TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 176, February 1973

INFINITE COMPOSITIONS OF MÖBIUS TRANSFORMATIONS(1)

BY

JOHN GILL

ABSTRACT. A sequence of Möbius transformations $\{t_n\}_{n=1}^{\infty}$, which converges to a parabolic or elliptic transformation t, may be employed to generate a second sequence $\{T_n\}_{n=1}^{\infty}$ by setting $T_n = t_1 \circ \cdots \circ t_n$. The convergence behavior of $\{T_n\}$ is investigated and the ensuing results are shown to apply to continued fractions which are periodic in the limit.

This paper treats the convergence behavior of sequences of Möbius transformations $\{T_n(z)\}$ which are generated in the following way:

Let $t_n(z) = (a_n z + b_n)/(c_n z + d_n)$, where $t = \lim t_n$ is either parabolic or elliptic. Set $T_1(z) = t_1(z)$, $T_n(z) = T_{n-1}(t_n(z))$, $n = 2, 3, \cdots$.

Our approach is essentially the same as that of Magnus and Mandell [1], who investigated the cases in which the t_n and t are hyperbolic or loxodromic, and in which the t_n and t are all elliptic. They established conditions on the fixed points $\{u_n\}$ and $\{v_n\}$ of $\{t_n\}$ that insure behavior of $\{T_n(z)\}$ very much like that observed in the special case $t_n = t$ for all n [2]. Convergence is in the extended plane, so that divergence is of an oscillatory nature only.

The present paper consists of results concerning the two remaining possible combinations of t_n and t:

(1) t_n any type and t parabolic, and (2) t_n elliptic or loxodromic and t elliptic. The principal result obtained in the investigation of case (2) is an extension and sharpening of the main theorem in [1].

The parabolic case. First consider the case in which $t = \lim t_n$ is parabolic, with a finite fixed point v. Some conditions on the rates at which u_n and v_n approach v are necessary, as the following example illustrates.

Example 1. Let $t_n = [n/(n+1)]^s z + 1$, where $s = 1 + iy$, $y \neq 0$. Then $t = z + 1$, which is parabolic with fixed point $v = \infty$. We have

$$T_n(z) = z/(n+1)^s + \zeta_n(s),$$

where $\zeta_n(s)$ is the truncated Riemann-Zeta function.

Received by the editors February 1, 1972.

AMS (MOS) subject classifications (1970). Primary 40A20.

Key words and phrases. Hyperbolic, loxodromic, elliptic Möbius transformations, parabolic Möbius transformations, continued fractions periodic in the limit.

(1) This research was supported in part by the United States Air Force Office of Scientific Research Grant 70-1922.

It can be shown, [3, p. 235], that $\zeta_n(s)$ oscillates finitely as $n \to \infty$ for the prescribed values of s.

Set $X(z) = z/(z-1)$. Then $X^{-1} \circ t_n \circ X(z) = t_n^*(z)$ and $t_n(z)$ are the same type of transformation [1], and $t^* = X^{-1} \circ t \circ X$ has the fixed point $v^* = 1$. Obviously

$$T_n^*(z) = t_1^* \circ \cdots \circ t_n^*(z) = X^{-1} \circ T_n \circ X(z)$$

has the same convergence behavior as $T_n(z)$.

Theorem 1. *Let $\{t_n\}$ be a sequence of Möbius transformations converging to a parabolic transformation t, having a finite fixed point v. If there exists an ordering of u_n and v_n, the fixed points of t_n, such that $\Sigma|u_n - v_n|$ and $\Sigma n|v_{n+1} - v_n|$ both converge, then the sequence $\{T_n(z)\}$ converges in the extended plane for every z.*

Proof. Assume the t_n's and t have been normalized so that $a_n d_n - b_n c_n = ad - bc = 1$, and that $a + d = 2$.

