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BOUNDARY VALUES OF SOLUTIONS OF ELLIPTIC EQUATIONS SATISFYING H^p CONDITIONS

BY

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ABSTRACT. Let A be an elliptic linear partial differential operator with C^∞ coefficients on a manifold Ω with boundary Γ. We study solutions of $Au = \sigma$ which satisfy the H^p condition that $\sup_{0<t<1} \|u(\cdot, t)\|_p < \infty$, where we have chosen coordinates in a neighborhood of Γ of the form $\Gamma \times [0,1]$ with Γ identified with $t = 0$. If A has a well-posed Dirichlet problem such solutions may be characterized in terms of the Dirichlet data $u(\cdot, 0) = f_0$, $(\partial/\partial t)^j u(\cdot, 0) = f_j$, $j = 1, \cdots, m-1$ as follows: $f_0 \in L^p$ (or $\mathfrak{M}$ if $p = 1$) and $f_j \in \Lambda(-j; p, \infty)$, $j = 1, \cdots, m$. Here Λ denotes the Besov spaces in Taibleson's notation. If $m = 1$ then u has nontangential limits almost everywhere.

1. **Introduction.** One of the oldest results in the theory of elliptic partial differential equations states that for a harmonic function in the unit disc the following two conditions are equivalent (see [14]):

(i) $\|u(re^{i\theta})\|_p \leq C$ independent of r, $0 < r < 1$;

(ii) u is the Poisson integral of an L^p function (bounded measure if $p = 1$) on the circle. Furthermore, u has nontangential limits almost everywhere.

In recent years the theory of boundary value problems for elliptic equations has been developed intensely. Nevertheless, most of these developments do not contain the above result as a special case. Thus we propose to study the following problem:

Let A be an elliptic partial differential operator on a compact manifold Ω with boundary Γ. Assume a neighborhood of Γ can be coordinatized by $[0, 1] \times \Gamma$. For solutions of $Au = 0$, what is the relationship between the condition $\|u(t, \cdot)\|_p \leq C$ for $0 < t < 1$ and conditions on the boundary values of u?

We will assume that A has a well-posed Dirichlet problem, so that modulo finite dimensional spaces the function u is determined uniquely by its boundary values and the boundary values of a finite number of normal derivatives of u. The answer to our question is that the condition $\|u(t, \cdot)\|_p \leq C$ for $0 < t < 1$ and fixed p, $1 \leq p \leq \infty$, is equivalent to:

(a) $$u_{|\Gamma} \in L^p \quad (\text{or } \mathfrak{M} \text{ if } p = 1),$$

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(b) $$(\partial/\partial t)^k u_{|\Gamma} \in \Lambda(-k; p, \infty),$$

for $k \geq 1$. The Besov spaces $\Lambda(-k; p, \infty)$ will be defined below. Notice that the conditions on the boundary values of u and its normal derivatives are of an entirely different nature, since L^p is properly contained in $\Lambda(0; p, \infty)$. In the case of the Laplacian condition (b) does not arise since no normal derivatives occur in the Dirichlet problem.

Our proof will utilize both the classical Fourier analysis technique of Agmon-Douglis-Nirenberg [1] and the pseudodifferential operators approach of Seeley [8]. We will use the following characterization of Besov spaces due to Peetre [6].

Let σ, τ be C^∞ functions on $(0, \infty)$ chosen so that $\tau \equiv 1$ on $(0, 4)$ and has support in $(0, 8)$, while $\sigma \equiv 1$ on $(1,2)$ with support in $(\frac{1}{2}, 4)$. For a distribution $f \in \mathcal{S}'(R^n)$ we define

$$\|f: \Lambda(\alpha; p, q)\| = \|\mathcal{F}^{-1}(\tau(|\xi|)\hat{f}(\xi))\|_p + \left(\int_0^1 \|\mathcal{F}^{-1}(\sigma(t|\xi|)\hat{f}(\xi)|\xi|^\alpha)\|_p^q \frac{dt}{t}\right)^{1/q}$$

(if $q = \infty$ replace the last term by $\sup_{0<t<1} \|\mathcal{F}^{-1}(\sigma(t|\xi|)\hat{f}(\xi)|\xi|^\alpha)\|_p$). The space of all distrubutions with finite norm $\|f: \Lambda(\alpha; p, q)\|$ coincides with the usual Besov spaces $\Lambda(\alpha; p, q)$ (see Taibleson [12, I] for other definitions). Since these spaces are known to be locally invariant under diffeomorphism we may define the spaces $\Lambda(\alpha; p, q)$ of distributions on a compact manifold Γ in the usual manner.

Before dealing with the general case it is instructive to consider one example. Let $\Omega = (0, 1) \times R^n$ and $A = (\partial/\partial t + (-\Delta_x)^{1/2})^2$. This is not a special case of the above since Ω is not compact and A is only a pseudodifferential operator, but it exhibits the basic features we are interested in. Any solution of $Au = 0$ which satisfies temperate growth conditions at infinity can be written

(*) $$u(x, t) = \mathcal{F}^{-1}((1 + t|\xi|)e^{-t|\xi|}\hat{f}(\xi) + te^{-t|\xi|}\hat{g}(\xi))$$

and satisfies boundary conditions

$$u(x, 0) = f(x), \qquad \frac{\partial u}{\partial t}(x, 0) = g(x).$$

Our first goal is to show that if $f \in L^p$ (or $\mathfrak{M}$ if $p = 1$) and $g \in \Lambda(-1; p, \infty)$ then $\|u(\cdot, t)\|_p \leq C$. Now a version of Sobolev's inequality shows that $\mathcal{F}^{-1}((1 + t|\xi|)e^{-t|\xi|}) \in L^1$. Since the L^1 norm is preserved by dilation on the Fourier transform side we have $\|\mathcal{F}^{-1}((1 + t|\xi|)e^{-t|\xi|})\|$, constant, independent of t, so $\|\mathcal{F}^{-1}((1 + t|\xi|)e^{-t|\xi|}\hat{f}(\xi))\|_p \leq A\|f\|_p$.

To handle the other term of (*) we write

$$\hat{g}(\xi) = \tau(|\xi|)\hat{h}(\xi) + \int_0^1 \sigma^2(s|\xi|)\hat{h}(\xi)\frac{ds}{s} \qquad \text{for } h \in \Lambda(-1; p, \infty)$$

(h is "essentially" g). The term $\mathcal{F}^{-1}(te^{-t|\xi|}\tau(\xi)\hat{h}(\xi))$ is handled as before. Finally,

$$\left\|\mathcal{F}^{-1}\left(te^{-t|\xi|}\int_0^1 \sigma^2(s|\xi|)\hat{h}(\xi)\frac{ds}{s}\right)\right\|_p$$
$$\leq \int_0^1 \|\mathcal{F}^{-1}(\sigma(s|\xi|)t|\xi|e^{-t|\xi|})\|_1 \cdot \|\mathcal{F}^{-1}(\sigma(s|\xi|)|\xi|^{-1}\hat{h}(\xi))\|_p \frac{ds}{s}$$

so it suffices to show

$$\int_0^1 \|\mathcal{F}^{-1}(\sigma(s|\xi|)t|\xi|e^{-t|\xi|})\|_1 \frac{ds}{s}$$

is bounded independent of t. If we extend the integral to infinity, dilate $\xi \to t^{-1}\xi$ and make a change of variable $s \to ts$ we can dominate this by

$$\int_0^\infty \|\mathcal{F}^{-1}(\sigma(s|\xi|)|\xi|e^{-|\xi|})\|_1 \frac{ds}{s}$$

which is finite because

$$\|\mathcal{F}^{-1}((\sigma|\xi|)|\xi|e^{-|\xi|})\|_1 \leq e^{-1/(2s)}, \qquad s \to 0,$$
$$\leq s^{-1}, \qquad s \to \infty.$$

