

LOCALLY B^* -EQUIVALENT ALGEBRAS. II(1)

BY

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ABSTRACT. Let A be a locally B^* -equivalent Banach $*$ -algebra. Then A possesses a unique norm $|\cdot|$ with the property that $|a^*a| = |a|^2$ for all $a \in A$. Let B be the B^* -algebra which is the completion of A in the norm $|\cdot|$. In this paper it is shown that there exists a closed B^* -equivalent $*$ -ideal of A which contains the maximal GCR ideal of B . In particular, when B is a GCR algebra, then $A = B$.

1. Introduction. Let A be a Banach $*$ -algebra. A is B^* -equivalent if A is $*$ -isomorphic to some B^* -algebra B . Then by [6, Theorem (4.1.20)] and the Closed Graph Theorem, the $*$ -isomorphism between A and B is a homeomorphism. Thus A and B have the same algebraic and topological structure (but not necessarily the same geometric structure). We call A locally B^* -equivalent if for every selfadjoint element $a \in A$, the closed $*$ -subalgebra of A generated by a is B^* -equivalent. Whether or not every locally B^* -equivalent Banach $*$ -algebra is B^* -equivalent is not known. Some partial results on this question have been obtained in [2].

Now assume that A is a locally B^* -equivalent Banach $*$ -algebra. Then A possesses a unique norm $|\cdot|$ with the property that $|a^*a| = |a|^2$ for all $a \in A$. Let B be the B^* -algebra which is the completion of A in the norm $|\cdot|$. In this paper, which is a sequel to [2], we prove that there exists a closed B^* -equivalent $*$ -ideal of A which contains the maximal GCR ideal of B (Theorem 2). Thus if B is a GCR algebra, then $A = B$.

H. Behncke has considered some questions concerning locally B^* -equivalent algebras in [3]. However, there is almost no overlap of his paper with either this paper or [2].

2. Preliminary remarks. Throughout this paper A is a locally B^* -equivalent Banach $*$ -algebra with norm $\|\cdot\|$. By [2, Proposition 2.1], there is a unique norm $|\cdot|$ on A with the B^* -property (that is, $|a^*a| = |a|^2$ for all $a \in A$). Throughout this paper B is the completion of A with respect to the norm $|\cdot|$. Then A is B^* -equivalent if and only if $A = B$.

By [6, Corollary (4.1.16)], there exists a constant $M > 0$ such that $|a| \leq M\|a\|$ for all $a \in A$. Since A is locally B^* -equivalent, the norms $\|\cdot\|$ and $|\cdot|$

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are equivalent on the closed $*$ -subalgebra of A generated by a selfadjoint element of A . Therefore, if $a = a^* \in A$, there exists $K > 0$ such that $\|a^n\| \leq K|a^n|$ for all $n \geq 1$. Then

$$\|a^n\|^{1/n} \leq K^{1/n}|a^n|^{1/n} = K^{1/n}|a|.$$

It follows that the spectral radius of a selfadjoint element a is $|a|$.

The norms $\|\cdot\|$ and $|\cdot|$ are equivalent on every maximal commutative $*$ -subalgebra of A by [2, Proposition 2.2]. This implies a fact that we use repeatedly: if $\{a_n\}$ is a sequence of selfadjoint elements of A and $a_n a_m = 0$ whenever $n \neq m$, then there exists $K > 0$ such that $\|a_n\| \leq K|a_n|$ for all $n \geq 1$ (proof—the sequence $\{a_n\}$ is contained in some maximal commutative $*$ -subalgebra of A).

3. Preliminary results. In this section we prove some basic lemmas. If I is a $|\cdot|$ -closed ideal of A , we let $|\cdot|_q$ denote the usual quotient norm on A/I ,

$$|a + I|_q = \inf\{|a - b| \mid b \in I\}.$$

Lemma 1. *Let I be a closed ideal of B . Then*

- (1) $A \cap I$ is $|\cdot|$ -dense in I , and
- (2) if $\|\cdot\|$ and $|\cdot|$ are equivalent on $A \cap I$, then $I \subset A$.

