KRULL DIMENSION IN POWER SERIES RINGS

BY

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ABSTRACT. Let $R$ denote a commutative ring with identity. If there exists a chain $P_0 \subset P_1 \subset \cdots \subset P_n$ of $n + 1$ prime ideals of $R$, where $P_n \neq R$, but no such chain of $n + 2$ prime ideals, then we say that $R$ has dimension $n$. The power series ring $R[[X]]$ may have infinite dimension even though $R$ has finite dimension.

1. Introduction. We shall write $\dim R = n$ to denote that $R$ has dimension $n$. Seidenberg, in [6] and [7], has investigated the theory of dimension in rings of polynomials. In particular, he has shown in [6] that if $\dim R = n$, then $n + 1 \leq \dim R[X] \leq 2n + 1$, where $X$ is an indeterminate over $R$. One might now ask whether it is also true that $n + 1 \leq \dim R[[X]] \leq 2n + 1$. It is easy to show that $n + 1 \leq \dim R[[X]]$ when $\dim R = n$. In [3] Fields has considered the theory of dimension in power series rings over valuation rings. Using results obtained by Fields, Arnold and Brewer have noted in [1] that $\dim V[[X]] \geq 4$ for any rank one nondiscrete valuation ring $V$. Thus, if $\dim R = n$, then $2n + 1$ is not, in general, an upper bound for $\dim R[[X]]$. In this paper we show that we may have $\dim R[[X]] = \infty$ even though $R$ has finite dimension. Our main result is Theorem 1, which gives sufficient conditions on a ring $R$ in order that $\dim R[[X]] = \infty$. In fact, the conditions given insure the existence of an infinite ascending chain of prime ideals in $R[[X]]$.

Throughout this paper, $R$ denotes a commutative ring with identity, $\omega$ is the set of natural numbers, and $\omega_0$ is the set of nonnegative integers. If $f(X) = \sum_{i=0}^{\infty} a_i X^i \in R[[X]]$, then we denote by $A_f$ the ideal of $R$ generated by the coefficients of $f(X)$. For an ideal $A$ of $R$, we let $A[[X]] = \{f(X) = \sum_{i=0}^{\infty} a_i X^i \mid a_i \in A \}$ for each $i \in \omega_0$ and we define $AR[[X]]$ to be the ideal of $R[[X]]$ which is generated by $A$. Thus, $AR[[X]] = \{f(X) \mid A_f \subseteq B$ for some finitely generated ideal $B$ of $R$, with $B \subseteq A\}$. We shall say that the ideal $A$ is an ideal of strong finite type (or an SFT-ideal) provided there is a finitely generated ideal $B \subseteq A$ and $k \in \omega$ such that $a^k \in B$ for each $a \in A$. If each ideal of $R$ is an SFT-ideal, then we say that $R$ satisfies the SFT-property. Throughout, our notation and terminology are essentially that of [4].

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Main Theorem. Let $R$ be a ring which does not satisfy the SFT-property. If $M$ is an ideal of $R$ which is not an SFT-ideal, then we may choose a sequence {$a_i^{k+1}$} of elements of $M$ so that $a_k^{k+1} \notin (a_0, \ldots, a_k)$ for each $k \in \omega$. Set $A_k = (a_0, \ldots, a_k)$ and let $A = \bigcup_{k=0}^{\infty} A_k$. For each $m \in \omega$, we now choose a sequence {$a_m,i^m$} of elements of $A$ as follows. For $m = 1$, we take $a_1,i = a_i$ for each $i \in \omega$. Having defined the sequence {$a_m,i^m$} for $1 \leq m < n$, we define the sequence {$a_n,i^n$} by taking $a_n,i = a_{n-1,i+1}$ for each $i \in \omega$. For each $n \in \omega$ we set $f(n) = \sum_{i=0}^{\infty} a_n,i X^i$.