For the converse we must show that if u is given by (*) and $\|u(\cdot, t)\|_p \leq C$ then f and g lie in the appropriate space. But $f = \lim_{t\to 0} u(\cdot, t)$ in the distribution sense so $f \in L^p$ (or $\mathfrak{M}$ if $p = 1$). To handle the term involving g we proceed indirectly via Green's theorem (a direct approach will work in this case, but it does not appear to generalize). If $v \in C^2$ and vanishes near $t = 1$ we have

$$(Au, v) = (u, A^*v) - 2\int(-\Delta_x)^{1/2} u(x, 0)v(x, 0)\,dx - \int u_t(x, 0)v(x, 0)\,dx$$
$$+ \int u(x, 0)v_t(x, 0)\,dx$$

where $A^* = (\partial/\partial t - (-\Delta_x)^{1/2})^2$. Since $Au = 0$ and we may suppose $u(x, 0) = 0$ by subtracting off the term involving f, we have $\int u_t(x, 0)v(x, 0)\,dx = (u, A^*v)$. Using the fact that $\Lambda(-1; p, \infty)$ is essentially the dual space of $\Lambda(1; p', 1)$ (small modifications are necessary if $p = 1$) we are reduced to showing that for a dense subspace of $\Lambda(1; p', 1)$ we may solve the equation $v(x, 0) = \phi(x)$ with

$$\int_0^1 \|A^*v(\cdot, t)\|_{p'}\,dt \leq A\|\phi: \Lambda(1; p', 1)\|.$$

We may do this explicitly by setting $v(x, t) = \mathcal{F}^{-1}(\psi(t)e^{-t|\xi|}\hat{\phi}(\xi))$ for $\psi \in C^\infty$, $\psi \equiv 1$ near $t = 0$ and $\psi \equiv 0$ near $t = 1$. Notice that A^*v is made up of terms like $\mathcal{F}^{-1}(\psi(t)|\xi|^2 e^{-t|\xi|}\hat{\phi}(\xi))$. Writing

$$\hat{\phi}(\xi) = \tau(|\xi|)\hat{h}(\xi) + \int_0^1 |\xi|^{-1}\hat{h}(\xi)\sigma^2(s|\xi|)\frac{ds}{s}$$

for $h \in \Lambda(0; p', 1)$ this becomes

$$\mathcal{F}^{-1}(\psi(t)\tau(|\xi|)e^{-t|\xi|}|\xi|^2\hat{b}(\xi)) + \int_0^1 \mathcal{F}^{-1}(\psi(t)|\xi|e^{-t|\xi|}\sigma^2(s|\xi|)\hat{b}(\xi))\,\frac{ds}{s}.$$

The first term is handled easily; the second term can be dominated by

$$\int_0^1\int_0^1 \|\mathcal{F}^{-1}(|\xi|e^{-t|\xi|}\sigma(s|\xi|))\|_1 \|\mathcal{F}^{-1}(\sigma(s|\xi|)\hat{b}(\xi))\|_{p'}\,\frac{ds}{s}\,dt.$$

Extending the integrations to infinity we find this is an integral transform of $\|\mathcal{F}^{-1}(\sigma(s|\xi|)\hat{b}(\xi))\|_{p'}\, s^{-1} \in L^1$ with kernel homogeneous of degree -1. Thus we have the desired estimate if $\int_0^\infty \|\mathcal{F}^{-1}(|\xi|e^{-|\xi|}\sigma(s|\xi|))\|_1 s^{-1}ds < \infty$, a condition we have already verified.

The above special case gives in outline the approach we shall follow. The crux of the argument is to show that in general the solution to the Dirichlet problem is given by a formula similar to (*). We then utilize the following lemma, which is an extension of one version of Sobolev's inequality.

Lemma 1. *Suppose $q(x, \xi)$ defined on $R^n \times R^n$ has compact support in x and satisfies*

$$\int \left|\left(\frac{\partial}{\partial\xi}\right)^\alpha\left(\frac{\partial}{\partial x}\right)^\beta q(x, \xi)\right|^{1+\epsilon} d\xi \le A$$

for $|\alpha| \le n$, $|\beta| \le n$, and some ϵ in $0 < \epsilon < 1$. Then the operator

$$Qf(x) = \int q(x, \xi)e^{ix\cdot\xi}\hat{f}(\xi)\,d\xi$$

is bounded on L^p, $\|Qf\|_p \le c(A)\|f\|_p$, $1 \le p \le \infty$, and $\|Q\mu\|_1 \le C(A)\|\mu\|$ for $\mu \in \mathfrak{M}$.

A refinement of this lemma will enable us to generalize Fatou's theorem: a solution of $Au = 0$ which satisfies $\|u(t, \cdot)\|_p \le C$ has nontangential limits almost everywhere as $t \to 0$ provided A is second order and has a well-posed Dirichlet problem. Widman [13] has proved the same conclusion with weaker assumptions on the smoothness of coefficients for operators with symmetric second order terms.

2. **Statement of results.** Let us describe in detail our assumptions. Ω is a C^∞ compact manifold with boundary Γ realized as submanifolds of a compact manifold M without boundary (we may take for M the double of Ω). A neighborhood of Γ in M is coordinatized by $[-1, 1] \times \Gamma$ with $(0, 1] \times \Gamma$ in Ω and $0 \times \Gamma = \Gamma$. We assume that A is an elliptic pseudodifferential operator on M of order m (in the sense of Kohn-Nirenberg) which is a differential operator in the normal variable in $[-1, 1] \times \Gamma$ and such that $Au|_\Omega$ depends only on $u|_\Omega$. Any uniformly elliptic partial differential operator on Ω with coefficients in $C^\infty(\bar{\Omega})$ can be extended to such an operator.

Let $\Gamma_1, \cdots, \Gamma_a$ be the connected components of Γ, and denote by $\sigma_m(A)(x, t, \xi, \tau)$ the top order symbol of A in $[-1, 1] \times \Gamma$ in local coordinates. Here x is a local coordinate for Γ, t is the normal coordinate, ξ is dual to x and

τ is dual to t. We say A *has a well-posed Dirichlet problem* if for each component Γ_j there exists an integer m_j in $0 \leq m_j \leq m$ such that the polynomial $\sigma_m(A)(x, 0, \xi, \tau)$ in τ has exactly m_j roots in $\operatorname{Im} \tau > 0$ for all $x \in \Gamma_j$ and $\xi \neq 0$. If A is a partial differential operator and $\dim \Omega \geq 3$ this can only happen if m is even and $m_j = m/2$; by allowing A to be a more general pseudodifferential operator this need not be the case (the example in the introduction has $m_1 = 2$, $m_2 = 0$).

The Dirichlet problem is then

$$Au = 0 \quad \text{in } \Omega,$$
$$(\partial/\partial t)^k u(x, 0) = f_k^{(j)}(x), \qquad x \in \Gamma_j,\ k = 0, \cdots, m_j - 1.$$

For simplicity of notation we will only deal with the case when Γ has only one component. All our arguments go over unchanged in the general case.