Proof. Since I is a closed ideal in the B^* -algebra B , I is a $*$ -ideal (see [6, Theorem (4.9.2)]). Then $A \cap I$ is a $|\cdot|$ -closed (and therefore also $\|\cdot\|$ -closed) $*$ -ideal of A . Let J be the $|\cdot|$ -closure of $A \cap I$ in B . Then $A \cap J = A \cap I$. The quotient algebra $A/A \cap I$ is naturally embedded in the quotient algebras B/I and B/J by the maps $a + A \cap I \rightarrow a + I$ and $a + A \cap I \rightarrow a + J$, respectively. Define two norms on $A/A \cap I$ by

$$|a + A \cap I|_1 = |a + I|_q \quad \text{and} \quad |a + A \cap I|_2 = |a + J|_q.$$

Since $|a + I|_q$ and $|a + J|_q$ are B^* -norms, then $|\cdot|_1$ and $|\cdot|_2$ are norms on $A/A \cap I$ that have the B^* -property. By [2, Proposition 2.3], $A/A \cap I$ is locally B^* -equivalent. Then by [2, Proposition 2.1], $|a + A \cap I|_1 = |a + A \cap I|_2$ for all $a \in A$. It follows that $|a + I|_q = |a + J|_q$ for all $a \in A$. Now assume $t \in I$, and choose a sequence $\{t_n\}$ in A such that $|t_n - t| \rightarrow 0$. Then $|(t_n - t) + I|_q \rightarrow 0$, and since $t \in I$, it follows that $|t_n + I|_q \rightarrow 0$. Therefore $|t_n + J|_q \rightarrow 0$. This means that there exists $\{s_n\} \subset J$ such that $|t_n - s_n| \rightarrow 0$. Then $|t - s_n| \rightarrow 0$, so that $t \in J$. This proves (1).

Now assume that $\|\cdot\|$ and $|\cdot|$ are equivalent on $A \cap I$. If $t \in I$, by (1) we can choose $\{t_n\} \subset A \cap I$ such that $|t_n - t| \rightarrow 0$. Then the sequence $\{t_n\}$ is $|\cdot|$ -Cauchy, and therefore $\|\cdot\|$ -Cauchy. Therefore there exists $s \in A$ such that $\|t_n - s\| \rightarrow 0$. Then $|t_n - s| \rightarrow 0$ so that $s = t$.

If $a = a^* \in A$, let $C(a)$ be the closed $*$ -subalgebra of A generated by a . Let $\Phi_{C(a)}$ be the carrier space of $C(a)$, and if $b \in C(a)$, let \hat{b} denote the Gelfand transform of b . Since A is locally B^* -equivalent, $b \rightarrow \hat{b}$ is a bicontinuous $*$ -isomorphism of $C(a)$ onto the algebra of all complex continuous functions that vanish at infinity on $\Phi_{C(a)}$. Now let D be a $*$ -subalgebra of A . By [2, Lemma 2.5], if there exists $K > 0$ such that $K|a| \geq \|a\|$ for all $a = a^* \in D$, then the $\|\cdot\|$ -closure of D is B^* -equivalent. We use these results in the proof of the next lemma.

Lemma 2. *Let I be a $*$ -subalgebra of A with the properties:*

- (1) *if $t^* = t \in I$, then $C(t) \subset I$,*
- (2) *if $t \in I$, then $(1 - t^*)I(1 - t) \neq \{0\}$.*

Then there exists $b \in I$ such that the $\|\cdot\|$ -closure of $(1 - b^)I(1 - b)$ is B^* -equivalent.*