Definition 1. Suppose that $g(X) \in R[[X]]$, $g(X) = \sum_{i=0}^{\infty} b_i X^i$, and let $n, m, \mu, r$ be integers such that $m \geq n \geq 1$, and $r > 0$. We shall say that the tuple $(g, m, \mu, r)$ has property $(n)$ if for $i > r$ there exists an integer $t_i$ such that the following hold, where we assume that $a_m,i = a_n,k_i = a_{n+1,s_i}$:

(i) $b_{t_i} = a_m,i + \alpha$ for some $\alpha \in A_{s_i-1}$.
(ii) $t_i \leq \mu k_i$.
(iii) $k_i \in A_{s_i-1}$ for $0 < t_i \leq k_i$.

For $n \in \omega$, we set $S_n = \{g(X) \in R[[X]] | (g, m, \mu, r)$ has property $(n)$ for some $m, \mu \in \omega$ and $r \in \omega\}$. $S_n$ is nonempty since $(1, 1, n, 1, 0)$ satisfies property $(n)$.

Lemma 1. If $n, n_1 \in \omega$ are such that $n \geq n_1$, then $S_n \subseteq S_{n_1}$.

Proof. Suppose that $g(X) \in S_n$ and that $(g, m, \mu, r)$ has property $(n)$. We wish to see that $(g, m, \mu, r)$ also has property $(n_1)$. But properties (i) and (iii) of Definition 1 already hold since they are independent of the choice of $n$. To see that (ii) holds, suppose that $i > r$ and that $a_m,i = a_n,k_i = a_{n+1,s_i}$. Then $k_i \leq v_i$, and hence $t_i \leq \mu k_i \leq \mu v_i$. It follows that $g(X) \in S_{n_1}$.

Lemma 2. For each $n \in \omega$, $S_n$ is a multiplicatively closed subset of $R[[X]]$.

Proof. Let $g(X) \in R[[X]]$, $g(X) = \sum_{i=0}^{\infty} b_i X^i$. We first show that if $(g, m, \mu, r)$ has property $(n)$ and if $m_1 \geq m$, then $(g, m_1, \mu, r)$ also has property $(n)$. Thus, suppose that $i \geq r$ and that $a_{m_1,i} = a_m,i = a_{m+1,s_i}$. Since $j_i \geq i \geq r$, there exists an integer $t_{j_i}$ such that:

(i) $b_{t_{j_i}} = a^\mu_{m,i,j_i} + \alpha$ for some $\alpha \in A_{s_i-1}$.
(ii) $t_{j_i} \leq \mu k_i$.
(iii) $k_i \in A_{s_i-1}$ for $0 \leq \lambda < t_{j_i}$.

Taking $r_i = t_{j_i}$ and using the fact that $a_{m_1,i} = a_m,i$, we see that $r_i$ satisfies properties (i), (ii) and (iii) of Definition 1 so $(g, m_1, \mu, r)$ has property $(n)$.

