ON THE ZEROS OF POWER SERIES WITH
HADAMARD GAPS–DISTRIBUTION IN SECTORS(1)

BY

I-LOK CHANG

ABSTRACT. We give a sufficient condition for a power series with
Hadamard gaps to assume every complex value infinitely often in every
sector of the unit disk.

I. Introduction. Let

\[ f(z) = c_0 + \sum_{k=1}^{\infty} c_k z^{n_k} \]

be a power series convergent in \(|z| < 1\), with Hadamard gaps, \(n_{k+1}/n_k > q > 1\),
\(k \geq 1\). Given a complex number \(c\), we are interested in the distribution of the
zeros of \(f(z) - c\). We shall discuss the problem in term of the zeros of \(f\),
replacing the constant term \(c_0\) of (1) by \(c_0 - c\) if necessary.

It has been shown that

(i) \(f\) has infinitely many zeros in the unit disk if \(\sum_{k=0}^{\infty} |c_k| = \infty\) and
\(q \geq q_0\), where \(q_0\) is about 100 [5].

(ii) \(f\) has infinitely many zeros in any sector \(\theta_2 < \arg z < \theta_1\), \(|z| < 1\), if
\(\lim_{k \to \infty} |c_k| > 0\) [2].

It remains undetermined whether \(f\) has zeros in the unit disk, or perhaps in any
sector, if \(\sum_{k=0}^{\infty} |c_k| = \infty\), \(\lim_{k \to \infty} c_k = 0\), and \(1 < q < q_0\). We prove

Theorem 1. Let \(f(z) = c_0 + \sum_{k=1}^{\infty} c_k z^{n_k}\) be a power series convergent in
\(|z| < 1\), with

(i) \(n_{k+1}/n_k > q > 1\) \((k \geq 1)\),

(ii) \(\lim_{k \to \infty} c_k = 0\),

(iii) \(\sum_{k=0}^{\infty} |c_k|^{2+\epsilon} = \infty\) for some positive \(\epsilon\).

Then \(f\) has infinitely many zeros in any sector \(\theta_2 < \arg z < \theta_1\), \(|z| < 1\).
meromorphic in sectors. The basic idea of this formula goes back to V. P. Petrenko [3]. The following lemma can be found in [2].

**Lemma 1.** Suppose \( f(z) \) is meromorphic in the sector \(|\arg z| \leq \pi/\nu \ (\nu > 1)\), \(|z| \leq R\). Let \( z = t (0 < t < R) \) be a regular point of \( f \) on the real axis, where \( f(t) \neq 0 \). For \( z \neq t, R^2/t, \) define

\[
a(z) = a(R, z, t) = \log \left| \frac{R^2 - tz}{R(z - t)} \right|
\]

and

\[
\Lambda(R, z, t) = a(z) - a(-|z|).
\]

If we write

\[
I_1(R, t, \nu) = \int_0^R \left\{ \int_{-\pi/\nu}^{\pi/\nu} \log |f(re^{i\theta})| \, d\theta \right\} \xi_1(R, r, t, \nu) \, dr,
\]

\[
I_2(R, t, \nu) = \int_{-\pi/\nu}^{\pi/\nu} \log |f(re^{i\theta})| \xi_2(R, \theta, t, \nu) \, d\theta,
\]

where

\[
\xi_1(R, r, t, \nu) = \frac{\nu^2}{2\pi} \frac{\nu - \nu^2}{(\nu + t^\nu)(\nu^2 + t^\nu)^2},
\]

\[
\xi_2(R, \theta, t, \nu) = \frac{\nu}{\pi} \frac{R^\nu t^\nu (R^\nu - t^\nu)(1 + \cos \nu \theta)}{(R^\nu + t^\nu)(R^\nu + t^\nu - 2R^\nu t^\nu \cos \nu \theta)}
\]

then

\[
\log |f(t)| = I_1(R, t, \nu) + I_2(R, t, \nu) + \sum_{b_i} A(R^\nu, t^\nu, b_i^\nu)
\]

(2)

\[
- \sum_{a_i} A(R^\nu, t^\nu, a_i^\nu)
\]

where the summation is taken over the zeros \( \{a_i\} \) and the poles \( \{b_i\} \) of \( f \) which lie in the interior of the sector.