The following results are essentially contained in Seeley [8] and [9].

(A) The Dirichlet problem has a unique solution modulo finite dimensional spaces. More precisely, given any distributions $f_0, \cdots, f_{m_1-1}$ on Γ which are orthogonal to a finite number of C^∞ functions on Γ, there exists a C^∞ function u on Ω which solves the Dirichlet problem (the boundary values $(\partial/\partial t)^k u(x, 0)$ existing as weak limits of $(\partial/\partial t)^k u(x, t)$ as $t \to 0^+$). Furthermore, u is the restriction to Ω of a distribution on M. The space of solutions of the homogeneous Dirichlet problem (all $f_k \equiv 0$) is finite dimensional and all such solutions are in $C^\infty(\overline{\Omega})$. Finally all solutions to the equation $Au = 0$ in Ω which are restrictions to Ω of distributions on M arise in the above manner.

(B) Let B be an almost-inverse of A; B is a pseudodifferential operator of order $-m$ such that $BA = I$ on the orthogonal complement of the kernel of A, and $AB = I$ on the range of A. Then the Dirichlet problem may be solved by "Poisson integrals" of the form

$$* \qquad u = \sum_{j=0}^{m-1} \sum_{k=0}^{m_1-1} B\left(\frac{\partial}{\partial t}\right)^j R^* T_{jk} f_k \quad \text{on } \Omega$$

where $R^*: \mathfrak{D}'(\Gamma) \to \mathfrak{D}'(M)$ is the adjoint of the restriction map $R: \mathfrak{D}(M) \to \mathfrak{D}(\Gamma)$ given by $R\phi(x) = \phi(x, 0)$ and T_{jk} is a pseudodifferential operator of order $m - k - j - 1$ on Γ.

(C) The top order symbols of T_{jk} satisfy the equations

$$\sum_{j=0}^{m-1} \frac{1}{2\pi i} \int \frac{(i\tau)^{j+p} d\tau}{\sigma_m(A)(x, 0, \xi, \tau)} \sigma_{m-k-j-1}(T_{jk})(x, \xi) = \delta_{kp}$$

for $p = 0, \cdots, m_1 - 1$, where the integral is over a contour surrounding the roots of $\sigma_m(A)$ in $\operatorname{Im} \tau > 0$.

Our first goal is to establish

Theorem 1. *Assume* $f_0 \in L^p(\Gamma)$ (*or* $f_0 \in \mathfrak{M}$ *if* $p = 1$) *and* $f_k \in \Lambda(-k; p, \infty)$

for $k = 1, \cdots, m_1 - 1$ *for fixed* p, $1 \le p \le \infty$. *Then any solution of the Dirichlet problem satisfies* $\|u(\cdot, t)\|_p \le C$. *If there are no nonzero solutions of the homogeneous Dirichlet problem then we have an estimate*

$$\|u(\cdot, t)\|_p \le c\left(\|f_0\|_p + \sum_{k=1}^{m_1-1} \|f_k : \Lambda(-k; p, \infty)\|\right).$$

Next we will establish a slightly more general converse:

Theorem 2. *Assume* Ω, Γ, A *as above except that* A *need not have a well-posed Dirichlet problem. Let* u *satisfy* $Au = 0$ *in* Ω *and* $\|u(\cdot, t)\|_p \le C$. *Then* $(\partial/\partial t)^k u(x, t)$ *converges weakly as* $t \to 0^+$ *to a distribution* f_k *for all* k *and* $f_0 \in L^p$ *(or* $\mathfrak{M}$ *if* $p = 1$*) and* $f_k \in \Lambda(-k; p, \infty)$ *for* $k \ge 1$, *with* $\|f_0\|_p \le cC$ *(or* $\|f_0\| \le cC$ *if* $p = 1$*).*

Finally, we will establish a version of Fatou's theorem:

Theorem 3. *Let the hypotheses of Theorem 2 be satisfied, and assume also that* A *has a well-posed Dirichlet problem with all* $m_j \le 1$. *Then for almost every point* $y \in \Gamma$ *we have* $u(x, t)$ *converging pointwise to* $f(y)$ *as* (x, t) *approaches* $(y, 0)$ *along any nontangential path.*

We will present the proofs in the next three sections. The final section is devoted to some generalizations.

3. **Poisson kernels and proof of Theorem 1.** We begin by establishing Lemma 1. It is a generalization of the fact that if a function has $n/p + \epsilon$ derivatives in L^p for $1 \le p \le 2$ then the function is the Fourier transform of an L^1 function.

Proof of Lemma 1. Let $Q(x, y) = \int q(x, \xi) e^{iy\cdot\xi}\, d\xi$. Then $Qf(x) = \int Q(x, x - y) f(y) dy = \int Q(x, y) f(x - y)\, dy$ so it suffices to show $\sup_x |Q(x, y)| \in L^1$. Now by Sobolev's inequality

$$\sup_x |Q(x, y)| \le c \sum_{|\beta| \le n} \int \left| \left(\frac{\partial}{\partial x}\right)^\beta Q(x, y) \right| dx.$$

Also $y^\alpha (\partial/\partial x)^\beta Q(x, y)$ for fixed x is the Fourier transform of $(\partial/\partial\xi)^\alpha (\partial/\partial x)^\beta q(x, \xi)$, so by the hypotheses and the Hausdorff-Young inequality, we have

$$\left(\int \left| y^\alpha \left(\frac{\partial}{\partial x}\right)^\beta Q(x, y) \right|^p dy\right)^{1/p} \le A$$

for some $p < \infty$. Since $q(x, \xi)$ has compact support in x we obtain

$$\left\| y^\alpha \sup_x |Q(x, y)| \right\|_p \le c \sum_{|\beta| \le n} \int \left(\int \left| y^\alpha \left(\frac{\partial}{\partial x}\right)^\beta Q(x, y) \right|^p dy\right)^{1/p} dx \le c' A$$

for all $|\alpha| \le n$. The result follows since $(1 + |y|^n)^{-1} \in L^{p'}$. Q.E.D.

In applying Lemma 1 we will use the following dilation principle: If $q(x, \xi)$ satisfies the hypotheses of Lemma 1, then the conclusion holds for $q(x, s\xi)$ for any positive s with the bounds independent of s. Indeed the operator associated with $q(x, s\xi)$ is $s^{-n}\int Q(x, s^{-1}y)f(x-y)\,dy$ and

$$\left\|\sup_x |s^{-n}Q(x, s^{-1}y)|\right\|_1 = \left\|\sup_x |Q(x, y)|\right\|_1.$$

Thus the same argument applies. This observation is important because the constant A in the hypotheses of Lemma 1 is not invariant under dilation.

Now let $\theta(\xi)$ be a patch function on R^n (θ vanishes near the origin and is $\equiv 1$ away from the origin) chosen so that $\theta(\xi)\tau(|\xi|) \equiv 0$.

