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INDUCED FLOWS

BY

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ABSTRACT. The construction of induced transformations is considered in the setting of topological dynamics. Sufficient conditions are given for induced flows to be topologically weakly mixing, and it is proved that Toeplitz flows and certain Sturmian flows satisfy these conditions and give rise to new and easily constructed classes of flows which have entropy zero and are uniquely ergodic, minimal, and topologically weakly mixing. An example is given of a weakly mixing minimal flow which is not topologically strongly mixing.

1. **Introduction.** S. Kakutani (oral communication) has constructed examples of induced measure-preserving transformations on Lebesgue spaces which are weakly mixing but not strongly mixing. By beginning with a doubly infinite sequence x on the symbols 0 and 1 and doubling the 1's in x, one obtains a new sequence $x^{\sim}$; Kakutani proved that if x is a particular Toeplitz sequence or one of many sequences of Sturmian type, then the orbit closure $X^{\sim}$ of $x^{\sim}$ under the shift σ is minimal and uniquely ergodic and the measure-preserving transformation $\sigma: X^{\sim} \to X^{\sim}$ is weakly mixing but not strongly mixing. Related examples are found in [14, §8]. It was already known to von Neumann [21, §VI] that one could construct a measure-preserving transformation with continuous spectrum from one with pure point spectrum by using induced transformations.

The concept of induced transformation [13] can be carried over from ergodic theory to topological dynamics. By a *flow* we mean a pair $\mathcal{X} = (X, T)$, where X (called the *phase space*) is a compact Hausdorff space and $T: X \to X$ is a homeomorphism. A flow $\mathcal{X} = (X, T)$ is said to be *minimal* if for each $x \in X$ the orbit $\mathcal{O}(x) = \{T^n x: n \in \mathbf{Z}\}$ is dense in X. If $\mathcal{X} = (X, T)$ is a minimal flow and X is the disjoint union of a finite number of closed sets, then both primitive flows $\mathcal{X}^u$ and derivative flows $\mathcal{X}_A$ of $\mathcal{X}$ (both called *induced flows*) can be defined. These induced flows are again minimal, and under certain conditions they are topologically weakly mixing.

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In §§2 and 3 we show how to construct the two types of induced flows and give sufficient conditions for them to be topologically weakly mixing. We also observe that inducing from a given flow preserves unique ergodicity, minimality, expansiveness, and equicontinuity, and allows an easy computation of the topological entropy of the resulting flow. In §4 we define two classes of examples, the Sturmian flows and the Toeplitz flows (these include the examples considered by Kakutani). We show that these flows satisfy the sufficient conditions of §§2 and 3, so the flows induced by them are uniquely ergodic, topologically weakly mixing, and minimal.

In §5 we show that the particular Toeplitz flow considered by Kakutani has a derivative which is topologically weakly mixing but not topologically strongly mixing. We also mention in §6 an application of the foregoing results to a question concerning the irregularities of distribution of the fractional parts of multiples of irrational numbers.

Before proceeding we recall some common definitions. Let $\mathfrak{X} = (X, T)$ and $\mathfrak{Y} = (Y, S)$ be flows. We say ϕ is a *homomorphism from* $\mathfrak{X}$ *to* $\mathfrak{Y}$, written $\phi: \mathfrak{X} \to \mathfrak{Y}$, if ϕ is a continuous map from X to Y such that $S \circ \phi = \phi \circ T$; ϕ is an *isomorphism* if $\phi: X \to Y$ is also one-to-one and onto. Let $\mathbf{K}$ denote the compact group of complex numbers with absolute value one, and if $\xi \in \mathbf{K}$ then define $T_\xi: \mathbf{K} \to \mathbf{K}$ by $T_\xi(\beta) = \xi\beta$ for $\beta \in \mathbf{K}$. Let $\mathfrak{K}_\xi$ denote the flow $(\mathbf{K}, T_\xi)$. A continuous function $f: X \to \mathbf{K}$ is called a *continuous eigenfunction of* $\mathfrak{X}$ *with eigenvalue* ξ if f is a homomorphism from $\mathfrak{X}$ to $\mathfrak{K}_\xi$, i.e. $f(Tx) = \xi f(x)$ for $x \in X$. The flow $\mathfrak{X} \times \mathfrak{Y}$ is defined to have phase space $X \times Y$ and homeomorphism $T \times S$, where $(T \times S)(x, y) = (Tx, Sy)$. $\mathfrak{X}$ is said to be (topologically) *ergodic* if whenever $A \subseteq X$, A is closed, and $T(A) = A$, then either $A = X$ or A is nowhere dense. $\mathfrak{X}$ is (topologically) *weakly mixing* if $\mathfrak{X} \times \mathfrak{X}$ is ergodic. It has been proved [22] that if $\mathfrak{X}$ is minimal then $\mathfrak{X}$ is weakly mixing if and only if it has no nontrivial equicontinuous homomorphic images. Now any nonconstant continuous eigenfunction of $\mathfrak{X}$ provides a nontrivial equicontinuous homomorphic image of $\mathfrak{X}$. Conversely, if $\mathfrak{Y} = (Y, T)$ is a nontrivial equicontinuous homomorphic image of $\mathfrak{X}$, then Y is a compact abelian group [4, Remark 4.6.2], and any nontrivial character of Y will give rise to a nonconstant continuous eigenfunction of $\mathfrak{X}$. Thus $\mathfrak{X}$ is weakly mixing if and only if every continuous eigenfunction is constant (cf. [16, Corollary 2.11]).

Two points $x, y \in X$ are said to be *positively* (or *negatively*) *proximal* in case there is $z \in X$ and a net $\{n_r\}$ of positive (or negative) integers such that $T^{n_r}x \to z$ and $T^{n_r}y \to z$. If x and y are both positively and negatively proximal, then they are said to be *doubly proximal*; if x and y are either positively or negatively proximal, then they are said to be *proximal*. The *ω-limit set of x with* *respect to* T is defined to be the set of limit points of $\{T^n x \mid n > 0\}$. Clearly x and y are positively proximal if and only if $\Delta \cap \Omega \neq \varnothing$, where Δ is the diagonal of $X \times X$ and Ω is the ω-limit set of (x, y) with respect to $T \times T$.

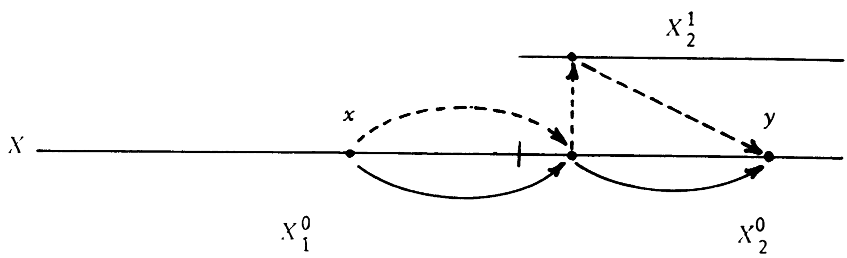
In the hope that the wide range of this paper may make it a suitable introduction to current work in the field, the authors have tried to keep the paper as self-contained as possible. For background information the most basic works are [4], [6], and [9]. We are grateful to Professors H. B. Keynes and N. G. Markley for several helpful conversations.