First observe that any t_n may be written implicitly

$$\frac{1}{t_n(z) - v_n} = \frac{k_n}{z - v_n} + q_n, \tag{1}$$

where

$$k_n = 1 \quad \text{if } t_n \text{ is parabolic,}$$
$$= (a_n - c_n u_n)/(a_n - c_n v_n) \quad \text{if } t_n \text{ is nonparabolic}$$

and

$$q_n = c_n \quad \text{if } t_n \text{ is parabolic,}$$
$$= (k_n - 1)/(v_n - u_n) \quad \text{if } t_n \text{ is nonparabolic.}$$

It may easily be shown that $\lim k_n = 1$ and $\lim q_n = c \neq 0$.

Next, set

$$Y_n(z) = 1/(z - v_n), \qquad K_n(z) = k_n \cdot z, \qquad Q_n(z) = q_n + z.$$

Then

$$t_n(z) = Y_n^{-1} \circ Q_n \circ K_n \circ Y_n(z).$$

Set

$$w_n(z) = Q_n \circ K_n \circ Y_n \circ Y_{n+1}^{-1}(z), \quad S_n(z) = Q_n \circ K_n \circ Y_n(z), \qquad n = h, h+1, \cdots,$$

where h will be chosen later. Thus

$$T_n(z) = T_{h-1} \circ Y_h^{-1} \circ w_h \circ \cdots \circ w_n \circ S_n(z).$$

Direct computation shows that $w_n(z) = (p_n z + q_n)/(r_n z + 1)$ where $r_n = v_{n+1} - v_n$ and $p_n = k_n + q_n r_n$.

Set $W_n^b(z) = w_b \circ \cdots \circ w_n(z)$, and consider the convergence behavior of $\{W_n^b \circ S_n(z)\}_{n=b+1}^{\infty}$ for a fixed value of b.

Let $W_n^b(z) = (A_n^b z + B_n^b)/(C_n^b z + D_n^b)$, where

$$A_n^b = p_n A_{n-1}^b + r_n B_{n-1}^b, \tag{2}$$

$$B_n^b = q_n A_{n-1}^b + B_{n-1}^b, \tag{3}$$

$$C_n^b = p_n C_{n-1}^b + r_n D_{n-1}^b, \tag{4}$$

$$D_n^b = q_n C_{n-1}^b + D_{n-1}^b. \tag{5}$$

It follows from (2) and (3) that

$$A_n^b = \prod_b^n p_i + \sum (\prod p_i) q_{k_1} r_{k_2} + \sum (\prod p_i) q_{k_1} r_{k_2} q_{k_3} r_{k_4} + \cdots + \sum (\prod p_i) q_{k_1} r_{k_2} \cdots q_{k_{2j-1}} r_{k_{2j}}, \tag{6}$$

where $b < k_1 < \cdots < k_l \leq b + m = n$, $1 < l \leq 2j$. The q- and r-factors alternate, and (Πp_i) designates finite p-products with $i \geq b$.

Lemma 1. *Suppose* $\{r_{b+k_j}\}_{j=1}^{l}$ *are the r-factors in a term of* A_n^b. *Then there are no more than s terms having this specific set of r-factors in* A_n^b, *where* $s \leq \Pi_{i=1}^{l} k_i$.

Proof. The proof is by induction on the auxiliary recurrence relations:

$$A_{b+m}^b = A_{b+m-1}^b + r_{b+m} B_{b+m-1}^b \quad \text{and} \quad B_{b+m}^b = A_{b+m-1}^b + B_{b+m-1}^b.$$

We observe that

$$p_i = k_i + q_i r_i = 1 + (v_i - u_i) q_i + q_i r_i,$$

so that, by hypothesis, Πp_i converges, and there exists a positive number M such that both $|\Pi p_i|$ and $|q_i|$ are less than M for i greater than some b.

Fix $\epsilon > 0$ and choose b so large that the following conditions are met, in addition to those described above: $|\Pi_b^n p_i - 1| < \epsilon/2$, for $n \geq b$, and $\Sigma_{m=1}^{\infty} m|r_{b+m}| < l/M$, where $l < \min\{1, M, \epsilon/(2M + \epsilon)\}$.