Lemma 2. *Let $q(x, t, \xi) \in C^\infty(R^n \times [0, 1] \times R^n)$ with compact support in x. Let $u(x, t) = \int e^{ix\cdot\xi}\hat{f}(\xi)\theta(\xi)q(x, t, \xi)\,d\xi$. Then*

(a) *if for some $\epsilon > 0, k \geq 0$ the estimate*

$$\left|\left(\frac{\partial}{\partial\xi}\right)^\alpha\left(\frac{\partial}{\partial x}\right)^\beta q(x, t, \xi)\right| \leq At|\xi|^{1-k-|\alpha|}e^{-\epsilon t|\xi|}$$

holds for all α, β with $|\alpha| \leq n$, $|\beta| \leq n$ and $|\xi| \geq 1$, then for any $f \in \Lambda(-k; p, \infty)$ we have

$$\|u(\cdot, t)\|_p \leq cA\,\|f\colon \Lambda(-k; p, \infty)\|;$$

(b) *the same conclusion follows from the estimate*

$$\left|\left(\frac{\partial}{\partial\xi}\right)^\alpha\left(\frac{\partial}{\partial x}\right)^\beta q(x, t, \xi)\right| \leq A|\xi|^{-1-k-|\alpha|}e^{-\epsilon t|\xi|}$$

with ϵ, k, α, β, ξ as before;

(c) *if for some $\epsilon > 0$ the estimates*

$$\left|\left(\frac{\partial}{\partial x}\right)^\beta q(x, t, \xi)\right| \leq A\,e^{-\epsilon t|\xi|},$$

$$\left|\left(\frac{\partial}{\partial\xi}\right)^\alpha\left(\frac{\partial}{\partial x}\right)^\beta q(x, t, \xi)\right| \leq At|\xi|^{1-|\alpha|}e^{-\epsilon t|\xi|}, \qquad \alpha \neq 0,$$

hold for all α, β with $|\alpha| \leq n$, $|\beta| \leq n$ and $|\xi| \geq 1$, then for any $f \in L^p$ (or $\mathfrak{M}$ if $p = 1$) we have

$$\|u(\cdot, t)\|_p \leq cA\,\|f\|_p \text{ (or } \|f\| \text{ if } p = 1).$$

Proof. Let $r(x, t, \xi) = (1 + |\xi|^2)^{k/2}q(x, t, \xi)\theta(\xi)$ and $v(x, t) = \int e^{ix\cdot\xi}\hat{g}(\xi)r(x, t, \xi)\,d\xi$. To establish (a) and (b) it suffices to show $\|v(\cdot, t)\|_p \leq cA\,\|g\colon \Lambda(0; p, \infty)\|$. Now it is easy to see that every $g \in \Lambda(0; p, \infty)$ may be written

$$\hat{g}(\xi) = \tau(|\xi|)\hat{h}(\xi) + \int_0^1 \sigma^2(s|\xi|)\hat{h}(\xi)\frac{ds}{s} \qquad \text{for some } h \in \Lambda(0; p, \infty)$$

with comparable norm. The first term does not contribute to v since $\theta(\xi)\tau(|\xi|) \equiv 0$. To estimate the remaining term we apply Lemma 1 to $r(x, t, \xi)\sigma(s|\xi|)$. Denote by $R(s, t)$

the norm of the operator $f \to \int e^{ix\cdot\xi}\hat{f}(\xi)r(x, t, \xi)\sigma(s|\xi|)\,d\xi$ as an operator on L^p. Then

$$\left\|\int_0^1\int e^{ix\cdot\xi}\hat{h}(\xi)r(x, t, \xi)\sigma^2(s|\xi|)\,d\xi\,\frac{ds}{s}\right\|_p$$
$$\leq \int_0^1 R(s, t)\,\|\mathcal{F}^{-1}(\sigma(s|\xi|)\hat{h}(\xi))\|_p\,\frac{ds}{s}.$$

Since $\|\mathcal{F}^{-1}(\sigma(s|\xi|)\hat{h}(\xi))\|_p \leq \|h\colon \Lambda(0; p, \infty)\|$ it suffices to show

$$\int_0^1 R(s, t)\frac{ds}{s} \leq c$$

independent of t to establish (a) and (b).

For (a) we apply Lemma 1 to $r(x, t, (s/t)\xi)\sigma(|\xi|)$. From the hypotheses of (a) we have

$$\left|\left(\frac{\partial}{\partial\xi}\right)^\alpha\left(\frac{\partial}{\partial x}\right)^\beta\left(r\left(x, t, \frac{s}{t}\xi\right)\sigma(|\xi|)\right)\right|$$
$$\leq c\sum_{\gamma\leq\alpha}\left(\frac{s}{t}\right)^{|\gamma|} t\left(\frac{s}{t}|\xi|\right)^{1-|\gamma|} e^{-\epsilon s|\xi|} \leq cse^{-\epsilon s/2}$$

on the support of $\sigma(s|\xi|)$. By the dilation principle and Lemma 1 this means $R(t/s, t) \leq cse^{-\epsilon s/2}$, hence

$$\int_0^1 R(s, t)\frac{ds}{s} \leq \int_0^\infty R\left(\frac{t}{s}, t\right)\frac{ds}{s} < \infty.$$

For (b) we have

$$\left|\left(\frac{\partial}{\partial\xi}\right)^\alpha\left(\frac{\partial}{\partial x}\right)^\beta(r(x, t, s\xi)\sigma(|\xi|))\right|$$
$$\leq c\sum_{\gamma\leq\alpha} s^{|\gamma|}(s|\xi|)^{-1-|\gamma|}e^{-\epsilon st|\xi|} \leq cs^{-1}$$

on the support of $\sigma(|\xi|)$. Thus by the dilation principle and Lemma 1 we have $R(s^{-1}, t) \leq cs^{-1}$, hence

$$\int_0^1 R(s, t)\frac{ds}{s} \leq c\int_0^1 ds \leq c.$$

Finally (c) follows by a direct application of Lemma 1 and the dilation principle since

$$|\xi|^{1-n}e^{-\epsilon|\xi|} \in L^{1+\delta}. \quad \text{Q.E.D.}$$

Proof of Theorem 1. It suffices to prove the estimate for solutions given by Poisson kernels (*) because the solutions of the homogeneous Dirichlet problem are all C^∞ up to the boundary. Thus we must estimate

$$u_k = \sum_{j=0}^{m-1} B\left(\frac{\partial}{\partial t}\right)^j R^*T_{jk}f_k$$

for $f_0 \in L^p$ (or $\mathfrak{M}$ if $p = 1$) and $f_k \in \Lambda(-k; p, \infty)$ for $k \geq 1$. We localize to coor-

dinate neighborhoods by means of a partition of unity $\{\phi_i\}$ on Γ and $\{\psi_i\}$ in $C^\infty(\bar{\Omega})$ such that $\psi_i\phi_i = \phi_i$ on Γ. It then suffices to estimate

$$\sum_{j=0}^{m-1} \psi_i B\left(\frac{\partial}{\partial t}\right)^j R^* T_{jk}(\phi_i f),$$

for the other terms are in $C^\infty(\bar{\Omega})$ by the quasi-locality of pseudodifferential operators.

We introduce local coordinates (x, t) on the support of ψ_i in such a way that t is the normal coordinate to Γ. We then expand the pseudodifferential operators B and T_{jk} into a series of homogeneous terms plus a remainder. By taking sufficiently many terms the remainders may be made smoothing of high order so that all terms involving remainders may be handled by crude estimates as in Seeley [8]. We shall show that the highest order terms are covered by (a) and (c) of Lemma 2, while the lower order terms are covered by (b).