Proof. Suppose that there is no $b \in I$ such that the $\|\cdot\|$ -closure of $(1 - b^*)I(1 - b)$ is B^* -equivalent. Then the $\|\cdot\|$ -closure of I is not B^* -equivalent, so that there exists $b_1^* = b_1 \in I$ such that $|b_1| < \frac{1}{2}\|b_1\|$. Choose $g_1^* = g_1 \in C(b_1)$ such that \hat{g}_1 has compact support in $\Phi_{C(b_1)}$ and $|g_1| < \frac{1}{2}\|g_1\|$. Then choose $k_1^* = k_1 \in C(b_1)$ such that $\hat{k}_1 \equiv 1$ on the support of \hat{g}_1 . Note that $g_1(1 - k_1) = (1 - k_1)g_1 = 0$. Let $I_1 = (1 - k_1)I(1 - k_1)$. By assumption the $\|\cdot\|$ -closure of I_1 is not B^* -equivalent. Then choose $b_2^* = b_2 \in I_1$ such that $|b_2| < \frac{1}{4}\|b_2\|$. Choose as before $g_2^* = g_2$ and $k_2^* = k_2$ in $C(b_2)$ such that $|g_2| < \frac{1}{4}\|g_2\|$ and $g_2(1 - k_2) = (1 - k_2)g_2 = 0$. Since $g_1b_2 = b_2g_1 = 0$, then $g_1k_2 = k_2g_1 = 0$ and $g_1g_2 = g_2g_1 = 0$. Let $I_2 = (1 - k_2)(1 - k_1)I(1 - k_1)(1 - k_2)$. If $u = (1 - k_2)v(1 - k_1)(1 - k_2)$ for some $v \in I$, then $g_1u = ug_1 = 0$ and $g_2u = ug_2 = 0$. As before choose $g_3 = g_3^*$ and $k_3 = k_3^*$ in I such that $|g_3| < (\frac{1}{2})^3\|g_3\|$, $(1 - k_3)g_3 = g_3(1 - k_3) = 0$, and $g_i g_j = 0$ if $i \neq j$. Continuing in this fashion we can construct a sequence of self-adjoint elements $\{g_k\} \subset I$ such that $g_i g_j = 0$ whenever $i \neq j$ and $|g_k| < (\frac{1}{2})^k\|g_k\|$. This contradicts the fact that A is locally B^* -equivalent.

Lemma 3. *Let $\{I_\lambda\}$, $\lambda \in \Lambda$, be a collection of closed $*$ -ideals of A , and let $I = \bigcup_{\lambda \in \Lambda} I_\lambda$. If I_λ is B^* -equivalent for each $\lambda \in \Lambda$, and I is a $*$ -subalgebra of A , then the $\|\cdot\|$ -closure of I is B^* -equivalent.*

Proof. Assume that there exists an element $t \in I$ such that $(1 - t^*)I(1 - t) = \{0\}$. Then for any $a \in I$, $(1 - t^*)a^*a(1 - t) = 0$, so that $a(1 - t) = 0$. There exists $\mu \in \Lambda$ such that $t \in I_\mu$. Then $a = at \in I_\mu$. Then $I = I_\mu$, which proves the lemma in this case.

Now assume that $(1 - t^*)I(1 - t) \neq \{0\}$ whenever $t \in I$. By Lemma 2 there exists $b \in I$ such that the $\|\cdot\|$ -closure of $(1 - b^*)I(1 - b)$ is B^* -equivalent. Assume that $\{b_n\}$ is a $\|\cdot\|$ -Cauchy sequence in the $\|\cdot\|$ -closure of I . For each n ,

$$b_n = [(1 - b^*)b_n(1 - b)] + [(1 - b^*)b_n b + b^* b_n].$$

There exists $\mu \in \Lambda$ such that $b \in I_\mu$. Then $\{(1 - b^*)b_n b + b^* b_n\}$ is a $|\cdot|$ -Cauchy sequence in I_μ , and $\{(1 - b^*)b_n(1 - b)\}$ is a $|\cdot|$ -Cauchy sequence in the $\|\cdot\|$ -closure of $(1 - b^*)I(1 - b)$. Since both these $|\cdot|$ -Cauchy sequences converge to some element of A , then $\{b_n\}$ converges to some element of A . It follows that the $\|\cdot\|$ -closure of I is B^* -equivalent.