Without loss of generality, we may assume that \( f(0) = 1 \) (consider \( f(z)/c_p z^p \) if necessary). Suppose now that \( f \) has no zero in some sector, which we may assume to be the sector \(|\arg z| \leq \pi/\nu_0, |z| < 1\), where \( \nu_0 > 1 \). We shall show that this leads to the conclusion

\[
\lim_{R \to 1} \sup \{I_1(R, 2\nu_0) + I_2(R, 2\nu_0)\} = \infty
\]

(3)

whereas (2) now reduces to the contradictory result

\[
I_1(R, 2\nu_0) + I_2(R, 2\nu_0) = \log |f(t)|.
\]

In the next section, we derive estimates which will be used to establish (3) in §IV.
III. Lower bounds for $|f(z)|$. Transform the domain of $f$ to the right half-plane with the change of variable $z = e^{-w}$, and write (1) as

$$F(w) = f(e^{-w}) = c_0 + \sum_{k=1}^{\infty} c_ke^{-nkw}$$

**Lemma 2.** There exist a subsequence $\{c_{k(i)}\}$ of the coefficients $\{c_k\}$ of (4) and positive constants $U_0(q), u_0(q), p_0(q)$ such that the derivatives of $F$ satisfy

$$F^{(p)}(w) = (-n_{k(i)})^pc_{k(i)}e^{-n_{k(i)}w} + R_i(w),$$

$$|R_i(w)| \leq \frac{1}{2}|c_{k(i)}|^p n_{k(i)} e^{-n_{k(i)} Re(w)}$$

whenever $p \geq p_0(q)$, and $Re(w)$ is in the range $u_0(q)/n_{k(i)} < Re(w) < u_0(q)/n_{k(i)}$.

**Proof.** Consider the sequence $\{d_k\}$, where

$$d_0 = \max\{|c_0|, |c_1|, |c_2|, \ldots\},$$

$$d_k = \max\{|1/2d_{k-1}, |c_k|, |c_{k+1}|, \ldots\} \quad (k \geq 1),$$

one verifies readily that

(a) $d_k > 0$ for all $k$,

(b) $1/2 \leq d_{k+1}/d_k \leq 1$, and also

(c) $d_k \geq |c_k|$, with equality occurring infinitely often.

If in (c), equality occurred finitely often, then $d_{k+1} = 1/2d_k$ for $k \geq k_0$. In this case

$$\sum_{k=k_0}^{\infty} |c_k| \leq \sum_{k=k_0}^{\infty} d_k = d_{k_0} \sum_{j=0}^{\infty} \left(\frac{1}{2}\right)^j < \infty,$$

contradicting the assumptions that $\sum_{k=0}^{\infty} |c_k|^{2+\epsilon} = \infty$ and $\lim_{k \to \infty} c_k = 0$.

Let $\{c_{k(i)}\}$ be the subsequence of $\{c_k\}$ satisfying $d_{k(i)} = |c_{k(i)}|$, $i = 1, 2, \ldots$. Differentiating $F(w)$ $p$ times, (4) becomes

$$F^{(p)}(w) = \sum_{k=1}^{\infty} \delta_k a_k(w)$$

where $\delta_k = (-1)^pc_{k}/d_k$, and $a_k(w) = (n_{k}^p d_k \exp(-n_{k}w))$. We can find, for each $k(i)$, and for sufficiently large $p$, a set of $w$ such that

$$|a_{k+1}(w)/a_k(w)| > 5 \quad \text{for } k < k(i),$$

$$|a_{k+1}(w)/a_k(w)| < 1/5 \quad \text{for } k \geq k(i).$$
I-LOK CHANG

For, (5) holds if
\[
\text{Re}(n_{k+1}w) < (p \log t_k - \log 10)/(1 - 1/t_k)
\]
where \( t_k = n_{k+1}/n_k > q \). For sufficiently large \( p \),
\[
f(t) = (p \log t - \log 10)/(1 - 1/t)
\]
is a positive increasing function of \( t \) in \( t > q \). Therefore, (5) holds if \( p > p_0 \),
and if
\[
\text{Re}(w) < (1/n_{k(i)})(p \log q - \log 10)/(1 - 1/q).
\]

