \cdots, P_n$ are all finitely generated. So, by hypothesis dom dim $P_n \geq n$ and it is straightforward to see that the same inductive proof as used in the sufficiency proof of (2.1) may be applied to obtain codom dim ${}_RQ \geq n$.

For ${}_RP$ finitely generated projective with minimum injective resolution

$$0 \longrightarrow P \longrightarrow E_1 \longrightarrow \cdots \longrightarrow E_n \longrightarrow \cdots,$$

we note that (see Müller [14, Lemma 1]) E_1 is finitely generated by a result of Faith and Walker [2, Proposition 2.4] since it is projective. By induction we get that $E_1, \cdots, E_n$ are all finitely generated.

Finally, we assume (d) and let ${}_RP$ be finitely generated with minimum injective resolution

$$0 \longrightarrow P \longrightarrow E_1 \longrightarrow \cdots \longrightarrow E_n \longrightarrow \cdots.$$

Since E_n is finitely generated by hypothesis, codom dim $E_n \geq n$ and one shows inductively as in the necessity proof of (2.1) that E_n is projective. Hence, (b) is equivalent to (d).

3. **Codominant dimension of generalized uniserial rings.** Recall that a generalized uniserial ring R is an artinian ring for which each primitive one-sided ideal (i.e., each Re and eR where e is a primitive idempotent) has a unique composition series. In the case where the ring is indecomposable, one can relate these indecomposable projectives in the following manner.

Theorem 3.1 (Kupisch [10]). *If R is an indecomposable generalized uniserial ring, then any basic set of primitive idempotents $e_1, \cdots, e_n$ for R can be indexed so that*

(a) $Re_i/Je_i \cong Je_{i+1}/J^2e_{i+1}$, *for* $i = 1, \cdots, n-1$; *and* $Re_n/Je_n \cong Je_1/J^2e_1$ *unless* $Je_1 = 0$;

(b) *the composition length of* $Re_i = c(Re_i) \geq 2$, *for* $i = 2, \cdots, n$;

(c) $c(Re_{i+1}) \leq c(Re_i) + 1$ *for* $i = 1, \cdots, n-1$;

(d) $c(Re_1) \leq c(Re_n) + 1$.

In this setting, $Re_1, \cdots, Re_n$ is called a left *Kupisch series* for R and $c_1 = c(Re_1), \cdots, c_n = c(Re_n)$ is referred to as the corresponding admissible

Kupisch sequence for R. Observe that in general $c_1, \cdots, c_n$ is unique to within an n-cycle. We say that Re_i is a *chain end* provided $c(Re_{[i+1]}) \le c(Re_i)$, where $[i]$ denotes the least strictly positive remainder of i modulo n. It can be shown that the chain ends are precisely the indecomposable injective projective R-modules (see Jans [6, p. 1107]).

In light of the work done by Kupisch, Murase, and Fuller with a generalized uniserial ring's Kupisch series, it is natural to attempt to compute the codominant dimension (or equivalently the dominant dimension) of such a ring from its Kupisch sequence. Indeed, Fuller showed how this sequence of integers characterizes the global dimension of the ring. In this section we prove a similar characterization for codominant dimension.

First we state a somewhat technical but useful result due to Kupisch [10] and Fuller [3].

Lemma 3.2. *Let R be an indecomposable generalized uniserial ring with Kupisch series $Re_1, \cdots, Re_n$ and $J^k e_i \ne 0$. Then,*

(a) $J^k e_i \cong Re_{[i-k]}/J^{c_i-k} \cdot e_{[i-k]}$;

(b) $J^k e_i / J^{k+1} e_i \cong Re_{[i-k]}/Je_{[i-k]}$;

(c) $E(Re_i/Je_i) \cong Re_{[i+d_i-1]}/J^{d_i} \cdot e_{[i+d_i-1]}$

where $d_i = c(e_i R)$.

Proposition 3.3. *The codominant dimension of an indecomposable generalized uniserial ring is completely determined by the ring's Kupisch sequence.*

Proof. Let $Re_1, \cdots, Re_n$ be a Kupisch series for R and $c_1, \cdots, c_n$ its corresponding Kupisch sequence of integers. In light of (1.9) it will suffice to show that $\operatorname{codom\,dim} E(Re_i/Je_i)$ is completely determined by $c_1, \cdots, c_n$ for $i = 1, \cdots, n$.

With this in mind, we fix i, set $\Omega_i = \inf\{j\colon Re_{k_j}$ is not a chain end$\}$, and inductively define two sequences $\{s_j\}, \{k_j\}$ by setting

$$s_0 = c(e_i R), \qquad k_0 = [i + s_0 - 1]$$

and

$$s_j = c_{k_{j-1}} - s_{j-1}, \qquad k_j = [k_{j-1} - s_{j-1}].$$

We shall show that $\operatorname{codom\,dim} E(Re_i/Je_i)$ is equal to ∞ or Ω_i depending on whether or not $s_j \le 0$ for some $j \le \Omega_i$.