Now let $g(X), h(X) \in S_n$, where $g(X) = \sum_{i=0}^{\infty} b_i X^i$ and $h(X) = \sum_{i=0}^{\infty} c_i X^i$, and suppose that $(g, m_1, \mu_1, r_1)$ and $(b, m_2, \mu_2, r_2)$ satisfy property $(n)$. By the preceding remarks, we may assume that $m_1 = m_2$ and, clearly, we may assume that $r_1 = r_2$. Set $m = m_1 = m_2$ and $r = r_1 = r_2$. We wish to show that $(g, h, m, \mu_1 + \mu_2, r)$
has property (n). Suppose that \( i \geq r \) and that \( a_{m, i} = a_{n, k} = a_1, s_i \). By assumption there exist integers \( t_i \) and \( r_i \) such that \( b_{t_i} = a_{m, i} + \alpha \) and \( c_{r_i} = a_{m, i} + \beta \) for some \( \alpha, \beta \in A_{S_{i-1}} \). Moreover, \( b_{\lambda}, c_\delta \in A_{S_{i-1}} \) for \( 0 \leq \lambda < t_i \) and \( 0 \leq \delta < r_i \). If \( g(X)b(X) = \sum_{j=0}^{\infty} \xi_j X^j \), then
\[
\xi_{t_i+r_i} = b_{t_i}c_{r_i} + \sum_{\lambda+\delta=t_i+r_i; 0 \leq \lambda < t_i; 0 \leq \delta < r_i} b_{\lambda}c_\delta.
\]
But if \( \lambda \neq t_i \) and \( \delta \neq r_i \), then either \( \lambda < t_i \) or \( \delta < r_i \). Consequently, either \( b_{\lambda} \in A_{S_{i-1}} \) or \( c_\delta \in A_{S_{i-1}} \). Since \( b_{t_i}c_{r_i} = a_{m, i} + \alpha a_{m, i}^2 + \alpha a_{m, i} + \beta a_{m, i} + \alpha \beta \), it follows that
\[
\xi_{t_i+r_i} = a_{m, i}^2 + \gamma \text{ for some } \gamma 
\]
and some \( \gamma \in A_{S_{i-1}} \). By assumption, we have \( t_i < \mu_1 k_i \) and \( r_i < \mu_2 k_i \). Therefore, \( t_i + r_i \leq (\mu_1 + \mu_2) k_i \). Finally, if \( 0 \leq \lambda < t_i + r_i \), then \( \xi_\lambda = \sum_{j=0}^{\infty} b_{j}c_{\lambda-j} \in A_{S_{i-1}} \) since either \( j < t_i \) or \( \lambda - j < r_i \).

**Lemma 3.** Let \( n, \nu \in \omega \) be such that \( n > \nu \). If \( g(X) \in S_n \), then \( g(X) + b(X)f(\nu)(X) \in S_n \) for arbitrary \( b(X) \in R[[X]] \).

**Proof.** Suppose that \( g(X) = \sum_{i=0}^{\infty} b_i X^i \) and that \( (g, m, \mu, r) \) has property (n). Let \( \eta = \min \{ j \in \omega \mid a_{m, j} = a_{n, k} = \mu \} \) and set \( r_1 = \max \{ r, \eta \} \). If \( q(X) = g(X) + b(X)f(\nu)(X) = \sum_{i=0}^{\infty} \xi_i X^i \), then we wish to show that \( (q, m, \mu, r_1) \) satisfies property (n). Thus, suppose that \( i \geq r_1 \) and that \( a_{m, i} = a_{n, k} = a_1, s_i \). By assumption, there exists an integer \( t_i \) such that \( b_{t_i} = a_{m, i}^2 + \alpha \) for some \( \alpha \in A_{S_{i-1}} \) and such that \( t_i < \mu k_i \leq k_i^2 \). Since \( \lambda \geq k_i^2 + 1 \), it follows that \( a_{\nu, j} \in A_{S_{i-1}} \) for \( 0 \leq j \leq t_i \). Consequently, if \( b(X) = \sum_{j=0}^{\infty} c_j X^j \) and \( b(X)f(\nu)(X) = \sum_{j=0}^{\infty} g_j X^j \), then \( \gamma_{t_i} = \sum_{j=0}^{t_i} a_{\nu, j} c_{t_i-j} \in A_{S_{i-1}} \). Therefore, \( \xi_{t_i} = b_{t_i} + \gamma_{t_i} = a_{m, i}^2 + \alpha + \gamma_{t_i} \) and (i) of Definition 1 is satisfied. We already have that \( t_i \leq \mu k_i \) so (ii) is also satisfied. To see that (iii) holds, suppose that \( 0 \leq \delta < t_i \). By assumption, we have that \( b_\delta \in A_{S_{i-1}} \). Also, \( g_\delta = \sum_{j=0}^{\infty} a_{\nu, j} c_{t_i-j} \in A_{S_{i-1}} \), since \( j \leq \delta < t_i \leq k_i^2 \) implies that \( a_{\nu, j} \in A_{S_{i-1}} \). Consequently, \( \xi_\delta = b_\delta + g_\delta \in A_{S_{i-1}} \) and our proof is complete.

We now state our main result.

**Theorem 1.** Let \( R \) be a commutative ring with identity. The following conditions are equivalent and imply that \( R[[X]] \) has infinite dimension.