Similarly (6) holds, if
\[
\text{Re}(n_kw) > (p \log t_k + \log 5)/(t_k - 1).
\]
The right-hand side of this inequality is bounded above by \( (p \log q + \log 5)/(q - 1) \), so
that (6) holds if
\[
\text{Re}(w) > (1/n_{k(i)})(p \log q + \log 5)/(q - 1).
\]

We note that if \( u = (p \log q + \log 5)/(q - 1) \), and \( U = (p \log q - \log 10)/(1 - 1/q) \),
then for large \( p \), \( U/u = q(1 + O(1/p)) > c > 1 \). Thus (5) and (6) hold simultane-
ously, if \( p > p_1 \), and \( \text{Re}(w) \) satisfies
\[
(7) \quad \frac{u}{n_k(i)} < \text{Re}(w) < U/n_k(i).
\]

If \( w \) is in the range of (7), we have then
\[
F^{(p)}(w) = \delta_{k(i)}n_k(i)(w) + \sum_{k \geq k(i)} \delta_k a_k(w)
\]
\[
= \delta_{k(i)}n_k(i)(w) + R_i(w)
\]
where \( |\delta_{k(i)}| = 1 \), and
\[
|R_i(w)| \leq \sum_{1 \leq k < k(i)} |a_{k(i)}(w)|(5)^{k-k(i)} + \sum_{k > k(i)} |a_{k(i)}(w)|(5)^{k(i)-k}
\]
\[
\leq 2|a_{k(i)}(w)| \sum_{j=1}^{\infty} (5)^{-j} = \frac{|a_{k(i)}(w)|}{2}.
\]

Lemma 3. Let \( F(w) \) be holomorphic in \( |w - w_0| < R \). If for some \( p \),
\[
|F^{(p)}(w)| \geq m > 0 \quad \text{and} \quad \sup_{|w - w_0| < R} |F^{(p)}(w)| = M, \text{then the image of } |w - w_0| < R \text{ under } F \text{ covers the disk}
\]
\[
\{z: |z - F(w_0)| < K_p R^p m^p + 1, M - |F^{(p)}(w)|\}
\]
where \( K_p \) is a positive constant depending on \( p \) only [1].
We infer from Lemma 2 and Lemma 3 the following

**Lemma 4.** If the function $f$ of Theorem 1 has no zero in the sector $|\arg z| \leq \pi/\nu_0$, $|z| < 1$, then there exist positive constants $U_1$, $u_1$, and $L$, depending on $q$ only, such that $|f(z)| > L|c_{k(i)}|$ in

$$S_i: \exp(-U_1/n_{k(i)}) < |z| < \exp(-u_1/n_{k(i)}),$$

$$|\arg z| < \pi/\nu \quad (\nu = 2\nu_0).$$

Here $\{k(i)\}$ is the sequence defined by $|c_{k(i)}| = d_{k(i)}$.

We next estimate the size of the set of points where

$$|f(z)| \left(\frac{1}{2} \left|c_0\right|^2 + \sum_{k=1}^{\infty} \left|c_k\right|^2 |z|^{2n_k}\right)^{-\frac{1}{2}}$$

is bounded away from zero. The following result is due to R. Salem and A. Zygmund. The basic idea of the proof can be found in [4]. Define

$$\Lambda(r) = \left\{\frac{1}{2} \left|c_0\right|^2 + \sum_{k=1}^{\infty} \left|c_k\right|^2 |r|^{2n_k}\right\}^{\frac{1}{2}}.$$

**Lemma 5.** If $f$ satisfies the conditions of Theorem 1, then, in any measurable subset $E \subset [0, 2\pi]$, the linear measure

$$m\{\theta \in E \mid |f(re^{i\theta})|\Lambda(r)^{-1} \leq y\}$$

tends to $(m(E)/2\pi)\int_0^{2\pi} e^{-r^2/2} dr = m(E)(1-e^{-y^2/2})$ as $r \to 1$.