Consequently, by the preceding remarks and Lemma 1,

$$\left| A_n^b - \prod_b^n p_i \right| \leq \sum |(\prod p_i) q_{k_1} r_{k_2}| + \cdots + \sum |(\prod p_i) q_{k_1} \cdots r_{k_{2j}}|$$
$$< M^2 (l/M) + \cdots + M^{j+1} (l/M)^j < \epsilon/2.$$

Hence $|A_n^b - 1| \leq |\Pi_b^n p_i - 1| + \epsilon/2 < \epsilon$.

In an entirely similar manner it may be shown that $|C_n^b| < \epsilon$, for a sufficiently large b.

(2) and (3) give

$$A_{b+m}^b - k_{b+m}A_{b+m-1}^b = q_{b+m}r_{b+m}A_{b+m-1}^b + r_{b+m}B_{b+m-1}^b,$$

from which we obtain

$$A_{b+m}^b - A_{b+m-1}^b = (k_{b+m} - 1)A_{b+m-1}^b + r_{b+m}B_{b+m}^b. \tag{7}$$

Summing both sides of (7),

$$A_{b+m}^b - p_b = \sum_{j=1}^m (k_{b+j} - 1)A_{b+j-1}^b + \sum_{j=1}^m r_{b+j}B_{b+j}^b. \tag{8}$$

Upon summing, (3) gives

$$B_{b+m}^b = q_b + \sum_{j=1}^m q_{b+j}A_{b+j-1}^b. \tag{9}$$

Combine (8) and (9) to obtain

$$A_{b+m}^b = p_b + \sum_{j=1}^m (k_{b+j} - 1)A_{b+j-1}^b + \sum_{j=1}^m r_{b+j}\left(q_b + \sum_{i=1}^m q_{b+i}A_{b+i}^b\right). \tag{10}$$

Thus, from (10), if $|q_{b+n}| < M$ and $|A_m^b| < 3$,

$$|A_{b+m+1}^b - A_{b+m}^b| < 3|k_{b+m+1} - 1| + M|r_{b+m+1}|[1 + 3(m+2)]$$
$$< 3[|k_{b+m+1} - 1| + M(m+3)|r_{b+m+1}|].$$

Therefore

$$|A_{b+m+n}^b - A_{b+m}^b| \leq \sum_{j=1}^n |A_{b+m+j}^b - A_{b+m+j-1}^b|$$
$$\leq 3M\left[\sum_{j=1}^n |v_{b+m+j} - u_{b+m+j}| + \sum_{j=1}^n (m+j+2)|r_{b+m+j}|\right].$$

The last expression on the right may be made arbitrarily small by choosing m sufficiently large and n a positive integer. The Cauchy criterion is satisfied and we have

$$\lim_{n\to\infty} A_n^b = l(A, b) \approx 1. \tag{11}$$

Similarly,

$$\lim_{n\to\infty} C_n^b = l(C, b) \approx 0. \tag{12}$$

It is obvious, from (9), that

$$\lim_{n\to\infty} B_n^b = \infty. \tag{13}$$

Also,

$$A_n^h D_n^h - B_n^h C_n^h = \det W_n^h = \prod_h^{n-2} (\det w_j) = \prod_h^{n-2} k_j = \prod_h^{n-2} [1 + q_j(v_j - u_j)].$$

The hypothesis implies the convergence of this product to some number close to one, as $n \to \infty$. Hence

$$\lim_{n\to\infty} (D_n^h/B_n^h) = l_h \not\approx 0. \tag{14}$$

It is now possible to complete the proof of Theorem 1 for $z \neq v$. We have, from (11), (12), (13), and (14),

$$\lim_{n\to\infty} [W_n^h \circ S_n(z)] = \lim_{n\to\infty} \frac{(A_n^h/B_n^h)S_n(z) + 1}{(C_n^h/B_n^h)S_n(z) + (D_n^h/B_n^h)} = \frac{1}{l_h}.$$

Thus, $\lim_{n\to\infty} T_n(z) = T_{h-1} \circ Y_h(1/l_h)$, $z \neq v$.