4. **Locally B^* -equivalent algebras of vector valued functions.** Let Ω be a compact Hausdorff space and denote by $C_R(\Omega)$ the algebra of all continuous real-valued functions on Ω . For each $\omega \in \Omega$, let B_ω be a B^* -algebra with norm $|\cdot|_\omega$. Define $C(\Omega, B_\omega)$ to be the algebra of all functions f defined on Ω such that $f(\omega) \in B_\omega$ for all $\omega \in \Omega$ and such that the function $\omega \rightarrow |f(\omega)|_\omega$ is in $C_R(\Omega)$. The algebra $C(\Omega, B_\omega)$ is a B^* -algebra in the norm

$$|f| = \sup_{\omega \in \Omega} |f(\omega)|_\omega.$$

Throughout this section we assume that B is a closed $*$ -subalgebra of $C(\Omega, B_\omega)$ which is closed under multiplication by functions in $C_R(\Omega)$.

Theorem 1. Assume that A is a dense $*$ -subalgebra of B such that A is a locally B^* -equivalent Banach algebra. Furthermore, assume that if $\omega_1, \dots, \omega_n$ are any n distinct points in Ω and b_1, \dots, b_n are such that $b_k \in B_{\omega_k}$, $1 \leq k \leq n$, then there exists $g \in A$ such that $g(\omega_k) = b_k$, $1 \leq k \leq n$.

Then $A = B$.

We proceed with the proof of Theorem 1 by establishing several lemmas. If V is a subset of Ω , let $J_V = \{f \in B \mid f(\omega) = 0 \text{ whenever } \omega \notin V\}$. We say that $\|\cdot\|$ and $|\cdot|$ are equivalent on A at $\omega \in \Omega$ if there exists an open neighborhood V of ω such that $\|\cdot\|$ and $|\cdot|$ are equivalent on the ideal $A \cap J_V$.

Lemma 4. The norms $\|\cdot\|$ and $|\cdot|$ are equivalent on A at all but at most a finite number of points of Ω .

Proof. Suppose that there is an infinite sequence of distinct points $\{\omega_n\} \subset \Omega$ such that $\|\cdot\|$ and $|\cdot|$ are not equivalent on A at ω_n for all $n \geq 1$. By choosing a suitable subsequence of $\{\omega_n\}$ if necessary, we may assume that there exists a disjoint sequence of open subsets of Ω , $\{U_n\}$, such that $\omega_n \in U_n$ for $n \geq 1$. Then for each $n \geq 1$ there exists $g_n^* = g_n \in J_{U_n} \cap A$ such that $n|g_n| < \|g_n\|$. Since $g_n g_m = g_m g_n = 0$ when $n \neq m$, this contradicts the assumption that A is locally B^* -equivalent.

If $|\cdot|$ and $\|\cdot\|$ are equivalent on A at every point in Ω , let $I = B$. In the case where there exist points in Ω at which $|\cdot|$ and $\|\cdot\|$ are not equivalent on A , then by Lemma 4 there must be only a finite number of such points. Let

$\{\omega_1, \dots, \omega_n\}$ be this collection of points. Define I in this case to be the set of all $f \in B$ such that $f(\omega_k) = 0, 1 \leq k \leq n$.

Lemma 5. $I \subset A$.

Proof. For each $\omega \in \Omega$ such that $\omega \neq \omega_k, 1 \leq k \leq n$, there exists an open neighborhood U of ω in Ω such that $|\cdot|$ and $\|\cdot\|$ are equivalent on $A \cap J_U$. It follows from Lemma 1 that $J_U \subset A$. Let K be a closed subset of Ω disjoint from $\{\omega_1, \dots, \omega_n\}$ (when $I = B$, let $K = \Omega$). Choose $\{U_1, \dots, U_m\}$ a finite open cover of K such that $|\cdot|$ and $\|\cdot\|$ are equivalent on $A \cap J_{U_k}, 1 \leq k \leq m$. Then, as noted above, $J_{U_k} \subset A, 1 \leq k \leq m$. Let $U_0 = \Omega \setminus K$. Choose $f_k \in C_R(\Omega)$ such that $f_k(\omega) = 0$ whenever $\omega \notin U_k, 0 \leq k \leq m$, and such that $1 = \sum_{k=0}^m f_k$. If $g \in J_K$, then $f_0 g = 0$. Then $g = \sum_{k=0}^m f_k g = \sum_{k=1}^m f_k g$. By hypothesis B is closed under multiplication by functions in $C_R(\Omega)$. Therefore $f_k g \in J_{U_k} \subset A$ for $1 \leq k \leq m$. (In the case where $I = B$, the proof is now complete.) There exist constants M_k such that $\|b\| \leq M_k |b|$ whenever $b \in J_{U_k}, 1 \leq k \leq m$. Then