**Lemma 6.** For any measurable subset $E \subset [0, 2\pi]$, and any positive $\delta < 1$, there is $r_0$ such that whenever $r \geq r_0$,

$$m\{\theta \in E \mid |f(re^{i\theta})|\Lambda^{-1}(r) > \delta\} \geq m(E)(1-\delta).$$

**Proof.** By Lemma 5, for $r < 1$,

$$m\{\theta \in E \mid |f(re^{i\theta})|\Lambda^{-1}(r) \leq y\} = m(E) - \{\theta \in E \mid |f(re^{i\theta})|\Lambda^{-1}(r) > y\} \to m(E)e^{-1/2}ey^2 \quad (r \to 1).$$

Set $\gamma = \delta$. Since $\exp(-\delta^2/2) > 1 - \delta^2/2 > 1 - \delta$, (8) is proved.

IV. Lower bounds for $I_1(R) + I_2(R)$. In the following derivations, we shall use letters $K_1$, $K_2$, $K_3$, ... for positive constants which depend on $f$, $t$ and $v$, but not on $R$.

With the notations of Lemma 1,

$$I_2(R, t, v) \geq \int_{-\pi/2}^{\pi/2} \log^+ |f(Re^{i\theta})|\xi_2 d\theta - \int_{-\pi/v}^{\pi/v} \log^+ |1/(Re^{i\theta})|\xi_2 d\theta.$$
In the first integral of the right-hand side $\xi_2 \geq K_1$ for all $R$ sufficiently close to 1. Choose $\delta$ in the interval $0 < \delta < 1/2$. By Lemma 6, if $R \in S_i$ ($i \geq i_0$), $\log^+|f| \geq \log A(R) + \log \delta$ in a subset of measure $> \pi/2\nu$ of $(-\pi/2\nu, \pi/2\nu)$.

In the second integral $0 \leq \xi_2 \leq K_2$. By Lemma 6, $\log^+|1/(Re^{i\theta})| = 0$ outside a set of $\theta$ of measure $< K_3 \delta$. In this set, by Lemma 4, $\log^+|1/(Re^{i\theta})| < -\log (L|c_{k(i)}|)$.

Therefore, for all large $i$ and $R \in S_i$,

$$I_2 \geq K_4 \log A(R) + K_4 \log \delta + K_3 \delta \log |c_{k(i)}| - K_6$$

(9)

Next we find a lower bound for $I_1(R, t, \nu)$. From Lemma 1,

$$I_1 = 2\pi \int_0^R \xi_1(R, r, t, \nu) \left\{ \frac{1}{2\pi} \int_{-\pi/\nu}^{\pi/\nu} \log |(re^{i\theta})| d\theta \right\} dr$$

and we see that $\xi_1$ satisfies $0 \leq \xi_1 \leq K_8(R - r)$. By the first fundamental theorem of Nevanlinna,

$$\frac{1}{2\pi} \int_{-\pi/\nu}^{\pi/\nu} \log^+ |(re^{i\theta})| d\theta \leq T(r, f) = m(r, f).$$

By the inequality of the arithmetic and geometric mean

$$m(r, f) \leq K_9 \log A(r) \leq K_9 \log A(R) \quad (r \leq R).$$

Therefore, if $0 < s < R$,

$$I_1 \geq 2\pi \int_0^s \xi_1 \left\{ \frac{1}{2\pi} \int_{-\pi/\nu}^{\pi/\nu} \log |(re^{i\theta})| d\theta \right\} dr$$

(10)

$$- 2\pi \int_s^R \xi_1 \left\{ \frac{1}{2\pi} \int_{-\pi/\nu}^{\pi/\nu} \log^+ |(re^{i\theta})| d\theta \right\} dr$$

$$\geq D(s) - K_{10} \int_s^R (R - r)A(R) dr \geq D(s) - K_{11}(R - s)^2 A(R).$$

By choosing $s$ sufficiently close to 1, we can make

$$K_7 - K_{11}(R - s)^2 > K_7 - K_{11}(1 - s)^2 > \frac{1}{2} K_7.$$

Combining (9) and (10)

$$I_1 + I_2 \geq D(s_0) + \frac{1}{2} K_7 \log A(R) + K_5 \delta \log |c_{k(i)}| + K_4 \log \delta \quad (s_0 < R).$$