We divide numerator and denominator of $W_n^h \circ S_n(v)$ by $S_n(v)$ and find, after some computation, that

$$\lim_{n\to\infty} T_n(v) = T_{h-1} \circ Y_h(1/l_h).$$

Corollary 1. *Let $\{t_n\}$ be a sequence of normalized Möbius transformations converging to t, which is parabolic and has a finite fixed point. If $t_n(z) = (a_n z + b_n)/(c_n z + d_n)$, then the convergence of the following four series imply the convergence of $\{T_n(z)\}$ for every z: $\Sigma n|\sqrt{[(a_{n+1} + d_{n+1})^2 - 4]}|$, $\Sigma n|a_{n+1} - a_n|$, $\Sigma n|c_{n+1} - c_n|$, $\Sigma n|d_{n+1} - d_n|$.*

The following example shows that the hypotheses of Theorem 1, although sufficient, are not necessary.

Example 2. Let

$$t_n(z) = [(v_n + 1)z - v_n^2]/[z + (1 - v_n)],$$

where $v_1 = 0$ and $v_n = \Sigma_{k=1}^{n-1} (-1)^k/k$ for $n \geq 2$. Then $\lim v_n = v = -\log 2$, and both t_n and t are parabolic. An intricate investigation, somewhat similar to the proof of Theorem 1, shows that $\{T_n(z)\}$ converges for every $z \neq v$.

The elliptic case. We next consider the case in which $t = \lim t_n$ is elliptic.

Theorem 2. *Let $\{t_n\}$ be a sequence of Möbius transformations having fixed points $\{u_n\}$ and $\{v_n\}$, chosen so that $|k_n| \leq 1$. Let $t = \lim t_n$ be an elliptic transformation having finite fixed points u and v.*

(i) *If $\Sigma|u_n - u_{n-1}| < \infty$, $\Sigma|v_n - v_{n-1}| < \infty$, and $\Pi k_n \to 0$, then $\{T_n(z)\}$ converges for every z except perhaps $z = v$.*

(ii) *If $\Sigma|u_n - u_{n-1}| < \infty$, $\Sigma|v_n - v_{n-1}| < \infty$, and $\Pi|k_n|$ converges, then*

$\{T_n(z)\}$ diverges by oscillation for $z \neq u$, v and converges to distinct values for $z = u$ and $z = v$.

Proof. Set $Y_n(z) = (z - u_n)/(z - v_n)$, $K_n(z) = k_n z$, $w_{n-1}(z) = K_{n-1} \circ Y_{n-1} \circ Y_n^{-1}(z)$, $S_n(z) = K_n \circ Y_n(z)$, and $W_n^b(z) = w_b \circ \cdots \circ w_{n-1}(z) = (A_n^b z + B_n^b)/(C_n^b z + D_n^b)$. Then

$$t_n(z) = Y_n^{-1} \circ K_n \circ Y_n(z),$$

and

$$T_n(z) = T_{b-1} \circ Y_b^{-1} \circ W_n^b \circ S_n(z).$$

As before, $w_n(z) = (p_n z + q_n)/(r_n z + 1)$, where $p_n = k_n(v_{n+1} - u_n)/(v_n - u_{n+1})$, etc.

We choose a positive ϵ and find an b such that $|A_n^b - \Pi_b^n p_j| < \epsilon$ and $|C_n^b| < \epsilon$ for $n > b$. Thus $\lim_{n\to\infty} B_n^b = l(B, b) \approx 0$ and $\lim_{n\to\infty} D_n^b = l(D, b) \approx 1$.

The following formula is established by induction:

$$A_n^b = \prod_b^n p_j + \sum_{m=b}^{n-2} \left(\prod_{m+1}^n p_j \right) r_{m+1} B_m^b + r_n B_{n-1}^b. \tag{15}$$

We observe that $\Pi_b^n |p_j| = \Pi_b^n |k_j| \cdot \Pi_b^n (1 + s_j)$, where $\Sigma |s_j| < \infty$. Therefore, in case (i), $\Pi_b^n |p_j| \to 0$, as $n \to \infty$. The three terms in (15) tend to zero, as $n \to \infty$. Hence, $\lim_{n\to\infty} A_n^b = 0$. In similar fashion, $\lim_{n\to\infty} C_n^b = 0$.