2. The primitive of a flow. In this section and the following one we deal with the topological analogues of the ergodic-theoretic constructions discussed in [13].

Let $\mathfrak{X} = (X, T)$ be a flow, N a positive integer, and $u: X \to \{1, 2, \cdots, N\}$ a continuous function ($\{1, \cdots, N\}$ has the discrete topology). We will use the function u to define a new flow $\mathfrak{X}^u = (X^u, T^u)$, called *the primitive of* $\mathfrak{X}$ *with respect to* u. For $n = 1, 2, \cdots, N$, let $X_n^0 = u^{-1}\{n\}$. For each $n = 2, \cdots, N$ and $i = 1, 2, \cdots, n - 1$, let X_n^i be a homeomorphic copy of X_n^0, and suppose $\phi_n^i: X_n^i \to X_n^{i+1}$ is a homeomorphism for $n = 2, \cdots, N$ and $i = 0, 1, \cdots, n - 2$. Let ϕ_1^0 be the identity map on X_1^0, and for $n = 2, \cdots, N$ let $\phi_n^{n-1}: X_n^{n-1} \to X_n^0$ be the inverse of the map $\phi_n^{n-2} \circ \cdots \circ \phi_n^1 \circ \phi_n^0$. Let X^u denote the discrete union of the sets X_n^i for $n = 1, 2, \cdots, N$ and $i = 0, 1, \cdots, n - 1$. We define $T^u: X^u \to X^u$ by

$$T^u x = \begin{cases} \phi_n^i x & \text{if } x \in X_n^i \text{ and } i < n - 1, \\ \\ T(\phi_n^{n-1} x) & \text{if } x \in X_n^i \text{ and } i = n - 1. \end{cases}$$

Then T^u is a homeomorphism and clearly $\mathfrak{X}^u$ is minimal if and only if $\mathfrak{X}$ is minimal. The formation of $\mathfrak{X}^u$ in the case $N = 2$ is illustrated by the following diagram, in which the action of T on X is represented by solid arrows and the action of T^u on X^u by dashed arrows.



Note that $y = T^2 x = (T^u)^3 x$.

Theorem 2.1. $\mathfrak{X}^u$ *has a continuous eigenfunction with eigenvalue* ξ *if and only if there is a continuous function* $f: X \to \mathbf{K}$ *such that* $f(Tx)/f(x) = \xi^{u(x)}$ *for all* $x \in X$.

Proof. If g is a continuous eigenfunction of $\mathfrak{X}^u$ with eigenvalue ξ, then let f be the restriction of g to $X \subseteq X^u$. It is easily checked that $f(Tx)/f(x) = \xi^{u(x)}$ for all $x \in X$.

Conversely, given a continuous function $f\colon X \to \mathbf{K}$ with $(f \circ T)/f = \xi^u$, define $g\colon X^u \to \mathbf{K}$ by

$$g(x) = \begin{cases} \xi^i f((\phi_n^{i-1} \circ \cdots \circ \phi_n^1 \circ \phi_n^0)^{-1}x) & \text{if } x \in X_n^i \text{ and } i > 0, \\ f(x) & \text{if } x \in X_n^0 \subseteq X. \end{cases}$$

Then clearly g is continuous and $g(Tx) = \xi g(x)$ for all $x \in X^u$.

We can now state a condition which will ensure topological weak mixing of the primitive of a minimal flow.

Theorem 2.2. *Let $\mathfrak{X} = (X, T)$ be a flow, N a positive integer, and $u\colon X \to \{1, 2, \cdots, N\}$ a continuous function. Suppose there is a pair of doubly proximal points $x, y \in X$ such that $|u(x) - u(y)| = 1$ and $u(T^n x) = u(T^n y)$ for $n \neq 0$. If g is a continuous eigenfunction of $\mathfrak{X}^u$ with eigenvalue ξ, then $\xi = 1$.*

Proof. By Theorem 2.1 it suffices to prove that if $f\colon X \to \mathbf{K}$ is continuous and $f \circ T/f = \xi^u$, then $\xi = 1$. Now for $n > 1$,

$$\frac{f(T^n x)}{f(x)} = \frac{f(T^n x)}{f(T^{n-1}x)} \frac{f(T^{n-1}x)}{f(T^{n-2}x)} \cdots \frac{f(Tx)}{f(x)}$$

$$= \frac{f(T^n y)}{f(T^{n-1}y)} \frac{f(T^{n-1}y)}{f(T^{n-2}y)} \cdots \frac{f(Ty)}{f(y)} \xi^{u(y)-u(x)} = \frac{f(T^n y)}{f(y)} \xi^{u(y)-u(x)}.$$

By letting n tend to infinity along the net of positive integers mentioned in the definition of doubly proximal, we conclude that $f(y) = \xi^{u(y)-u(x)} f(x)$. A similar calculation shows that, for $n < 0$,

$$f(x)/f(T^n x) = f(y)/f(T^n y);$$

then using the fact that x and y are negatively proximal, we conclude that $f(x) = f(y)$. If follows that $1 = \xi^{u(y)-u(x)} = \xi^{\pm 1}$, and hence $\xi = 1$.

Remarks. The rather special hypothesis of this theorem can be weakened in several ways.

2.1. Suppose that there are a pair of doubly proximal points $x, y \in X$ and nonnegative integers r, s such that $u(T^n x) = u(T^n y)$ unless $-r \leq n \leq s$. If $|\sum_{n=-r}^{s} [u(T^n x) - u(T^n y)]| = 1$, then it still follows that $\mathfrak{X}^u$ has no continuous eigenfunctions with eigenvalues other than 1.

2.2. Suppose there are a pair of doubly proximal points $x, y \in X$ and a number M such that $0 < |\sum_{i=0}^{n} [u(T^i x) - u(T^i y)]| < M$ for all n. Then a modification of the proof of Theorem 2.2 shows that if g is a continuous eigenfunction of $\mathfrak{X}^u$

with eigenvalue ξ, then ξ is a kth root of unity for some k with $0 < k < M$.

3. **The derivative of a flow.** As is the case with the induced transformations of ergodic theory, the above construction of the primitive of a flow determined by a function can be reversed.