$$\|g\| \leq \sum_{k=1}^m \|f_k g\| \leq \sum_{k=1}^m M_k |f_k g| \leq \left(\sum_{k=1}^m M_k \|f_k\|_{\infty} \right) |g|.$$

Therefore $|\cdot|$ and $\|\cdot\|$ are equivalent on the closed $*$ -ideal $J_K \subset A$. Let $I_0 = \bigcup J_K$ where the union is over all closed subsets K of Ω that are disjoint from $\{\omega_1, \dots, \omega_n\}$. By Lemma 3, $|\cdot|$ and $\|\cdot\|$ are equivalent on the $\|\cdot\|$ -closure of I_0 . But I_0 is $|\cdot|$ -dense in I . Therefore the same proof as given for part (2) of Lemma 1 shows that $I \subset A$.

Now we complete the proof of Theorem 1. Assume that $f \in B$. Let $\{\omega_1, \dots, \omega_n\}$ be as above. By hypothesis there exists $g \in A$ such that $g(\omega_k) = f(\omega_k), 1 \leq k \leq n$. Then $f - g \in I$. But by Lemma 5, $I \subset A$, and therefore $f \in A$.

5. The main result. Assume that B is a CCR (liminaire) algebra with Hausdorff structure space Ω . To each point $P \in \Omega$ associate the B^* -algebra $B_P = B/P$. Each of the algebras B_P is $*$ -isomorphic to the algebra of all compact operators on some Hilbert space. By [5, Lemma 4.3] Ω is locally compact in the Jacobson topology. If Ω is not compact, then compactify it by adding a point at ∞, P_∞ , and associate with P_∞ the algebra $B_{P_\infty} = \{0\}$. Each $b \in B$ determines a function $b(P)$ defined on Ω by

$$b(P) = b + P \in B_P.$$

Let $|b(P)|_P$ be the quotient norm of $b + P$ in B/P ($b \in B, P \in \Omega$). Then $|b| = \sup\{|b(P)|_P \mid P \in \Omega\}$ for all $b \in B$. Furthermore, by [5, Theorem 4.1], $P \rightarrow |b(P)|_P$ is a continuous function on Ω . If P_1, \dots, P_n are n distinct points of Ω , and b_1, \dots, b_n are n elements of B , then there exists $b \in B$ such that

$b(P_k) = b + P_k = b_k + P_k$ for $1 \leq k \leq n$. This follows from [4, Proposition 4.2] for the case $n = 2$, but the result is easily extended to all positive integers n . Then [5, Theorem 3.3] applies to B , so that B is closed under multiplication by functions in $C_R(\Omega)$.