Now since B is an almost-inverse of A the expansion of its symbol may be computed explicitly in terms of the symbol of A. The highest order term is $(\sigma_m(A)(x, t, \xi, \tau))^{-1}$ and the lower order terms are of the form $E(x, t, \xi, \tau)(\sigma_m(A)(x, t, \xi, \tau))^{-r}$ for $r \geq 1$ with E a symbol of order $\leq (r-1)m-1$ which is a polynomial in τ. Seeley [8] shows that the resulting terms in $B(\partial/\partial t)^j R^*$ can be expressed by contour integrals. Thus the terms we are considering (modulo remainder terms which are easily handled) have the form

$$\int e^{ix\cdot\xi}\theta(\xi)(\phi_i f_k)\hat{}(\xi)q(x, t, \xi)\,d\xi$$

where $q(x, t, \xi)$ is

$$\frac{\psi_i(x, t)}{2\pi i}\sum_{j=0}^{m-1}\int\frac{e^{it\tau}(i\tau)^j\sigma_{m-k-j-1}(T_{kj})(x, \xi)}{\sigma_m(A)(x, t, \xi, \tau)}\,d\tau$$

for the highest order term and

$$\frac{\psi_i(x, t)}{2\pi i}\int\frac{e^{it\tau}(i\tau)^j E(x, t, \xi, \tau)}{(\sigma_m(A)(x, t, \xi, \tau))^r}\,d\tau$$

for the lower order terms. Here E is a symbol of order $\leq rm - k - j - 2$ which is a polynomial in τ. The integration is over a contour surrounding the roots of $\sigma_m(A)$ that lie in $\operatorname{Im}\tau > 0$. By the ellipticity of A we can take the contour to lie in $\operatorname{Im}\tau > \epsilon|\xi|$ uniformly in x, t.

Now the estimate for the lower order terms is easy. Notice that applying $(\partial/\partial\xi)^\alpha(\partial/\partial x)^\beta$ produces a similar expression where the order of E is $\leq rm - k - j - 2 - |\alpha|$. The term $e^{it\tau}$ is dominated by $e^{-\epsilon t|\xi|}$ as τ varies over the contour, while the remainder of the integral is homogeneous in ξ of degree $\leq -k - 1 - |\alpha|$. Thus the expression is dominated by $c|\xi|^{-k-1-|\alpha|}e^{-\epsilon t|\xi|}$ where c is independent of x and t because of compactness. Lemma 2(b) gives the desired estimates for these terms (recall $L^p \subseteq \Lambda(0; p, \infty)$).

Next consider the highest order terms for $k \geq 1$. By paragraph (C) of §2

they vanish to order k at $t = 0$. Since the expression is clearly C^∞ in x, ξ, t away from $\xi = 0$ the vanishing to order k persists after x and ξ differentiation. For the sake of clarity let us consider the case $\alpha = 0$, $\beta = 0$ (the others are analogous). We then make a change of variable $\tau \to \tau|\xi|$ to obtain

$$q(x, t, \xi) = \frac{\psi_i(x, t)}{2\pi i} |\xi|^{-k} \sum_{j=0}^{k-1} \frac{e^{it|\xi|\tau}(i\tau)^j \sigma_{m-k-j-1}(T_{kj})(x, \xi')}{\sigma_m(A)(x, t, \xi', \tau)} \, d\tau$$

where $\xi' = \xi/|\xi|$. Let us write

$$g(s, t) = \sum_{j=0}^{m-1} \int \frac{e^{is\tau}(i\tau)^j \sigma_{m-k-j-1}(T_{kj})(x, \xi')}{\sigma_m(A)(x, t, \xi', \tau)} \, d\tau.$$

We know $g(0, 0) = 0$, and it is clear that g is C^∞ in s and t, so we have $|g(s, t)| \le c(s + t)$. We also have $|g(s, t)| \le ce^{-\epsilon s}$, and by compactness we may take the constant c uniformly for all x and ξ'. Thus we have $|g(t|\xi|, t)| \le c' t|\xi| e^{-\epsilon t|\xi|}$ for $|\xi| \ge 1$ so $q(x, t, \xi)$ satisfies the hypotheses of Lemma 2 (a).

Finally consider the highest order terms for $k = 0$. By paragraph (C) of §2 we know that (ignoring the ψ_i term) they tend to a constant as $t \to 0$. Thus applying $(\partial/\partial\xi)^\alpha(\partial/\partial x)^\beta$ for $\alpha \ne 0$ results in terms vanishing as $t \to 0$ and we may apply the argument above for $k \ge 1$. The estimate $|(\partial/\partial x)^\beta q(x, t, \xi)| \le Ae^{-\epsilon t|\xi|}$ is established by an argument identical to the one given above for the lower order terms, and so Lemma 2(c) applies. Q.E.D.

4. **Green's theorem and proof of Theorem 2.** The proof of Theorem 2 is based on a duality argument via Green's theorem. Let $\lambda(\alpha; p, q)$ denote the closure of $\mathfrak{D}$ in $\Lambda(\alpha; p, q)$.

Lemma 3. *$\Lambda(\alpha; p, q)$ is the dual space of $\lambda(-\alpha; p', q')$ for all real α, $1 \le p, q \le \infty$.*

We omit the proof which is a simple exercise using the definition of Λ given (cf. Taibleson [12, II]). We note the following characterizations of $\lambda(\alpha; p, q)$:

(i) if $p, q < \infty$ then $\lambda(\alpha; p, q) = \Lambda(\alpha; p, q)$;

(ii) if the underlying space is Euclidean space and $p = \infty$, $q < \infty$, $\lambda(\alpha; p, q)$ consists of those distributions in $\Lambda(\alpha; p, q)$ for which $\mathcal{F}^{-1}(\tau(|\xi|)f(\xi))$ and $\mathcal{F}^{-1}(\sigma(s|\xi|)f(\xi))$ vanish at infinity;

(iii) if the underlying space is Euclidean space and $q = \infty$, then $\lambda(\alpha; p, q)$ consists of those distributions in $\Lambda(\alpha; p, q)$ satisfying $\lim_{s\to 0}\|\mathcal{F}^{-1}(\sigma(s|\xi|)|\xi|^\alpha \hat{f}(\xi))\|_p = 0$ and if $p = \infty$ also the conditions of (ii);

(iv) if the underlying space is a compact manifold then $\lambda(\alpha; \infty, q) = \Lambda(\alpha; \infty, q)$ for $q < \infty$ and $\lambda(\alpha; p, \infty)$ consists of all distributions in $\Lambda(\alpha; p, \infty)$ which locally satisfy the condition of (iii).

Again we omit the details which are fairly routine using the definitions and

the fact that the Fourier transform of a distribution with compact support is a continuous function.