Let $\mathfrak{X} = (X, T)$ be a flow, and suppose A is an open-closed subset of X such that for each $x \in A$ there are a positive integer n and a negative integer m for which $T^n x \in A$ and $T^m x \in A$ (such a set A will be called *doubly recurrent*). We will define a new flow $\mathfrak{X}_A = (A, T_A)$, called the *derivative of* $\mathfrak{X}$ *with respect to* A, by letting $T_A: A \to A$ be the "first return time" transformation; that is, T_A maps each point x of A to the first point of $Tx, T^2x, \cdots$ which is in A. More precisely, let $A_1^0 = A \cap T^{-1}A$, $A_2^0 = A \cap T^{-2}A - A_1^0, \cdots, A_n^0 = A \cap T^{-n}A - (A_1^0 \cup \cdots \cup A_{n-1}^0), \cdots$. Then $A \subset \bigcup_{n=1}^\infty A_n^0$; since A is compact, there is a smallest N such that $A_n = \emptyset$ for $n > N$; and A is the disjoint union of $A_1^0, A_2^0, \cdots, A_N^0$. Now we define $T_A: A \to A$ by $T_A | A_n^0 = T^n$, $n = 1, 2, \cdots, N$. Again T_A is a homeomorphism, and $\mathfrak{X}_A$ is minimal if and only if $\mathfrak{X}$ is minimal.

Let us define $u_A: A \to \{1, 2, \cdots, N\}$ by $u_A | A_n^0 \equiv n$ for $n = 1, 2, \cdots, N$. Then the primitive $(\mathfrak{X}_A)^{u_A}$ of $\mathfrak{X}_A$ with respect to the function u_A is isomorphic with $\mathfrak{X}$. Similarly, if we start with a flow $\mathfrak{X} = (X, T)$ and a continuous function u from X into a finite set, then, regarding X as a subset of X^u, we see that $(\mathfrak{X}^u)_X$ is isomorphic with $\mathfrak{X}$. For a given flow $\mathfrak{X}$, a primitive or derivative of $\mathfrak{X}$ is called a *flow induced by* $\mathfrak{X}$.

Theorem 3.1. *$\mathfrak{X}_A$ has a continuous eigenfunction with eigenvalue ξ if and only if there are a continuous function $f: X \to \mathbf{K}$ and $\xi_1, \xi_2, \cdots, \xi_N \in \mathbf{K}$ such that $\xi_n^n = \xi$ for $n = 1, 2, \cdots, N$ and $f \circ T/f = \xi_n$ on $A_n^0 \cup TA_n^0 \cup \cdots \cup T^{n-1}A_n^0$, $n = 1, 2, \cdots, N$.*

Proof. Similar to the proof of Theorem 2.1.

Theorem 3.2. *Let $\mathfrak{X} = (X, T)$ be a minimal flow and A an open-closed subset of X (so A is automatically doubly recurrent). Suppose there is a pair of doubly proximal points $x \in A$ and $y \in X - A$ such that, for $n \neq 0$, $T^n x \in A$ if and only if $T^n y \in A$. If g is a continuous eigenfunction of $\mathfrak{X}_A$ with eigenvalue ξ, then $\xi = 1$.*

Proof. Suppose $f: X \to \mathbf{K}$ is continuous and $\xi_1, \xi_2, \cdots, \xi_N \in \mathbf{K}$ are such that $\xi_n^n = \xi$ and $f \circ T/f = \xi_n$ on $A_n^0 \cup TA_n^0 \cup \cdots \cup T^{n-1}A_n^0$ for $n = 1, 2, \cdots, N$; because of Theorem 3.1 it suffices to prove that $\xi = 1$.

Let $A_n^i = T^i A_n^0$ for $i = 1, 2, \cdots, n-1$, $n = 2, 3, \cdots, N$. Suppose $x \in A_m^0$; then we must have $y \in A_{m+k}^k$ for some k with $1 \le k \le N - m$. Because $T^j x \in A$ if and only if $T^j y \in A$ for $j > 0$, for $j > m$ the two points $T^j x$ and $T^j y$ must lie in the same cell $A_{n(j)}^i$ of X. Similarly, for $j < -k$ the points $T^j x$ and $T^j y$ lie in the same cell $A^i_{n(j)}$ of X. Therefore

$$\frac{f(T^j x)}{f(T^{j-1}x)} = \xi_{n(j)} = \frac{f(T^j y)}{f(T^{j-1}y)} \quad \text{for } j > m \text{ or } j < -k.$$

On the other hand,

$$\frac{f(T^m x)}{f(T^{m-1}x)} \cdots \frac{f(Tx)}{f(x)} = \xi^m_m = \xi, \quad \text{but} \quad \frac{f(T^m y)}{f(T^{m-1}y)} \cdots \frac{f(Ty)}{f(y)} = \xi^m_{m+k};$$

and

$$\frac{f(T^{-k+1}x)}{f(T^{-k}x)} \cdots \frac{f(x)}{f(T^{-1}x)} = \xi^k_k = \xi, \quad \text{but} \quad \frac{f(T^{-k+1}y)}{f(T^{-k}y)} \cdots \frac{f(y)}{f(T^{-1}y)} = \xi^k_{m+k}.$$

(The latter statement holds because $T^k y \in A^0_{m+k} \subset A$ and $T^j y \notin A$ for $0 < j < k$, so $T^{-k}x \in A$ and $T^{-j}x \notin A$ for $0 < j < k$; therefore, since $T^k(T^{-k}x) = x \in A$, in fact $T^{-k}x \in A^0_k$. It follows that $T^{-k+1}x \in TA^0_k = A^1_k$, $T^{-k+2}x \in A^2_k, \cdots,$ $T^{-1}x \in A^{k-1}_k$.)

As in the proof of Theorem 2.2 we may compute that, for large enough j,

$$\frac{f(T^j x)}{f(x)} = \frac{\xi}{\xi^m_{m+k}} \frac{f(T^j y)}{f(y)} \quad \text{and} \quad \frac{f(x)}{f(T^{-j}x)} = \frac{\xi}{\xi^k_{m+k}} \frac{f(y)}{f(T^{-j}y)}.$$

Using the fact that x and y are doubly proximal, we conclude that

$$f(y) = \xi f(x)/\xi^m_{m+k} \quad \text{and} \quad f(y) = \xi^k_{m+k} f(x)/\xi.$$

Therefore $\xi^2 = \xi^k_{m+k}\xi^m_{m+k} = \xi^{m+k}_{m+k} = \xi$ and $\xi = 1$.

Remarks. 3.1. If $\mathfrak{X}$ is a minimal flow and f is a continuous eigenfunction of $\mathfrak{X}$ with eigenvalue 1, then f is constant. Thus if $\mathfrak{X}$ is minimal and $(u)(A)$ satisfies the conditions of (Theorem 2.2) (Theorem 3.2), then $(\mathfrak{X}^u)(\mathfrak{X}_A)$ is topologically weakly mixing and minimal. We will consider specific examples in the next section.