1. \( R \) does not satisfy the SFT-property.
2. There exists an ideal \( A \) of \( R \) such that \( A[[X]] \notin \sqrt{AR[[X]]} \).
3. There exists a prime ideal \( P \) of \( R \) such that \( P[[X]] \notin \sqrt{PR[[X]]} \).

**Proof.** Assume that (1) holds. We shall first prove that \( \dim R[[X]] = \infty \). Let the ideal \( A \) be as previously defined. We wish to see that \( AR[[X]] \cap S_1 = \emptyset \). Thus, let \( g(X) \in AR[[X]] \cap S_1 \). Then \( A_g \subseteq C \) for some finitely generated ideal \( C \) of \( R \), where \( C \subseteq A \). Consequently, there exists \( k \in \omega_0 \) such that \( A_k \subseteq A_k \). Suppose that \( (g, m, \mu, r) \) has property (1) and that \( r \) has been chosen so that if \( i \geq r \) and \( a_{m, i} = a_1, s_i \), then \( s_i > \max \{ \mu, k \} \). If \( t_i \) is such that \( b_{t_i} = a_{m, i}^2 + \alpha \) for some \( \alpha \in A_{S_{i-1}} \), then we have \( a_{m, i}^\mu + \alpha \in A_k \subseteq A_{S_{i-1}} \).
Therefore, \( a_{m,i}^{s_i} \in A_{s_i-1} \), a contradiction since \( a_{m,i}^{s_i} \notin A_{s_i-1} \) and \( s_i > \mu \).

(Since \( f(1) \in S_1 \), it follows that \( f(1) \in A[[X]] - \sqrt{AR[[X]]} \). Thus we see that \( 1 \) implies \( 2 \).) But \( S_1 \cap AR[[X]] = \emptyset \) implies the existence of a prime ideal \( P \) of \( R[[X]] \) such that \( AR[[X]] \subseteq P \) and \( P \cap S_1 = \emptyset \). Suppose there exists a chain \( P_1 \subseteq \cdots \subseteq P_n \) of prime ideals of \( R[[X]] \) such that \( P \cap S_1 = \emptyset \), and let \( C_n = P_n / (f(n)(X)) \). If \( g(X) \in S_{n+1} \), then by Lemma 3, \( g(X) + b(X)f(n)(X) \in S_{n+1} \subseteq S_n \) for arbitrary \( b(X) \in R[[X]] \). It follows that \( g(X) + b(X)f(n)(X) \notin P_n \) and hence that \( g(X) \notin C_n \). Thus, \( C_n \cap S_{n+1} = \emptyset \) and there exists a prime ideal \( P_{n+1} \) such that \( P_n \subseteq P_{n+1} \subseteq S_{n+1} = \emptyset \). We see by induction that \( \dim R[[X]] = \infty \).

To see that \( 2 \) implies \( 3 \), we note that if \( A[[X]] \nsubseteq \sqrt{AR[[X]]} \), then there exists a prime ideal \( Q \) of \( R[[X]] \) such that \( AR[[X]] \subseteq Q \) but \( A[[X]] \nsubseteq Q \). If \( P = Q \cap R \), then \( P \supseteq A \) and hence \( P[[X]] \supseteq A[[X]] \). Therefore, \( Q \supseteq PR[[X]] \) but \( Q \nsubseteq P[[X]] \). It follows that \( P[[X]] \nsubseteq \sqrt{PR[[X]]} \). In order to show that \( 3 \) implies \( 1 \), we require the following lemma.

**Lemma 4.** Let \( A \) be an ideal of \( R \) and suppose that there exists \( k \in \omega \) such that \( a^k = 0 \) for each \( a \in A \). If \( f(X) \in A[[X]] \), then \( f(X) \) is nilpotent.