Now assume that A is a locally B^* -equivalent Banach algebra which is a dense $*$ -subalgebra of B . Let P_1, \dots, P_n be n distinct points of Ω . Let $B' = B_{P_1} \oplus \dots \oplus B_{P_n}$ (the direct sum of the quotient algebras B_{P_k} , $1 \leq k \leq n$). B' is a dual B^* -algebra. Let $I = A \cap P_1 \cap \dots \cap P_n$. The map $\pi: A/I \rightarrow B'$, defined for $a \in A$ by $\pi(a + I) = (a + P_1) \oplus \dots \oplus (a + P_n)$, is a $*$ -isomorphism of the locally B^* -equivalent Banach algebra A/I into B' . Let b_1, \dots, b_n be any n elements of B . Then as noted above, there exists $b \in B$ such that $b(P_k) = b + P_k = b_k + P_k$ for $1 \leq k \leq n$. Choose a sequence $\{a_j\} \subset A$ such that $|a_j - b| \rightarrow 0$. Then $\pi(a_j + I) \rightarrow (b_1 + P_1) \oplus \dots \oplus (b_n + P_n)$ in B' . This proves that $A' = \pi(A/I)$ is a dense locally B^* -equivalent Banach $*$ -subalgebra of the dual B^* -algebra B' . If $t = t^* \in A'$, then t has the same spectrum in A' and B' . Then by [1, Theorem 3.4], the spectrum of t in A' is at most countable. Therefore by [2, Theorem 4.1], $A' = B'$. It follows that if b_1, \dots, b_n are n elements of B , there exists $a \in A$ such that $a(P_k) = b_k + P_k$ for $1 \leq k \leq n$. Then A satisfies all the hypotheses of Theorem 1, so that $A = B$. We state this result as a proposition.

Proposition 6. *If B is a CCR algebra with Hausdorff structure space, then $A = B$.*

Lemma 7. *Let K be a closed B^* -equivalent $*$ -ideal of A . Then there exists a closed B^* -equivalent $*$ -ideal I of A such that $K \subset I$ and A/I has no nonzero closed B^* -equivalent $*$ -ideals.*

Proof. Let \mathcal{I} be the collection of all closed B^* -equivalent $*$ -ideals of A that contain K . Then \mathcal{I} is nonempty, and \mathcal{I} is partially ordered by inclusion. Assume that $\{I_\lambda\}$, $\lambda \in \Lambda$, is a chain in \mathcal{I} . Then by Lemma 3 the $\|\cdot\|$ -closure of $\bigcup_{\lambda \in \Lambda} I_\lambda$ is in \mathcal{I} . By Zorn's Lemma \mathcal{I} contains a maximal element I . Assume that J is a closed B^* -equivalent $*$ -ideal of A/I . Let π be the natural quotient map of A onto A/I . By [2, Lemma 2.4], $\pi^{-1}(J)$ is a closed B^* -equivalent $*$ -ideal of A that contains I . Then by the maximality of I , $\pi^{-1}(J) = I$. This proves that J is the zero ideal in A/I .

Proposition 8. *If B is a GCR algebra, then $A = B$.*

Proof. By Lemma 7 A contains a closed B^* -equivalent $*$ -ideal I such that A/I has no nonzero closed B^* -equivalent $*$ -ideals. Then I is a closed $*$ -ideal of B , and A/I is a dense locally B^* -equivalent $*$ -subalgebra of B/I . By [4, Proposition 4.3.5], B/I is a GCR algebra. Let A_1 and B_1 denote the quotient

algebras A/I and B/I , respectively. Assume $B_1 \neq \{0\}$ (that is, $B \neq I$). Then B_1 contains a nonzero closed $*$ -ideal J which is a CCR algebra (see [4, p. 87]). Then J_1 , the first nonzero ideal in the composition series described in [5, Theorem 6.2], is a closed $*$ -ideal of B_1 which is a CCR algebra with Hausdorff structure space. Now $A_1 \cap J_1$ is dense in J_1 by Lemma 1. Then by Proposition 6, $J_1 \subset A_1$. This contradiction implies that $B = I$. Therefore $B = A$.

Now we prove the main result.

Theorem 2. *There exists a closed $*$ -ideal I of A with the properties:*

- (1) I is B^* -equivalent.
- (2) I contains the largest GCR ideal of B , and
- (3) A/I has no nonzero closed B^* -equivalent ideals.

Proof. Let J be the largest GCR ideal of B (see [5, Theorem 7.5]). By Lemma 1 $A \cap J$ is a dense $*$ -subalgebra of J . Furthermore $A \cap J$ is locally B^* -equivalent. Then by Proposition 8, $J = A \cap J \subset A$. Then Lemma 7 implies the existence of a closed $*$ -ideal I of A with properties (1)–(3).

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