3.2. A flow $\mathfrak{X} = (X, T)$ is said to be *uniquely ergodic* if there is a unique Borel probability measure μ on X which is invariant in the sense that $\mu(T^{-1}E) = \mu(E)$ for each Borel subset E of X. $\mathfrak{X}$ is *expansive* if there is an index α of the uniformity of X such that given $x, y \in X$ with $x \neq y$, there is an integer n for which $(T^n x, T^n y) \notin \alpha$. $\mathfrak{X}$ is *equicontinuous* if $\{T^n: n \in \mathbf{Z}\}$ is a uniformly equicontinuous family of maps $X \to X$. Straightforward arguments show that if $\mathfrak{X}$ is (uniquely ergodic) (expansive) (equicontinuous), then so is any flow induced by $\mathfrak{X}$.

3.3. For a given flow $\mathfrak{X}$, let $\mathfrak{M}(\mathfrak{X})$ denote the collection of all T-invariant Borel probability measures on X. It follows from [1] and [7] that, denoting by $h(\mathfrak{X})$ the topological entropy of a flow $\mathfrak{X}$ (see [2] for the definition), $h(\mathfrak{X}) = h(\mathfrak{X}_A)$ $\sup_{\mu \in M(X)} \mu(A)$ for any doubly recurrent open-closed subset A of X. From this it follows that if $h(\mathfrak{X}) = 0$ then $h(\mathfrak{X}^u) = 0$ and $h(\mathfrak{X}_A) = 0$ if $\mu(A) > 0$ for some $\mu \in \mathfrak{M}(\mathfrak{X})$.

4. **Examples.** If we are interested in finding flows which satisfy the hypotheses of Theorems 2.2 and 3.2, then zero-dimensional flows, especially the symbolic flows, are obvious candidates.

Let $S = \{0, 1\}^Z$ be the space of all bilateral sequences $x = \cdots x(-1)\,x(0)\,x(1)\,x(2) \cdots$ on the symbols 0 and 1. When $\{0, 1\}$ is given the discrete topology and S the corresponding product topology, S becomes a compact zero-dimensional metrizable space with compatible metric $d(x, y) = (k+1)^{-1}$, where $k = \inf\{|n|: x(n) \neq y(n)\}$. The *shift transformation* $\sigma: S \to S$ is defined by $(\sigma x)(n) = x(n+1)$. Then σ is a homeomorphism and $\mathcal{S} = (S, \sigma)$ is a flow. If $X \subset S$ is closed and $\sigma X = X$, then $\mathcal{X} = (X, \sigma)$ is also a flow; such flows $\mathcal{X}$ are called *symbolic flows*. For more details concerning symbolic flows, see [9, Chapter 12].

An *n-block*, for some positive integer n, is an element of $\{0, 1\}^n$. If $x \in S$ and B is an n-block, then the phrase B *appears at the kth place in* x means that $x(k)x(k+1) \cdots x(k+n-1) = B$. An n-block B is said to be *the initial n-block* of an element $x \in S$ if B appears at the 0th place in x.

Now let $x \in S$, let $X = \mathcal{O}(x)^-$ be the orbit closure of x under the shift σ, suppose $\mathcal{X} = (X, \sigma)$ is minimal, and let B be a block which appears at some place in x. Let $A = \{y \in X: B$ is an initial block of $y\}$, and define $u: X \to \{1, 2\}$ by $u\,|\,A \equiv 1$ and $u\,|\,(X - A) \equiv 2$. Then from the flow $\mathcal{X}$ we may form the induced flows $\mathcal{X}^u$ and $\mathcal{X}_A$. In the case when B is the 1-block 0, the transformation $T_A: A \to A$ consists of "shifting to the next zero;" the primitive flow $\mathcal{X}^u$ in this case is isomorphic to the symbolic flow $(\mathcal{O}(x')^-, \sigma)$, where x' is the sequence obtained from x by "doubling ones". ($\mathcal{X}_A$ can also be realized as a symbolic flow.)

Let B be the 1-block 0 and let u and A be as above. Then for certain choices of $x \in S$, $\mathcal{X} = (\mathcal{O}(x)^-, \sigma)$ will be minimal and will satisfy the conditions of Theorem 2.2 and Theorem 3.2. Such is the case if x is a Sturmian sequence (Example 1) or a Toeplitz sequence (Example 2). Thus in these cases the induced flows $\mathcal{X}^u$ and $\mathcal{X}_A$ are minimal and topologically weakly mixing. For a particular Toeplitz sequence x we will see (§5) that the derivative $\mathcal{X}_A$ is not topologically strongly mixing.

The Sturmian and Toeplitz flows, which are of fundamental importance in topological dynamics, have been discussed by many authors using a variety of approaches. For the sake of completeness and clarity we will give a detailed construction of these flows via elementary techniques, using a method similar to [25].

Example 1. Sturmian flows. Since we prefer additive notation in this example, we take the circle group $\mathbf{K}$ to be $[0, 1)$, with addition modulo one. Unless stated otherwise, all real numbers will be reduced modulo 1 and taken to be in $[0, 1)$. A metric in $\mathbf{K}$ is given by $d(\gamma, \gamma') = \min\{|\gamma - \gamma'|, 1 - |\gamma - \gamma'|\}$. Note that if $p, q \in \mathbf{K}$ with $p \leq q$, then there are two closed intervals with endpoints p and q, namely

$[p, q] = \{r \in \mathbf{K} \mid p \le r \le q\}$ and $[q, p] = \{r \in \mathbf{K} \mid q \le r < 1$ or $0 \le r \le p\}$. Intervals such as $[p, q)$ and $(q, p]$ are defined similarly. For a set A, χ_A will denote the characteristic function of A.

Let α be irrational and $\beta \in (0, 1)$. Define $f: \mathbf{K} \to \{0, 1\}$ by $f(\gamma) = \chi_{[0,\beta]}(\gamma)$, and for $n \in \mathbf{Z}$ define $x_0(n) = f(n\alpha)$, so $x_0 \in S$. The flow $\mathfrak{X} = (\mathcal{O}(x_0)^-, \sigma)$ is called a *Sturmian flow of type* (α, β). These flows are related to the Sturmian minimal sets [10]. Sturmian flows are special cases of the flows considered in [19], [24].

Let $T(\gamma) = \gamma + \alpha$ for $\gamma \in \mathbf{K}$, so $\mathfrak{K} = (\mathbf{K}, T)$ is a minimal flow. We now assume that $\beta \notin \mathbf{Z}\alpha$, because we are interested in this case for our applications. We note that if $\beta \in \mathbf{Z}\alpha$ then a Sturmian flow of type (α, β) is not minimal, though it would have been minimal if we had chosen, in place of $[0, \beta]$, an interval I such that $(\mathbf{Z}\alpha) \cap \partial I = \varnothing$, where ∂ denotes boundary.