Consequently,

$$\lim_{n\to\infty} T_n(z) = T_{b-1} \circ Y_b^{-1} \circ \lim_{n\to\infty} W_n^b(S_n(z)) = T_{b-1} \circ Y_b^{-1} \circ \frac{l(B, b)}{l(D, b)}$$

for $z \neq v$.

The hypotheses of case (ii), and the observed behavior of the coefficients of W_n^b provide a straightforward proof of the next lemma.

Lemma 2. *For a fixed $z \neq v$, there exist finite numbers M and b_0 such that $b > b_0$, $n \geq b$, $m \geq b - 1$ imply $|S_n(z)| < M$ and $|T_n^b(z) - v_m| > |u - v|/4(1 + M)$.*

Using (1) and the fact that

$$\frac{1}{t_{n+1}(z) - v_n} = \frac{1}{t_{n+1}(z) - v_{n+1}} + \frac{v_n - v_{n+1}}{(t_{n+1}(z) - v_n)(t_{n+1}(z) - v_{n+1})},$$

the following formula may be established by induction on n:

$$\frac{1}{T_n^b(z) - v_b} = \frac{\Pi_b^n k_j}{z - v_n} + \sum_{m=b}^{n-1} \left(\prod_b^m k_j \right) \frac{v_m - v_{m+1}}{(T_n^{m+1}(z) - v_m)(T_n^{m+1}(z) - v_{m+1})} + \sum_{m=b-1}^{n-1} \left(\prod_b^m k_j \right) \frac{k_{m+1} - 1}{v_{m+1} - u_{m+1}}, \tag{16}$$

where $\Pi_b^{b-1} k_j \equiv 1$.

We may rewrite (16) in the form

$$\frac{1}{T_n^h(z) - v_h} = \frac{(\Pi_h^n k_j)(z - u_n)}{(z - v_n)(v_n - u_n)}$$

$$(17) \qquad + \sum_h^{n-1} \left(\prod_h^m k_j\right) \frac{v_m - v_{m+1}}{(T_n^{m+1}(z) - v_m)(T_n^{m+1}(z) - v_{m+1})}$$

$$+ \sum_{h+1}^{n-1} \left(\prod_h^m k_j\right) \frac{v_{m+1} - v_m + u_m - u_{m+1}}{(v_m - u_m)(v_{m+1} - u_{m+1})} + \frac{k_h - 1}{v_h - u_h} - \frac{k_h}{v_{h+1} - u_{h+1}}.$$

Set

$$\prod_h^n k_j = \exp\left(i \sum_h^n \theta_j\right) \prod_h^n |k_j|,$$

$F = F(z) = (z - u)/(z - v)(v - u)$, $R = |F| \sin(|\theta'|/4)$, where $\arg k = \theta = \theta' \pmod{2\pi}$, $|\theta'| \le \pi$.

We choose h so large that the following conditions are satisfied, in addition to previous stipulations:

$$(18) \qquad |f_1| < \frac{R}{6}, \quad \text{where } F + f_1 = \frac{z - u_n}{(z - v_n)(v_n - u_n)},$$

$$(19) \qquad |f_2| < \frac{R}{6}, \quad \text{where } \frac{k_h - 1}{v_h - u_h} - \frac{k_h}{v_{h+1} - u_{h+1}} = f_2 + \frac{1}{u - v},$$

$$(20) \qquad |f_3| < \min\{1, R/6|F|\}, \quad \text{where } \prod_h^n |k_j| = 1 + f_3,$$

$$(21) \qquad \sum_h |v_{m+1} - v_m| < \frac{R|v - u|^2}{96(1 + M)^2},$$

$$(22) \qquad \sum_h |u_{m+1} - u_m| < \frac{R|v - u|^2}{48},$$

$$(23) \qquad |v_m - u_m| > \frac{|v - u|}{2}, \qquad m \ge h - 1.$$

Then, from (17), we obtain

$$(24) \qquad \frac{1}{T_n^h(z) - v_h} = |F| \exp\left[i\left(\arg F + \sum_h^n \theta_j\right)\right] + \frac{1}{u - v} + H(h, n),$$

where $|H(h, n)| < R$.