**Proof.** We first prove the existence of an integer \( m \) such that \( m\xi = 0 \) for all \( \xi \in A^m \). Suppose we have integers \( \mu, \nu_1, \ldots, \nu_t \) such that \( \mu\nu_1 \cdots \nu_t = 0 \) for all \( a_1, \ldots, a_t \in A \) (certainly this condition is satisfied if \( \mu = t = 1 \) and \( \nu_1 = k \)) and suppose that \( \nu_i \geq 2 \) for some \( i, 1 \leq i \leq t \). For convenience, we suppose that \( \nu_1 \geq 2 \). Now let \( b_0, b_1, \ldots, b_t \in A \). By assumption, we have that

\[
0 = \mu (b_0 + b_1)^{\nu_1} b_2^{\nu_2} \cdots b_t^{\nu_t} = \mu b_0^{\nu_1 - 2} (b_0 + b_1)^{\nu_1 - 2} b_2^{\nu_2} \cdots b_t^{\nu_t} = \sum_{j=0}^{\nu_1} \xi_j f^j,
\]

where \( \xi_j = \mu (\nu_1) b_0^{2\nu_1 - j - 2} b_1^{\nu_1 - 2} \cdots b_t^{\nu_t} \). If \( 0 \leq j \leq \nu_1 - 2 \), then \( 2\nu_1 - j - 2 > \nu_1 \), so that \( \xi_j = 0 \). Also, \( \xi_{\nu_1} = b_0^{\nu_1 - 2} (\mu b_1^{\nu_1} b_2^{\nu_2} \cdots b_t^{\nu_t}) = 0 \). It follows that \( 0 = \xi_{\nu_1 - 1} = \mu \nu_1 b_1^{\nu_1 - 1} b_2^{\nu_2 - 1} \cdots b_t^{\nu_t} \). By a finite number of repetitions of this procedure, we may find integers \( \mu, \nu_1, \ldots, \nu_t \) such that \( \mu a_1 \cdots a_t = 0 \) for all \( a_1, \ldots, a_t \in A \). If we set \( m = \mu \), then \( mA^m = (0) \). Now let \( f(X) \in A[[X]] \), \( f(X) = \sum_{i=0}^{\infty} a_i X^i \). Following a proof given by Fields [2, Theorem 1] we suppose that \( m \) is a prime integer. Then \( (f(X))p^k = (\sum_{i=0}^{\infty} a_i X^i)^{p^k} = 0 \). If \( m \) is not prime and \( m = p_1^{e_1} \cdots p_t^{e_t} \) is a prime factorization for \( m \), then let \( \phi_j : R[[X]] \rightarrow (R/p_j A^{p_j})[[X]] \) be the canonical homomorphism for \( 1 \leq j \leq t \). By the previous case for \( m \) a prime, we have \( 0 = [\phi_j(f(X))]p_j^{e_j} \), that is \( (f(X))p_j^{e_j} \in p_j A^{p_j}[[X]] \). If \( n = (p_1^{e_1} + \cdots + p_t^{e_t})m \), then

\[
(f(X))^n - \left( (f(X))p_1^{e_1} \cdots (f(X))p_t^{e_t} \right)^m \in [(p_1 A^{p_1})^e[[X]]] \cdots (p_t A^{p_t})^{e_d[[X]]} m \subseteq mA^m[[X]] = (0).
\]
To complete the proof of Theorem 1, suppose that $B$ is an ideal of $R$ which is an SFT-ideal. By definition, there exists $k \in \omega$ and a finitely generated ideal $C \subseteq B$ such that $b^k \in C$ for all $b \in B$. Setting $\overline{R} = R/C$ and $\overline{B} = B/C$, it follows from Lemma 4 that $f(X)$ is nilpotent for each $f(X) \in \overline{B}[[X]]$. Therefore, if $g(X) \in B[[X]]$, then $g(X) = \sqrt{C[[X]]} = \sqrt{CR[[X]]} \subseteq \sqrt{BR[[X]]}$. Consequently, if $P$ is a prime ideal of $R$ such that $P[[X]] \neq \sqrt{PR[[X]]}$, then $P$ is not an SFT-ideal. This proves that (3) implies (1) and the theorem follows.