Proof of Theorem 2. Since we know that the boundary values exist as weak limits, the fact that $f_0 \in L^p$ (or $\mathfrak{M}$ if $p = 1$) is obvious, and also that $\|f_0\|_p \leq C$. Thus it suffices to show $f_k \in \Lambda(-k; p, \infty)$ for $0 \leq k \leq m-1$ (higher values of k may be obtained by replacing A by A^N). To do this we introduce a smooth measure on Ω which agrees with $dxdt$ near the boundary, and we denote the inner product with respect to this measure by (u, v). Let A^* be the adjoint of A with respect to this inner product. It is clear that A^* is an operator of the same form as A. Let

$$F = \left(u(x, 0), \frac{\partial}{\partial t} u(x, 0), \cdots, \left(\frac{\partial}{\partial t}\right)^{m-1} u(x, 0)\right)$$

and

$$G = \left(\left(\frac{\partial}{\partial t}\right)^{m-1} v(x, 0), \left(\frac{\partial}{\partial t}\right)^{m-2} v(x, 0), \cdots, v(x, 0)\right).$$

Then Green's Theorem (see Seeley [8]) states that for $v \in C^\infty(\bar{\Omega})$ we have $(Au, v) = (u, A^*v) - \langle \mathfrak{A} F, G\rangle$ where $\langle\ \rangle$ denotes the inner product of vector-valued functions on Γ and $\mathfrak{A}$ is an $m \times m$ matrix of pseudodifferential operators on Γ. Furthermore we know that $\mathfrak{A}$ is lower triangular, its entries $\mathfrak{A}_{ij}$ have order $\leq j - i$ and the diagonal entries $\mathfrak{A}_{ii}$ have for highest order term the coefficient of $(\partial/\partial t)^m$ in A (with $t = 0$). This means that $\mathfrak{A}$ is an elliptic system of order zero. Using the fact that singular integrals preserve the $\Lambda(\alpha; p, q)$ classes on compact manifolds (this is proved by Taibleson [12, II] for the torus and is obvious from our definition) we conclude that in order to show $\|F: \Pi_{k=0}^{m-1} \Lambda(-k; p, \infty)\|$ is finite it suffices to show $\|\mathfrak{A} F: \Pi_{k=0}^{m-1} \Lambda(-k; p, \infty)\|$ is finite. In view of Green's Theorem and Lemma 3 this is accomplished by

Lemma 4. *Given $g_0, \cdots, g_{m-1} \in \mathfrak{D}(\Gamma)$ there exists $v \in C^\infty(\bar{\Omega})$ such that $(\partial/\partial t)^k v(x, 0) = g_{m-1-k}(x)$ and*

$$\int_0^1 \|A^* v(\cdot, t)\|_{p'}\, dt \leq c \sum_{k=0}^{m-1} \|g_k : \Lambda(k; p', 1)\|.$$

Proof. It suffices to construct v for $g_0, \cdots, g_{n-1}$ supported in a coordinate neighborhood of Γ. In that case we use the explicit construction

$$v(x, t) = \psi(x, t) \sum_{k=0}^{m-1} \mathfrak{F}^{-1}\left(\frac{t^k}{k!} \sum_{j=0}^{m-k-1} \frac{(t|\xi|)^j}{j!} e^{-t|\xi|} \hat{g}_{m-k-1}(\xi)\right)$$

where $\psi \equiv 1$ on the support of the g's but has compact support.

It is clear that v has the desired boundary values. Since A^* is a differential

operator in t we know that A^*v is made up of terms of the form $\int e^{ix\cdot\xi}\hat{g}_{m-k-1}(\xi)q(x, t, \xi)\,d\xi$ where $q(x, t, \xi) = \psi(x, t)E(x, \xi)t^r e^{-t|\xi|}$ and $E(x, \xi)$ is a symbol of order $\leq r + m - k$.

Now we proceed as in the proof of Theorem 1. We write

$$\hat{g}_{m-k-1}(\xi) = \tau(|\xi|)\hat{h}(\xi) + \int_0^1 |\xi|^{1-m-k}\sigma^2(s|\xi|)\hat{h}(\xi)\,\frac{ds}{s}$$

for $h \in \Lambda(0; p', 1)$. The term involving $\tau(|\xi|)\hat{h}(\xi)$ is easily handled. For the remaining term we have

$$\left\|\iint e^{ix\cdot\xi}q(x, t, \xi)|\xi|^{1-m-k}\sigma^2(s|\xi|)\hat{h}(\xi)d\xi\,\frac{ds}{s}\right\|_{p'}$$
$$\leq \int_0^1 R(s, t)\|\mathcal{F}^{-1}(\sigma(s|\xi|)\hat{h}(\xi))\|_{p'}\,\frac{ds}{s}$$

where $R(s, t)$ is the norm of the operator

$$\int e^{ix\cdot\xi}\,q(x, t, \xi)|\xi|^{1-m-k}\sigma(s|\xi|)\hat{f}(\xi)\,d\xi$$

acting on $L^{p'}$. Now we apply Lemma 1 as in the proof of Lemma 2 to show that $R(s, t)$ is homogeneous of degree -1 and $R(s, 1) \leq cs^{-r-1}e^{-1/s}$ Since

$$\int_0^1 \|\mathcal{F}^{-1}(\sigma(s|\xi|)\hat{h}(\xi)\|_{p'}\,\frac{ds}{s} \leq \|h\colon \Lambda(0; p', 1)\|$$

we have the desired estimate by virtue of the fact that $\int_0^\infty R(s, 1)\,ds/s < \infty$.

5. **Maximal inequalities and Fatou's Theorem.** Let $d(x, y)$ be a smooth metric on Γ. By a path in Ω approaching a point $(y, 0) \in \Gamma$ nontangentially we mean a curve lying in the region $\{(x, t)\colon d(x, y) \leq ct\}$ for some $c > 0$. Theorem 3 follows by standard functional analysis arguments from the following maximal inequality:

Lemma 5. *Under the hypotheses of Theorem* 3 *we have* $\sup\{|u(x, t)|\colon d(x, y) \leq ct\} \in L^p(\Gamma)$ *for any* $c > 0$ *if* $p > 1$ *and weak-*L^1 *if* $p = 1$.

Proof of Lemma 5. It suffices to establish the estimate for solutions of the form (*). We will do this by following the outline of the proof of Theorem 1 with a few refinements. Indeed we showed that $u(x, t)$ in local coordinates could be expressed as a sum of terms of the form

$$\int e^{ix\cdot\xi}\,\theta(\xi)(\phi_i f_0)\hat{}(\xi)q(x, t, \xi)d\xi$$

plus an innocuous remainder where $q(x, t, \xi)$ satisfies the hypotheses of Lemma 2(a), (b) or (c). We now use

Lemma 6. *Let* $q(x, t, \xi)$ *satisfy the hypotheses of Lemma* 2(a) *or* (c) *with* α *allowed to extend to* $|\alpha| \leq n + 1$. *Let* $r(x, t, \xi) = |\xi|^k q(x, t, \xi)$ (*in case* (c) $k = 0$). *Then*

$$\int e^{ix\cdot\xi} r(x, t, \xi)\theta(\xi)\hat{f}(\xi)\,d\xi = \int R(t, x, y)f(x-y)\,dy$$

where $|R(t, x, y)| \le ct^{-n}(1+|y/t|)^{-n-a}$ *for any* a *in* $0 < a < 1$.

Lemma 7. *Let* $q(x, t, \xi)$ *satisfy*

$$\left|\left(\frac{\partial}{\partial\xi}\right)^{\alpha}\left(\frac{\partial}{\partial x}\right)^{\beta} q(x, t, \xi)\right| \le A|\xi|^{-1-k-|\alpha|}e^{-\epsilon t|\xi|}$$

and

$$\left|\frac{\partial}{\partial t}\left(\frac{\partial}{\partial\xi}\right)^{\alpha}\left(\frac{\partial}{\partial x}\right)^{\beta} q(x, t, \xi)\right| \le A|\xi|^{-k-|\alpha|}e^{-\epsilon t|\xi|}$$

for some $k \ge 0$, $\epsilon > 0$, *and all* $|\alpha| \le n+1$, $|\beta| \le n$ *and* $|\xi| \ge 1$. *Then we have* $q(x, t, \xi) = q(x, 0, \xi)e^{-t|\xi|} + |\xi|^{-1}q'(x, t, \xi)$ *where* q' *satisfies the hypotheses of Lemma* 6.