Theorem 4.1. *There is a homomorphism* $\rho: \mathfrak{X} \to \mathfrak{K}$ *such that* (a) $\rho^{-1}\{\gamma\}$ *is a singleton unless* $\gamma \in E = (\mathbf{Z}\alpha) \cup (\beta + \mathbf{Z}\alpha)$; (b) *for each* $\gamma \in E$, $\rho^{-1}\{\gamma\}$ *consists of exactly two points*; (c) $\rho^{-1}\{0\} = \{x_0, x_1\}$, *where* $x_1(0) \neq x_0(0)$ *but* $x_1(n) = x_0(n)$ *for* $n \neq 0$; (d) *if* $\rho(x) = \rho(x')$, *then* x *and* x' *are doubly proximal. In addition,* (e) $\mathfrak{X}$ *is minimal.*

Proof. Define $\rho(\sigma^n x_0) = n\alpha$. To show that ρ can be continuously extended to $\mathcal{O}(x_0)^-$ it suffices to show that Cauchy sequences in $\mathcal{O}(x_0)$ are mapped to Cauchy sequences in $\mathbf{K}$. Therefore we assume that $n(k)$ is a sequence with $\{\sigma^{n(k)}x_0\}$ convergent to $x_1 \in S$ and let α_1, α_2 be cluster points of $\{n(k)\alpha\}$, and we will show $\alpha_1 = \alpha_2$. Now for $n \in \mathbf{Z}$ we have $f(\alpha_1 + n\alpha) = f(\alpha_2 + n\alpha) = x_1(n)$ if f is continuous at $\alpha_1 + n\alpha$ and $\alpha_2 + n\alpha$, i.e. unless $\alpha_1 + n\alpha$ or $\alpha_2 + n\alpha$ is 0 or β. Thus $f(\gamma) = f((\alpha_1 - \alpha_2) + \gamma)$ for a dense set of $\gamma \in \mathbf{K}$, which implies that $\alpha_1 - \alpha_2 = 0$.

Thus ρ can be extended to all of $\mathcal{O}(x_0)^-$. The extension is clearly unique, and it is a homomorphism from $\mathfrak{X}$ to $\mathfrak{K}$ since $\rho \circ \sigma = T \circ \rho$ on the dense set $\mathcal{O}(x_0)$.

(a) Now we show that ρ is one-to-one except at the orbits of the points of discontinuity of f, denoted E. We define a continuous function $e: \mathcal{O}(x_0)^- \to \{0, 1\}$ by $e(x) = x(0)$. Then $e(\sigma^n x_0) = x_0(n) = f(\rho(\sigma^n x_0))$; it follows that if $\rho(x)$ is a point of continuity of f, then $e(x) = f(\rho(x))$. Now suppose $\rho(x) = \rho(x') \notin E$, and let $n \in \mathbf{Z}$. Clearly $\rho(\sigma^n x) = \rho(\sigma^n x') \notin E$, and therefore $\rho(\sigma^n x)$ is a point of continuity of f. Now choose a sequence $\{n(k)\}$ such that $\sigma^n x = \lim_k \sigma^{n(k)} x_0$. Then $f(\rho(\sigma^n x)) = \lim_k f(\rho(\sigma^{n(k)} x_0)) = \lim_k x_0(n(k)) = x(n)$. Similarly, $f(\rho(\sigma^n x')) = x'(n)$, so $x(n) = x'(n)$, and thus $x = x'$, as desired.

(b) Since there are points of $\mathbf{K} - E$ arbitrarily close to $0 \in \mathbf{K}$ where f assumes the value 0, and points of $\mathbf{K} - E$ arbitrarily close to 0 where f assumes the value 1, and the continuous function $e: \mathcal{O}(x_0)^- \to \{0, 1\}$ coincides with $f \circ \rho$ on the set $\rho^{-1}(\mathbf{K} - E)$, $\rho^{-1}\{0\}$ contains two points x, x' such that $x(0) = 0$ and $x'(0) = 1$. We will show that $x(j) = x'(j)$ for $j \neq 0$, so $\rho^{-1}\{0\}$ contains exactly two points. Now $j\alpha = \rho(\sigma^j x) = \rho(\sigma^j x')$ is a point of continuity of f, so $x(j) = e(\sigma^j x) = f(\rho(\sigma^j x)) = f(\rho(\sigma^j x')) = e(\sigma^j x) = x'(j)$. Similarly, ρ is two-to-one at β. Since ρ is a homomorphism, ρ is two-to-one on E.

(c) This follows from the proof of (b).

(d) Assume $\rho(x) = \rho(x')$; we will show that x and x' are doubly proximal. Since $\{n\alpha: n > 0\}$ and $\{n\alpha: n < 0\}$ are dense, we can choose a sequence $\{n(k)\}$ of positive (or negative) integers such that $\lim_k [\rho(x) + n(k)\alpha] = \gamma \notin E$, and refine $\{n(k)\}$ so that $z = \lim_k \sigma^{n(k)}x$ and $z' = \lim_k \sigma^{n(k)}x'$ both exist. Then $\rho(z) = \rho(z') = \gamma \notin E$, so $z = z'$ as desired.

(e) It remains to show that $\mathfrak{X}$ is minimal. By Zorn's Lemma, there is a subflow $M \subseteq \mathcal{O}(x_0)^-$ such that (M, σ) is minimal. Since $\mathcal{K}$ is minimal, it follows that $\rho(M) = K$. Thus there is some point $m_0 \in M$ such that $\rho(m_0) = \rho(x_0)$. If $m_0 = x_0$ then we are done, for then $\mathcal{O}(x_0)^- \subseteq M$ so $\mathcal{O}(x_0)^- = M$. Suppose then that $m_0 = x_1$, where x_1 is as in part (c). We will show that $x_0 \in \mathcal{O}(x_1)^-$, so that $\mathcal{O}(x_0)^- \subseteq \mathcal{O}(m_0)^- = M$, as desired. From part (c) it is clear that $x_1(0) = 0$ and $x_1(n) = x_0(n)$ for $n \neq 0$. Now form a sequence $\{n(k)\}$ such that $n(k)\alpha \in (0, \beta)$ for all k, $\lim_{k\to\infty} n(k)\alpha = 0$, and $\{\sigma^{n(k)}x_1\}$ converges, say $\lim_{k\to\infty} \sigma^{n(k)}x_1 = x$. Now we need only show that $x = x_0$ to complete our proof. But

$$\rho(x) = \lim_{k\to\infty} \rho(\sigma^{n(k)}x_1) = \lim_{k\to\infty} \rho(\sigma^{n(k)}x_0) = \lim_{k\to\infty} n(k)\alpha = 0 = \rho(x_0),$$

and

$$\begin{aligned} x(0) &= \lim_{k\to\infty} \sigma^{n(k)}x_1(0) = \lim_{k\to\infty} x_1(n(k)) = \lim_{k\to\infty} x_0(n(k)) \\ &= \lim_{k\to\infty} \chi_{[0,\beta]}(n(k)\alpha) = \lim_{k\to\infty} 1 = 1. \end{aligned}$$

By (c) $x = x_0$ as desired.