The sum of the first two terms of (24) is a point on a circle C with center $1/(u - v)$ and radius $|F|$. Hence $1/(T_n^h(z) - v_h)$ lies in a disc $U(h, m)$ at radius R with center g_n on C. R has been chosen so that three tangent discs of radius

R with centers on C can be constructed if the centers of the two end discs are separated by a central angle of θ'.

Clearly, the sequence $\{1/(T_n^b(z) - v_b)\}_{n=b}^{\infty}$ diverges by oscillation, so that $\{T_n^b(z)\}_{n=b}^{\infty}$ must do likewise. The pattern of divergence bears a close resemblence to that observed when $t_n = t$ for all n. In this special case

$$\frac{1}{T_n(z) - v} = |F| \exp[i(\arg F + n\theta)] + \frac{1}{u - v}.$$

Convergence at $z = u$ is easily established, since $S_n(u) \to 0$. We return to the beginning of the proof of case (ii) and interchange the u_n's and v_n's, in order to show convergence at $z = v$. The development in [1] can be paraphrased to show that $\lim T_n(u) \neq \lim T_n(v)$.

Corollary 2. *If the transformations t_n converge to the elliptic transformation t, where $a_n d_n - b_n c_n = ad - bc = 1$ and $\Sigma|a_n - a_{n-1}|$, $\Sigma|b_n - b_{n-1}|$, $\Sigma|c_n - c_{n-1}|$, and $\Sigma|d_n - d_{n-1}|$ all converge, then $\{T_n(z)\}$*

(i) *converges for $z \neq v$, if $\Pi k_n \to 0$,*

(ii) *diverges for $z \neq u, v$, and converges to distinct values at u and v, if $\Pi|k_n|$ converges.*

Continued fractions may be interpreted as compositions of Möbius transformations, and may be written so as to display the fixed points. Set $t_n(z) = -u_n v_n/(-(u_n + v_n) + z)$, to obtain

$$\frac{-u_1 v_1}{-(u_1 + v_1)} + \frac{-u_2 v_2}{-(u_2 + v_2)} + \cdots, \tag{25}$$

whose nth approximant is $T_n(0)$.

The following two examples are applications of Theorems 1 and 2 to continued fractions which are periodic in the limit.

Example 3. Let $u_n = |u_n|\exp(i\theta_n)$, $v_n = |v_n|\exp(i\phi_n)$, where $\lim|u_n| = \lim|v_n| = c \neq 0$, $\lim\theta_n = \theta$, $\lim\phi_n = \phi$, $\theta \not\equiv \phi \pmod{2\pi}$. Then

$$\lim k_n = \lim|u_n/v_n|\exp[i(\theta_n - \phi_n)] = k = \exp[i(\theta - \phi)],$$

so that t is elliptic. Theorem 2, case (i) guarantees the convergence of (25), provided $|u_n|$ and $|v_n|$ are chosen so that $\Pi|u_n/v_n| \to 0$, (e.g., $|u_n| = 1 - 1/n^2$, $|v_n| = 1 + 1/n$).

Example 4. Let $u_n = c + \epsilon_n$, $v_n = c + \delta_n$, where $\lim\epsilon_n = \lim\delta_n = 0$, $c \neq 0$, $\Sigma|\epsilon_n - \delta_n| < \infty$, $\Sigma n|\delta_{n+1} - \delta_n| < \infty$, (e.g., $u_n = -\frac{1}{2} - i/n^2$, $v_n = -\frac{1}{2} + i/n^2$). Then t is parabolic, and Theorem 1 insures the convergence of (25).

BIBLIOGRAPHY

1. M. Mandell and A. Magnus, *On convergence of sequences of linear fractional transformations*, Math. Z. 115 (1970), 11–17. MR 41 #3621.

2. L. R. Ford, *Automorphic functions*, McGraw-Hill, New York, 1929.

3. T. J. I'A. Bromwich, *An introduction to the theory of infinite series*, Macmillan, London, 1947.

DEPARTMENT OF MATHEMATICS, COLORADO STATE UNIVERSITY, FORT COLLINS, COLORADO 80521

Current address: Department of Mathematics, Southern Colorado State College, Pueblo, Colorado 81005