Assuming Lemmas 6 and 7 let us complete the proof of Lemma 5. First we note that the terms that satisfy the hypotheses of Lemma 2(a) and (c) also satisfy the hypotheses of Lemma 6, while those that satisfy the hypotheses of Lemma 2(b) are covered by Lemma 7.

Thus the terms we are considering are of the form $\int e^{ix\cdot\xi} r(x, t, \xi)\theta(\xi)\hat{g}(\xi)d\xi$ where $g \in L^p$ or $g \in \mathfrak{M}$ and $r(x, t, \xi)$ satisfies the conclusion of Lemma 6. But the estimate of Lemma 6 implies

$$\sup_{\{d(x,y)\le ct\}} \left|\int e^{ix\cdot\xi} r(x, t, \xi)\theta(\xi)\hat{g}(\xi)\,d\xi\right| \le Mg(y)$$

where Mg is the Hardy-Littlewood maximal function (see Stein [10, Chapter III]). Since the maximal function is bounded on L^p for $p > 1$ and from $\mathfrak{M}$ to weak-L^1 we obtain the desired result. Q.E.D.

Proof of Lemma 6. We define $R(t, x, y) = \int e^{iy\cdot\xi}\theta(\xi)r(x, t, \xi)\,d\xi$, and verify immediately that

$$\int e^{ix\cdot\xi} r(x, t, \xi)\theta(\xi)\hat{f}(\xi)\,d\xi = \int R(t, x, y)f(x-y)\,dy.$$

Now by the change of variable $\xi \to \xi/t$ we see that

$$R(t, x, y) = t^{-n}\int e^{i(y/t)\cdot\xi}\theta(\xi/t)r(x, t, \xi/t)\,d\xi.$$

Let $h(x, \xi) = \theta(\xi/t)r(x, t, \xi/t)$. Then the estimate we need is $|\hat{h}(x, z)| \le C(1+|z|)^{-n-a}$ independent of t, where the Fourier transform is taken in the ξ variable.

Now the hypotheses of the lemma imply the estimate $|(\partial/\partial\xi)^{\alpha}(\partial/\partial x)^{\beta}h(x, \xi)| \le c|\xi|^{1-|\alpha|}e^{-\epsilon|\xi|}$ independent of t. In order to avoid the use of fractional derivatives we establish the estimate $|\hat{h}(x, z)^k| \le c(1+|z|)^{-k(n+a)}$ where k and a are chosen so that $k(n+a)$ is an integer. Now $\hat{h}(x, z)^k$ is the Fourier transform of $h^{(k)}(x, \xi)$, the convolution product of h with itself k times (in the ξ variable). The estimates for h easily imply

$$\left|\left(\frac{\partial}{\partial\xi}\right)^{\alpha}\left(\frac{\partial}{\partial x}\right)^{\beta} h^{(k)}(x,\ \xi)\right| \leq c|\xi|^{(n+1)k-n-|\alpha|}\, e^{-\epsilon|\xi|}$$

provided $|\alpha| \leq (n+1)k-1$ and $|\beta| \leq n$. Applying Sobolev's inequality we have

$$(1+|z|)^{(n+1)k-1}|\hat{h}(x,\ z)|^{k}$$
$$\leq c \sum_{|\beta|\leq n} \sum_{|\alpha|\leq(n+1)k-1} \iint \left|\left(\frac{\partial}{\partial\xi}\right)^{\alpha}\left(\frac{\partial}{\partial x}\right)^{\beta} h^{(k)}(x,\ \xi)\right| dx\,d\xi$$

which yields the desired estimate with $a = 1 - 1/k$. Q.E.D.

Proof of Lemma 7. From the hypotheses we have the estimate ($|\xi| \geq 1$)

$$\left|\left(\frac{\partial}{\partial\xi}\right)^{\alpha}\left(\frac{\partial}{\partial x}\right)^{\beta}(q(x,\ t,\ \xi) - q(x,\ 0,\ \xi)\, e^{-t|\xi|})\right| \leq A|\xi|^{-1-k-|\alpha|}\, e^{-\epsilon t|\xi|}.$$

On the other hand we claim

$$\left|\left(\frac{\partial}{\partial\xi}\right)^{\alpha}\left(\frac{\partial}{\partial x}\right)^{\beta}(q(x,\ t,\ \xi) - q(x,\ 0,\ \xi)\, e^{-t|\xi|})\right| \leq At|\xi|^{-k-|\alpha|}.$$

Indeed we have

$$q(x,\ t,\ \xi) - q(x,\ 0,\ \xi)e^{-t|\xi|} = q(x,\ t,\ \xi) - q(x,\ 0,\ \xi) + q(x,\ 0,\ \xi)(1 - e^{-t|\xi|}).$$

The estimate for the second term is easy. For the first term we use the mean value theorem:

$$\left|\left(\frac{\partial}{\partial\xi}\right)^{\alpha}\left(\frac{\partial}{\partial x}\right)^{\beta}(q(x,\ t,\ \xi) - q(x,\ 0,\xi))\right|$$
$$\leq t \sup_{0<s<t}\left|\frac{\partial}{\partial s}\left(\frac{\partial}{\partial\xi}\right)^{\alpha}\left(\frac{\partial}{\partial x}\right)^{\beta} q(x,\ s,\ \xi)\right| \leq At|\xi|^{-k-|\alpha|}.$$

Now for $|\xi| \geq 1$ we either have $t|\xi| \geq 1$ in which case

$$|\xi|^{-1-k-|\alpha|}\, e^{-\epsilon t|\xi|} \leq t|\xi|^{-k-|\alpha|}\, e^{-\epsilon t|\xi|}$$

or else $t|\xi| < 1$ in which case $t|\xi|^{-k-|\alpha|} \leq e^{\epsilon} t|\xi|^{-k-|\alpha|} e^{-\epsilon t|\xi|}$. Thus

$$\left|\left(\frac{\partial}{\partial\xi}\right)^{\alpha}\left(\frac{\partial}{\partial x}\right)^{\beta}(q(x,\ t,\ \xi) - q(x,\ 0,\ \xi)\, e^{-t|\xi|})\right| \leq At|\xi|^{-k-|\alpha|}\, e^{-\epsilon t|\xi|}$$

which gives the desired conclusion. Q.E.D.

Proof of Theorem 3. First let $p > 1$. We claim that for every $\epsilon > 0$ we can write $u(x,\ t) = u_{\epsilon}(x,\ t) + v_{\epsilon}(x,\ t)$ where $u_{\epsilon} \in C^{\infty}(\bar{\Omega})$ and $\|\sup\{|v_{\epsilon}(x,\ t)| : d(x,\ y) \leq ct\}\|_{q} \leq \epsilon$. Indeed as in Lemma 5 we need only consider the case when u is a Poisson integral given by (*). We then approximate f_0 by C^{∞} functions and take u_{ϵ} to be the Poisson integral of the approximating functions.

The existence of a.e. nontangential limits follows from the decomposition

$u = u_\epsilon + v_\epsilon$ by routine arguments. Let us say $u(x, t) \to g(x)$ in this sense. We want to show $g = f_0$. Now by the dominated convergence theorem $u(x, t) \to g(x)$ in L^p. But we also know that $u(x, t) \to f_0(x)$ in the distribution sense, so $f_0 = g$.

Finally, consider the case $p = 1$.