Corollary. *The flow $\mathfrak{X}$ is uniquely ergodic.*

Proof. $\mathcal{K}$ is uniquely ergodic because of the uniqueness of normalized Haar measure, which we denote by ν. If μ is a Borel probability measure on X which is invariant under σ then $\rho(\mu)$, defined by $\rho(\mu)(B) = \mu(\rho^{-1}(B))$, is an invariant measure on $\mathbf{K}$ so $\rho(\mu) = \nu$. Thus $\rho: (X, \mu) \to (\mathbf{K}, \nu)$ is a measure-theoretic isomorphism because it is one-to-one except on the countable, zero-measure set E. Therefore $\mathfrak{X}$ is uniquely ergodic.

From Theorem 4.1(c) and Theorems 2.2 and 3.2 we see that if we set $u(x) = 1 + e(x)$ and $A = u^{-1}\{1\}$, then the primitive $\mathfrak{X}^u$ and the derivative $\mathfrak{X}_A$ of a Sturmian flow $\mathfrak{X}$ of type (α, β), where α is irrational and $\beta \notin \mathbf{Z}\alpha$, are topologically weakly mixing and minimal. Previously Katok and Stepin [14] and Kakutani (oral communication) have proved that the primitive $\mathfrak{X}^u$ is measure-theoretically (and therefore topologically) weakly mixing for certain choices of α and β.

Example 2. Toeplitz flows. For this example $\mathbf{K}$ is replaced by the group of 2-adic integers, denoted G (the construction could as well be carried out with G equal to the inverse limit of any totally directed set of finite cyclic groups). G, being the inverse limit of cyclic groups of order 2^n, consists of sequences

$$g = (g_0, g_1, g_2, \cdots) \in \prod_{i=0}^{\infty} \{0, 1, \cdots, 2^i - 1\}$$

such that $g_i \equiv g_{i+1} \pmod{2^i}$ for $i = 0, 1, 2, \cdots$. When addition in G is defined coordinatewise (mod 2^i in the ith coordinate) and G is given the product topology, then G becomes a compact abelian group.

Let $\theta = (0, 1, 1, 1, \cdots) \in G$ (so $-\theta = (0, 1, 3, 7, \cdots)$), and for $g \in G$ define $Tg = g + \theta$. Then $\{T^n\theta : n \in \mathbf{Z}\}$ corresponds to the rational integers in G so it is dense in G [12, Chapter V. 5]; from this it follows readily that $\mathcal{G} = (G, T)$ is a minimal flow.

Define $r(g) = \min \{j : g_j = g_{j+1}\}$ for $g \neq -\theta$ and $r(-\theta) = 0$. Then r is continuous on $G - \{-\theta\}$. Let

$$\psi(n) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd,} \end{cases}$$

for $g \in G$ set $f(g) = \psi \circ r(g)$, and for $n \in \mathbf{Z}$ let $y_0(n) = f(n\theta)$, so $y_0 \in S$. Let $\mathcal{Y} = (\mathcal{O}(y_0)^-, \sigma)$.

Theorem 4.2. *There is a homomorphism $\rho : \mathcal{Y} \to \mathcal{G}$ such that* (a) *$\rho^{-1}\{g\}$ is a singleton unless $g \in E = \mathbf{Z}\theta$ (the orbit of the point of discontinuity of f);* (b) *for each $g \in E$, $\rho^{-1}\{g\}$ consists of exactly two points;* (c) *$\rho^{-1}\{-\theta\} = \{y_0, y_1\}$, where $y_1(0) \neq y_0(0)$ but $y_1(n) = y_0(n)$ for $n \neq 0$;* (d) *if $\rho(y) = \rho(y')$, then y and y' are doubly proximal. In addition, $\mathcal{Y}$ is minimal and uniquely ergodic.*

Proof. Similar to the proof of Theorem 4.1 and its corollary.

As before, let $u(y) = 1 + e(y)$ (recall $e(y) = y(0)$) for $y \in \mathcal{Y}$, and let $A = u^{-1}\{1\}$. Then the hypotheses of Theorems 2.2 and 3.2 are again satisfied, and it follows that the primitive $\mathcal{Y}^u$ and derivative $\mathcal{Y}_A$ are topologically weakly mixing and minimal.

Toeplitz sequences were first studied in [8] and were named in [11]. Kakutani (oral communication) proved that for this particular Toeplitz sequence y_0, the induced flow $\mathcal{Y}^u$ is uniquely ergodic and measure-theoretically weakly mixing, hence also topologically weakly mixing.

Remarks. 4.1. Note that the sequence y_0 has the following property. For a block $A = a_1 a_2 \cdots a_n$ of 0's and 1's, let $A^* = a_1 a_2 \cdots a_{n-1} a_n'$, where $0' = 1$

and $1' = 0$. Let $A_0 = 0$, and for $n > 0$ let $A_n = A_{n-1}A^*_{n-1}$. Then the initial block $y_0(0)y_0(1) \cdots y_0(2^n - 1)$ of y_0 coincides with A_n for $n \geq 0$.

4.2. The method of construction used in these two examples can of course be applied in other cases as well. For example, let $\mathcal{H} = (H, T)$ be any minimal flow and B a subset of H such that

$$E = \bigcup_{n \in \mathbf{Z}} T^n \partial B \subsetneq H$$

where ∂B denotes the boundary of B. We will construct a flow $\mathfrak{X} = (X, T)$ for which there is a homomorphism ρ: $\mathfrak{X} \to \mathcal{H}$ and for which one can prove a theorem similar to Theorem 4.1. Pick $h_0 \in H - E$ and define η: $\mathbf{Z} \to H$ by $\eta(n) = T^n(h_0)$. Let $C(\cdot)$ denote the set of bounded continuous real-valued functions on the space $(\cdot)$. We say a subset $\mathcal{B}$ of $C(\mathbf{Z})$ is *translation invariant* if $g \in \mathcal{B}$ implies $g_n \in \mathcal{B}$ for each $n \in \mathbf{Z}$ where $g_n(m) = g(n + m)$ for $m \in \mathbf{Z}$. Define $\mathcal{A} = \{g \circ \eta$: $g \in C(H)\}$, so that $\mathcal{A}$ is a uniformly closed, translation invariant subalgebra of $C(\mathbf{Z})$. Define f: $\mathbf{Z} \to \{0, 1\}$ by $f(n) = \chi_B(T^n h_0)$. Let $\mathcal{A}^*$ be the smallest translation-invariant, uniformly closed subalgebra of $C(\mathbf{Z})$ containing $\mathcal{A} \cup \{f\}$. By [4], [5], or [17] there is a flow $\mathfrak{X} = (X, T)$ and η^*: $\mathbf{Z} \to X$ such that $\mathcal{A}^* = \{g \circ \eta^*$: $g \in C(X)\}$, and a homomorphism ρ: $\mathfrak{X} \to \mathcal{H}$ such that $\eta = \rho \circ \eta^*$. Let $A = \rho^{-1}(B)$ and define u: $X \to \{1, 2\}$ by $u|A \equiv 1$ and $u|(X - A) \equiv 2$. If there is a point $x \in \partial B$ such that $T^n x \notin \partial B$ for all $n \neq 0$, then it can be shown that (X, T) has properties analogous to those mentioned in Theorem 4.1, and so the induced flows $\mathfrak{X}^u$ and $\mathfrak{X}_A$ have no nonconstant continuous eigenfunctions. This approach, in many disguises, is common in the literature: cf. [5, pp. 8–32], [4, Chapter 9], or [17] (a good exposition of these methods can be found in [3]).