Since $R \in S_i$, $A(R) \geq K_2 \sum_{k=0}^{\infty} |c_k|^2$, and thus

$$I_1(R) + I_2(R) \geq K_{13} \left\{ \log \left( \delta \sum_{k=0}^{\infty} |c_k|^2 \right) + \log \delta \right\}.$$

To show that
ON THE ZEROS OF POWER SERIES WITH HADAMARD GAPS

(11) \[ \limsup_{R \to 1} [I_1(R) + I_2(R)] = \infty \]

it is therefore enough to show that for some \( \delta \),

(12) \[ \limsup_{i \to \infty} |c_{k(i)}|^\delta \left( \sum_{k=0}^i |c_k|^2 \right) = \infty. \]

We prove first that if \( 0 < \delta < \epsilon/2 \) where \( \epsilon \) is the exponent of condition (iii) of Theorem 1, then

\[ W(\delta) = \limsup_{p \to \infty} |c_p|^\delta \left( \sum_{k=0}^p |c_k|^2 \right) \]

is infinite.

Suppose \( W(\delta) < \infty \), then for some \( K > 0 \), and all \( c_p \) with \( |c_p| < 1 \),

(13) \[ |c_p|^{2+\epsilon} < |c_p|^{2+2\delta} \leq K|c_p|^2 \left( \sum_{k=0}^p |c_k|^2 \right)^2. \]

Summing (13) over \( p \),

\[ \sum_{p=0}^\infty |c_p|^{2+\epsilon} \leq K \sum_{p=0}^\infty \left\{ |c_p|^2 \left( \sum_{k=0}^p |c_k|^2 \right)^2 \right\}. \]

The left-hand side of the inequality is infinite by assumption. The right-hand side is finite by a well-known theorem on divergent series, stating that if \( a_n > 0 \), and \( \sum_{n=0}^\infty a_n = \infty \), then for any positive \( \rho \),

\[ \sum_{p=0}^\infty \left\{ a_p \left( \sum_{n=0}^p a_n \right)^{1+\rho} \right\} < \infty. \]

\( W(\delta) \) must therefore be infinite.

Let \( S = |c_p|^\delta \left( \sum_{k=0}^p |c_k|^2 \right) \). We now prove (12) by showing that for at least one of the members of the sequence \( \{k(i)\} \) which are closest to \( p \),

\[ |c_{k(i)}|^\delta \left( \sum_{k=0}^i |c_k|^2 \right) > \frac{25}{3}. \]

The case \( p \in \{k(i)\} \) is trivial. Suppose that \( K < p \), and \( K' > p \) are the two members of \( \{k(i)\} \) which are closest to \( p \). If, for some \( k \) in \( K < k \leq p \), \( d_k = |c_k| \)

\( (l > k) \), then \( l \in \{k(i)\} \), and by the definition of \( K \) and \( K' \), we must have \( l = K' \)

and \( |c_p| < |c_{K'}| \), so that
The only other possibility is that $d_k = \frac{1}{2} d_{k-1}$ ($K < k < p$) and so $|c_k| \leq 2^{-k+K} d_k = 2^{-k+K} |c_K|$, 

$$
\sum_{k=0}^{p} |c_k|^2 \leq \sum_{k=0}^{K} |c_k|^2 + |c_K|^2 \left( \frac{1}{4} + \frac{1}{4} + \cdots \right),
$$

so 

$$
\sum_{k=0}^{K} |c_k|^2 \geq \sum_{k=0}^{p} |c_k|^2 \left( 1 - \frac{1}{3} \frac{\sum_{k=0}^{p} |c_k|^2}{\sum_{k=0}^{p} |c_k|^2} \right) \geq \frac{2}{3} \sum_{k=0}^{p} |c_k|^2
$$

if $p$ is so large that $|c_K| < 1$, $\sum_{k=0}^{p} |c_k|^2 > 1$. We have now 

$$
|c_K|^\delta \left( \sum_{k=0}^{K} |c_k|^2 \right) > \frac{2}{3} |c_p|^\delta \left( \sum_{p=0}^{p} |c_k|^2 \right) = \frac{25}{3}.
$$

This proves (11) and completes the proof of Theorem 1.

REFERENCES


DEPARTMENT OF MATHEMATICS, AMERICAN UNIVERSITY, WASHINGTON, D. C. 20016