We handle the absolutely continuous part of f_0 as before. For the singular part we use a splitting of the form $g_\epsilon + h_\epsilon$ where $\|h_\epsilon\| \le \epsilon$ and g_ϵ has support in a closed set F_ϵ of measure zero. It is easy to see from the estimate of Lemma 6 that the "Poisson integral" of g_ϵ has nontangential limit zero on the complement of F_ϵ. Using the maximal estimate for h_ϵ a routine argument shows that u has nontangential limits almost everywhere equal to the absolutely continuous part of f_0.

See [4], [5], and [13] for related results.

6. Generalizations.

1. Fatou's Theorem extends immediately to solutions of elliptic equations on noncompact manifolds with boundary, since both hypotheses and conclusions localize. It does not apply, however, to equations which degenerate on the boundary.

2. If the operator A does not have a well-posed Dirichlet problem it is still possible to obtain a version of Theorem 1 as follows.

Let D denote the space of m-tuples of distributions on Γ, to be regarded as Cauchy data for A. Seeley [8] shows that there exists an $m \times m$ matrix of pseudo-differential operators P^+ which is a projection onto the Cauchy data of solutions of $Au = 0$ (where u is the restriction to Ω of a distribution on M).

Corollary 1. *Let $f = (f_0, \cdots, f_{m-1}) \in D$. Then f is the Cauchy data for a solution of $Au = 0$ satisfying $\|u(\cdot, t)\|_p \le c$ if and only if $f_0 \in L^p$ (or $\mathfrak{M}$ if $p = 1$), $f_k \in \Lambda(-k; p, \infty)$ for $k = 1, \cdots, m-1$ and $P^+ f = f$.*

Proof. The only if part is clear from Theorem 2 and the above. Conversely, given that $P^+ f = f$ we know there exists a solution of $Au = 0$ with Cauchy data f. Now u also satisfies $A^*Au = 0$, and A^*A has a well-posed Dirichlet problem. Furthermore, the Dirichlet data for A^*A coincides with the Cauchy data for A. Thus by Theorem 1 we have $\|u(\cdot, t)\|_p \le c$. Q.E.D.

3. If we replace the condition $\|u(\cdot, t)\|_p \le c$ by the condition $\int_0^1 \|u(\cdot, t)\|_p^q \, dt \le c$ for $1 \le q < \infty$ we may prove variants of Theorems 1 and 2 where the conditions on the boundary values becomes $f_k \in \Lambda(-k - 1/q; p, q)$ for all k. The proofs are quite similar so we leave the details to the reader.

In the case $1 < p, q < \infty$ the result may be obtained more simply. Let $L^{p,q}(\Gamma \times (0, 1))$ denote the space of functions satisfying

$$\int_0^1 \|u(\cdot, t)\|_p^q \, dt < \infty$$

(so $L^{p,p}$ is the usual L^p space). Let $L_\alpha^{p,q}$ denote the image of $L^{p,q}$ under an

invertible elliptic pseudodifferential operator of order $-\alpha$ (so $L_\alpha^{p,p}$ is the usual L_α^p space). The space $L^{p,q}$ is known to be preserved by singular integral operators [2], so $L_\alpha^{p,q}$ is well defined. The condition $Au = 0$ and $u \in L_\alpha^{p,q}$ is then equivalent to $f_k \in \Lambda(\alpha - k - 1/q; p, q)$. This may be proved in exactly the same manner as Seeley [8] proves the case $p = q$. The one additional fact we need is the following generalization of the restriction theorem of Stein [11].

Theorem. *If $u(x, t) \in L_\alpha^{p,q}$ for $\alpha > k + 1/q$ then $(\partial/\partial t)^k u(x, 0) \in \Lambda(\alpha - k - 1/q; p, q)$. Conversely every element of $\Lambda(\alpha - k - 1/q; p, q)$ arises in this fashion.*

We omit the proof, which is a trivial modification of Stein's argument.

4. Theorems 1,2,3 may be extended to a class of semi-elliptic equations, including parabolic equations. Suppose $\Omega = (0, T) \times X$ so $\Gamma = \{0\} \times X \cup \{T\} \times X$. Let $A = \sum_{j=0}^m A_j(t)(\partial/\partial t)^j$ where $A_j(t)$ is a pseudodifferential operator on X of order $\mu(m - j)$ for some fixed $\mu > 0$. Let A be semi-elliptic:

$$\left| \sum_{j=0}^m (i\tau)^j \sigma_{\mu(m-j)}(A_j)(x, \xi, t) \right| \geq \epsilon > 0 \quad \text{for all } (x, t) \in \bar{\Omega}$$

and $|\xi|^2 + \tau^2 = 1$. Finally assume A has a well-posed Dirichlet problem, meaning

$$\sum_{j=0}^m (i\tau)^j \sigma_{\mu(m-j)}(A_j)(x, \xi, 0) = 0$$

has exactly m_1 roots in $\operatorname{Im} \tau > 0$ and

$$\sum_{j=0}^m (i\tau)^j \sigma_{\mu(m-j)}(A_j)(x, \xi, T) = 0$$

has exactly m_2 roots in $\operatorname{Im} \tau < 0$ for all x and ξ (obviously $m_1 + m_2 = m$). The Dirichlet problem is to solve $Au = 0$ specifying $(\partial/\partial t)^k u(x, 0)$ for $k = 0, \cdots, m_1 - 1$ and $(\partial/\partial t)^k u(x, T)$ for $k = 0, \cdots, m_2 - 1$.

In this context Theorems 1 and 2 hold with k replaced by μk. The modifications in the proof are slight. The generalizations of Seeley's results to this context are in Polking [7]. In the estimates of Lemma 2 we must replace $|\xi|$ by $|\xi|^\mu$. We leave the details to the interested reader. For the analogue of Theorem 3 we must replace nontangential limits by limits along paths lying in regions of the form $|d(x, y)|^\mu \leq ct$. Also, in Lemma 6 we must replace $0 < a < 1$ by $0 < a < \mu$.

5. We have based our presentation on a C^∞ theory of pseudodifferential operators so our results apply only to differential operators with C^∞ coefficients. On the other hand, our estimates only involve a small number of derivatives, so the results should hold with much weaker regularity assumptions on the coeffi-

cients of A. The problem of deciding how much smoothness we require seems unduly complicated. Perhaps an approach based on a more precise theory of pseudodifferential operators (such as Calderón [3]) would illuminate this question.

6. The hypothesis that all $m_j \leq 1$ in Theorem 3 is essential as the following example shows: Let

$$u(x, t) = \sum_{k=1}^{\infty} 2^k t \exp(-2^k t) \exp(2^k i x).$$

Then $\Delta^2 u = 0$ in $t > 0$ and u is bounded there (either by Theorem 2 or direct estimate). But $\lim_{t\to 0} u(x, t)$ fails to exist for almost every x. Indeed

$$u(x, 2^{-j}) = \sum_{m=-10}^{10} 2^m \exp(-2^m) \exp(2^m i(2^j x))$$

$$+ \text{ a uniformly small remainder.}$$

But for almost every x the sequence $2^j x \bmod 2\pi$ is everywhere dense (this follows from the ergodicity of $x \to 2x \bmod 1$) so that a different limit is obtained along subsequences for which $2^{j-10}x \equiv \epsilon \bmod 2\pi$ and $2^j x \equiv \pi + \epsilon \bmod 2\pi$.

The existence of similar examples was communicated to the author by S. Spanne prior to the discovery of the above example.

I am grateful to Professor H. Furstenberg for help with this example.

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