5. **Weak mixing without strong mixing.** For this section let x denote y_0, where y_0 is the particular Toeplitz sequence discussed in Example 2; x may also be defined by $x(n) = 0$ if and only if $n = (k \cdot 2^{2m}) - 1$ for some odd k and some $m = 0, 1, 2, \cdots$. Recall that the initial 2^n-block of x is A_n, where A_n is as in Remark 4.1.

Let $X = \mathcal{O}(x)^-$, $\mathfrak{X} = (X, \sigma)$, and $A = \{y \in X$: $y(0) = 0\}$. It follows from Example 2 that the derivative $\mathfrak{X}_A$ of $\mathfrak{X}$ with respect to A is topologically weakly mixing and minimal. Recall that a flow (Z, T) is said to be *topologically strongly mixing* if for any nonempty open sets $U, V \subset Z$ there is n_0 such that $T^n U \cap V \neq \emptyset$ whenever $|n| \geq n_0$. We will prove that $\mathfrak{X}_A$ is not topologically strongly mixing.

For any block B which appears in x, let $\eta(B)$ denote the collection of all (nonnegative) integers which have the following property: there is a block C such that the block BCB appears in x and the block BC contains n zeros. If $\mathfrak{X}_A$ is topologically strongly mixing, then for each block B which appears in x there is an n_0 such that $\eta(B) \supset \{n \in \mathbf{Z}: n \geq n_0\}$ (cf. [23, Theorem 1.2]). Fix odd $r \geq 3$. For each $n = 0, 1, 2, \cdots$ let p_n denote the number of zeros in the initial 2^n-block A_n of x. Then $p_0 = 1, p_1 = 1, p_2 = 3, p_3 = 5, p_4 = 11, \cdots$ and $p_n \to \infty$. We will prove that for odd $n > r$, $p_n - 1 \notin \eta(A_r)$; therefore $\mathfrak{X}_A$ cannot be topologically strongly mixing.

For the following three lemmas, let n denote an arbitrary nonnegative integer.

Lemma 5.1. *A_n appears at the $(k \cdot 2^{n+1})$th place in x for all $k \in \mathbf{Z}$.*

Proof. From Remark 4.1 it follows that the 2^{n+1}-block appearing at the $(k \cdot 2^{n+1})$th place in x is either A_{n+1} or A^*_{n+1}, both of which have A_n for their initial 2^n-block.

Lemma 5.2. *If A_n appears at the mth place in x then $m = k \cdot 2^n$ for some $k \in \mathbf{Z}$.*

Proof. For $n = 0$ there is nothing to prove. Suppose then that $n \geq 1$, so $A_n = A_{n-1}A^*_{n-1}$, and suppose that A_n appears at the mth place in x. Making the appropriate induction hypothesis, we may assume that $m = k \cdot 2^{n-1}$ for some $k \in \mathbf{Z}$. Suppose k is odd. Then A^*_{n-1} is the 2^{n-1}-block which appears at the $m + 2^{n-1} = (k+1)2^{n-1}$ place in x; by Lemma 5.1, since $k + 1$ is even, $A^*_{n-1} = A_{n-1}$, and this is impossible. Therefore k must in fact be even, and $m = k' \cdot 2^n$ for some $k' \in \mathbf{Z}$. The result then follows by induction.

Lemma 5.3. *If n is odd, $p_{n+1} = 2p_n + 1 = 1 + \sum_{i=0}^n p_i$; if n is even, $p_{n+1} = 2p_n - 1 = \sum_{i=0}^n p_i$. Thus (since r is odd) if $n > r$ then*

$$p_n - \sum_{i=r}^{n-1} p_i = \begin{cases} p_r & (n \text{ odd}), \\ 1 + p_r & (n \text{ even}). \end{cases}$$

Proof. That $p_{n+1} = 2p_n + (-1)^{n+1}$ is clear from the fact that $A_{n+1} = A_nA^*_n$ and $p_0 = 1$. The remaining formulas are then easily proved by induction.

Lemma 5.4. *$\eta(A_r)$ is contained in the collection of all numbers a of the form*

$$a = \epsilon_r p_r + \epsilon_{r+1}p_{r+1} + \cdots + \epsilon_m p_m,$$

where $m \geq r$, each ϵ_i is 1, 0, or -1 for $i = r, r+1, \cdots, m$, and $\epsilon_m = 1$.

Proof. Because of Lemma 5.2 it suffices to show that for each $n = 1, 2, \cdots$ the number of zeros in an initial $k \cdot 2^r$-block of x, for any $k = 1, 2, \cdots, 2^n - 1$, is a number of the form

$$\delta_r p_r + \delta_{r+1}p_{r+1} + \cdots + \delta_{r+n-1}p_{r+n-1},$$

where each δ_i is 0 or 1 for $i = r, r+1, \cdots, r+n-1$. We proceed by induction on n. The statement is clearly true for $n = 1$, since the number of zeros in the initial 2^r-block A_r of x is p_r by definition.

Suppose then that for each $k = 1, 2, \cdots, 2^n - 1$ the number of zeros in the initial $k \cdot 2^r$-block of x is a number of the specified form. If $k = 2^n$, then the number of zeros in the initial $k \cdot 2^r$-block of x is p_{n+r}, which is a number of the required form. If $k \in \{2^n + 1, 2^n + 2, \cdots, 2^{n+1} - 1\}$, then the initial $k \cdot 2^r$-block of x is A_{n+r} followed by the initial $(k - 2^n)2^r$-block of A^*_{n+r}. Since $k - 2^n < 2^n$ and this latter block is also an initial block of x, the induction hypothesis implies that it contains $\delta_r p_r + \delta_{r+1} p_{r+1} + \cdots + \delta_{r+n-1} p_{r+n-1}$ zeros for some choice of $\delta_i = 0$ or 1, $i = r, r+1, \cdots, r+n-1$. But then the initial $k \cdot 2^r$-block of x contains $\delta_r p_r + \delta_{r+1} p_{r+1} + \cdots + \delta_{r+n-1} p_{r+n-1} + p_{n+r}$ zeros, and this is a number of the required form.