If $\dim R = n$, then it is natural to ask whether the conditions given in Theorem 1 are necessary in order that $\dim R[[X]] = \infty$. Another interesting question which arises is whether the following conditions are equivalent:

1. $\dim R[[X]] \neq n + 1$.
2. $\dim R[[X]] = \infty$.

We show that both these questions can be answered affirmatively if $\dim R = 0$.

**Theorem 2.** Let $R$ be a commutative ring with identity and suppose that $\dim R = 0$. Then the following statements are equivalent:

1. $\dim R[[X]] \neq 1$.
2. $\dim R[[X]] = \infty$.
3. $R$ contains a maximal ideal $M$ such that $M[[X]] \neq \sqrt{MR[[X]]}$.

**Proof.** We have already seen that (3) implies (2) and clearly, (2) implies (1). Suppose that (1) holds and let $Q_0 \subseteq Q_1 \subseteq Q_2 \subseteq R[[X]]$ be a chain of prime ideals of $R[[X]]$. If $M = Q_0 \cap R$, then $M$ is a maximal ideal of $R$ so we have $M = Q_0 \cap R = Q_1 \cap R = Q_2 \cap R$. Now $Q_0 \nsubseteq M[[X]]$ since $R[[X]]/M[[X]] \cong (R/M)[[X]]$ is a rank one discrete valuation ring. But by [1, Proposition 1], either $Q_0 \subseteq M[[X]]$ or $Q_0 \nsubseteq M[[X]]$. Therefore, $MR[[X]] \subseteq Q_0 \subseteq M[[X]]$ and $M[[X]] \neq \sqrt{MR[[X]]}$.

3. **Examples.** We conclude by providing three examples of finite dimensional rings $R$ such that $\dim R[[X]] = \infty$.

**Example 1.** If $V$ is a rank one nondiscrete valuation ring, then $\dim V[[X]] = \infty$. More generally, if $V$ is a valuation ring which contains an idempotent prime ideal $P$, then $P$ is not an SFT-ideal so $\dim V[[X]] = \infty$.

**Example 2.** An integral domain $D$ is said to be almost Dedekind provided $D_M$ is a Noetherian valuation ring for each maximal ideal $M$ of $D$. Let $D$ be any almost Dedekind domain which is not Dedekind [4, p. 586], and let $M$ be a maximal ideal of $D$ which is not finitely generated. It follows from Theorem 29.4 of [4, p. 411] that $M$ is not the radical of a finitely generated ideal. Thus, $M$ is not an SFT-ideal and $\dim D[[X]] = \infty$. More generally, if $R$ is a commutative ring with identity which does not have Noetherian prime spectrum, then $\dim R[[X]] = \infty$.

This is an immediate consequence of Corollary 2.4 of [5] which states that a ring $R$ has Noetherian prime spectrum if and only if each prime ideal of $R$ is the radical of a finitely generated ideal. Example 1 and the following example illustrate...
that we may have $\dim R[[X]] = \infty$ even though $R$ has Noetherian prime spectrum.

Example 3. Let $\{Y_i\}_{i=0}^\infty$ be a collection of indeterminates over $\mathbb{Q}$, the field of rationals, and set $R = \mathbb{Q}[Y_0, Y_1, \ldots]/(Y_0^n, Y_1^n, \ldots)$, where $n$ is a positive integer and $n \geq 2$. We note that $\dim R = 0$ and that $M = (\bar{Y}_0, \bar{Y}_1, \ldots)$ is the unique proper prime ideal of $R$. If $f(X) = \sum_{i=0}^\infty \bar{Y}_i X^i$, then Fields proves in [2] that $f(X)$ is not nilpotent. If $g(X) \in MR[[X]]$, then $g(X) = \sum_{i=0}^t \bar{Y}_i b_i(X)$ for some $t \in \omega$ and $b_i(X) \in R[[X]]$. Since $\bar{Y}_i^n = 0$ for $0 \leq i \leq t$, it follows that $g(X)$ is nilpotent. Consequently, $f(X) \notin \sqrt{MR[[X]]}$ so, by Theorem 1, $\dim R[[X]] = \infty$.

BIBLIOGRAPHY


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