Fix $m < \Omega_i$. By (3.2)

$$E_R(Re_i/Je_i) \cong Re_{k_0}/J^{s_0} \cdot e_{k_0}$$

and

$$J^{s_j} \cdot e_{k_j} \cong Re_{k_{j+1}} / J^{s_{j+1}} \cdot e_{k_{j+1}}$$

provided $s_{j+1} > 0$. Moreover, in this case $Re_{k_{j+1}}$ is a projective cover for $J^{s_j} \cdot e_{k_j}$. Hence, assuming that $s_j > 0$ for $j = 1, 2, \cdots, m$,

$$\begin{array}{ccccc} Re_{k_m} & \longrightarrow & Re_{k_{m-1}} & \longrightarrow & \\ & \searrow & \nearrow & & \\ & & J^{s_{m-1}} \cdot e_{k_{m-1}} & & \end{array}$$

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & Re_{k_1} & \longrightarrow & Re_{k_0} & \longrightarrow & E_R(Re_i/Je_i) & \longrightarrow & 0 \\ & & & \searrow & \nearrow & & & & \\ & & & & J^{s_0} \cdot e_{k_0} & & & & \end{array}$$

affords the first $m + 1$ terms of a minimal projective resolution of $E_R(Re_i/Je_i)$. Since Re_{k_j} is a chain end for $j = 0, 1, \cdots, m$ and so, injective, codom dim $E_R(Re_i/Je_i) \geq m + 1$. Consequently, codom dim $E_R(Re_i/Je_i) \geq \Omega_i$.

Now if codom dim $E_R(Re_i/Je_i) > \Omega_i$ and $s_j > 0$ for each $j \leq \Omega_i$, then by considering the above minimal projective resolution of $E_R(Re_i/Je_i)$ we see that $Re_{k_{\Omega_i}}$ must be injective and hence, a chain end. This is impossible, however, so we conclude that codom dim $E_R(Re_i/Je_i) = \Omega_i$.

Finally, if $s_j \leq 0$ for some $j \leq \Omega_i$, then it is immediate that codom dim $E_R(Re_i/Je_i) = \infty$.

Lemma 3.4. *Let R_1, R_2 be left perfect rings and $M_1 \in {}_{R_1}\mathfrak{M}$, $M_2 \in {}_{R_2}\mathfrak{M}$. Then M_1 and M_2 are naturally $R_1 \oplus R_2$-modules and*

$$\text{codom dim }_{R_1 \oplus R_2} M_1 \oplus M_2 = \min_{i=1,2} \text{ codom dim }_{R_i} M_i.$$

Proof. Let

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow M_1 \longrightarrow 0$$

and

$$\cdots \longrightarrow P'_n \longrightarrow \cdots \longrightarrow P'_1 \longrightarrow M_2 \longrightarrow 0$$

be minimal projective resolutions of ${}_{R_1}M_1$ and ${}_{R_2}M_2$ respectively. Then,

$$\cdots \longrightarrow P_n \oplus P'_n \longrightarrow \cdots \longrightarrow P_1 \oplus P'_1 \longrightarrow M_1 \oplus M_2 \longrightarrow 0$$

is a minimal projective resolution of $M_1 \oplus M_2 \in {}_{R_1 \oplus R_2}\mathfrak{M}$, where each summand is now considered as an $R_1 \oplus R_2$-module. Further, $P_i \oplus P'_i$ is $R_1 \oplus R_2$ injective if and only if P_i is R_1 injective and P'_i is R_2 injective.

Proposition 3.5. *Let R_1 and R_2 be left perfect rings. Then,* l-codom dim $R_1 \oplus R_2 = \min_{i=1,2}$ l-codom dim R_i.

Proof. Each simple left $R_1 \oplus R_2$-module can be obtained from a simple R_i-module, say ${}_{R_i}S$, for either $i = 1$, or 2, by considering S as an $R_1 \oplus R_2$-module. Also, $E_{R_1 \oplus R_2}(S) \cong {}_{R_1 \oplus R_2}(E_{R_i}(S))$ where we consider $E_{R_i}(S)$ as an $R_1 \oplus R_2$-module. But then, by (3.4)

$$\text{codom dim } E_{R_1 \oplus R_2}(S) = \text{codom dim } E_{R_i}(S).$$

The desired result now follows from (1.9).

Corollary 3.6. *The codominant dimension of a generalized uniserial ring is completely determined by the admissible sequences of its indecomposable direct summands.*

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