Assuming $\epsilon_m \neq 0$, unless $\epsilon_m = 1$ it follows from Lemma 5.3 that $a < 0$; this is why we may assume that $\epsilon_m = 1$.

Theorem 5.1. *With* $\mathfrak{X}$ *and* A *as above,* $\mathfrak{X}_A$ *is not topologically strongly mixing.*

Proof. Let n be odd, $n > r$; as we have remarked, it is enough to show that $p_n - 1$ is not a number of the form $a = \epsilon_r p_r + \epsilon_{r+1} p_{r+1} + \cdots + \epsilon_m p_m$, where each ϵ_i is 1, 0, or -1 and $\epsilon_m = 1$, for any $m \geq r$. Because of Lemma 5.3 we may assume without loss of generality that $m \geq n$.

The smallest number of the above form is achieved by taking $\epsilon_i = -1$ for $i = r, r+1, \cdots, m-1$; from Lemma 5.3 we obtain

$$a = p_m - \sum_{i=r}^{m-1} p_i = \begin{cases} p_r & (m \text{ odd}), \\ 1 + p_r & (m \text{ even}). \end{cases}$$

Now if for some $k \geq n$ we change ϵ_k from -1 to 0, then a becomes either $p_k + p_r$ or $p_k + p_r + 1$, both of which are larger than p_n. Therefore in order to achieve $a = p_n - 1$, we must keep $\epsilon_k = -1$ for $k = n, n+1, \cdots, m-1$. Thus

$$a = p_m - \sum_{i=n}^{m-1} p_i + \epsilon_{n-1} p_{n-1} + \cdots + \epsilon_r p_r$$

$$= \epsilon_{n-1} p_{n-1} + \cdots + \epsilon_r p_r + \begin{cases} p_n & (m \text{ odd}), \\ 1 + p_n & (m \text{ even}). \end{cases}$$

The question then becomes whether we can ever have $b = \epsilon_{n-1} p_{n-1} + \cdots + \epsilon_r p_r$ equal to either -1 or -2. Let ϵ_q $(q \geq r)$ be the first nonzero coefficient in this expression, so $b = \epsilon_q p_q + \cdots + \epsilon_r p_r$. If $\epsilon_q = 1$, b is positive by Lemma 5.3.

If $\epsilon_q = -1$, the largest possible value of b is achieved by taking $\epsilon_k = 1$ for $k = r, r+1, \cdots, q-1$. This gives

$$b = -p_q + \sum_{i=r}^{q-1} p_i = \begin{cases} -p_r & (q \text{ odd}), \\ -p_r - 1 & (q \text{ even}), \end{cases}$$

so in any case $b \leq -5$. Therefore b can never be -1 or -2.

This example completes the proof that in the following diagram relating four properties of uniquely ergodic flows, none of the implications can be reversed, even if we were to restrict the statements to the class of minimal flows with metrizable phase spaces.

$$\begin{tikzcd}[column sep=small]
& \begin{array}{c}1\\ \text{measure-theoretic}\\ \text{strong mixing}\end{array} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
\begin{array}{c}2\\ \text{topological}\\ \text{strong mixing}\end{array} \arrow[dr, Rightarrow] & & \begin{array}{c}3\\ \text{measure-theoretic}\\ \text{weak mixing}\end{array} \arrow[dl, Rightarrow] \\
& \begin{array}{c}4\\ \text{topological}\\ \text{weak mixing}\end{array} &
\end{tikzcd}$$

That (3) does not imply (1) was proved by Kakutani in the examples mentioned above. Kolmogorov [18] shows that (4) does not imply (3) when the acting group is a one-parameter family of homeomorphisms. It follows from [15, Proposition 3.3] that (4) does not imply (3) for our setting of a cyclic group $\{T^n\}$ of homeomorphisms. Petersen [23] constructed an example for which (2) holds but not (1); and of course our present example disposes of the question whether (4) implies (2).

6. **Irregularities of distribution.** In this section notation is as in Example 1, Sturmian flows, and again all real numbers are assumed reduced modulo one so as to lie in $\mathbf{K} = [0, 1)$.

It is well known [20] that if α is irrational then the sequence $\{n\alpha : n \in \mathbf{Z}\}$ is uniformly distributed. In particular if I is an interval in $\mathbf{K}$ of length β, then $\lim_{n \to \pm\infty} (1/|n|) \sum_{i=0}^{n} \chi_I(i\alpha) = \beta$, i.e. $\{n\alpha\}$ hits I with probability β. Clearly if $\gamma \in \mathbf{K}$ then what we have said about $\{n\alpha\}$ is true for $\{\gamma + n\alpha\}$ (consider $I + \gamma$ in place of I). In particular if we pick $\gamma, \gamma' \in \mathbf{K}$ then each of the sequences $\{\gamma + n\alpha\}$ and $\{\gamma' + n\alpha\}$ hits I with probability β.

The Sturmian flows of Example 1 are closely related to the question of whether one of the two sequences $\{\gamma + n\alpha\}$, $\{\gamma' + n\alpha\}$ hits I infinitely more often than the other. More precisely, let

$$N(\gamma, n) = \sum_{i=0}^{n} \chi_I(\gamma + i\alpha);$$

then from the discussion above we know that

$$\lim_{n\to\pm\infty} \frac{1}{|n|} (N(\gamma, n) - N(\gamma', n)) = \beta - \beta = 0.$$

We wish to determine when the quantity $N(\gamma, n) - N(\gamma', n)$ is unbounded.

It is not hard to show that if $\gamma' = \gamma + k\alpha$, then $|N(\gamma, n) - N(\gamma', n)| \le k$ for $n \in \mathbf{Z}$. It can be proved that if $\beta \notin \mathbf{Z}\alpha$, then for a residual set of $(\gamma, \gamma') \in \mathbf{K} \times \mathbf{K}$ we have $(N(\gamma, n) - N(\gamma', n))$ unbounded in n. This is done by showing that if $\{x, y\} \subseteq X - \rho^{-1}(E)$, x and y are proximal under σ^u, and $y \neq \sigma^n x$ for any $n \in \mathbf{Z}$, then $N(\rho(x), n) - N(\rho(y), n)$ is unbounded in n. The assertion then follows from the known result [16, Remark 3.4] that, since $\mathfrak{X}^u$ is weakly mixing, its proximal relation is residual.

It is possible to prove a much stronger result [26]: If $2\beta \notin \mathbf{Z}\alpha$ or $2(\gamma - \gamma') \notin \mathbf{Z}\alpha$ then $N(\gamma, n) - N(\gamma', n)$ is unbounded whenever $\gamma - \gamma' \notin \mathbf{Z}\alpha$.

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