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ITERATED LIMITS IN $N^*(U^n)$

BY

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ABSTRACT. It is shown that if f is in $N^*(U^n)$, then the iterated limits of f are almost everywhere independent of the order of iteration. In fact, the iterated limit and the radial limit are equal almost everywhere.

1. Let $U^n = \{(z_1, \cdots, z_n) \in \mathbf{C}^n: |z_i| < 1\}$, $T^n = \{(w_1, \cdots, w_n): |w_i| = 1\}$, and let m_n denote normalized Lebesgue measure on T^n. If f is holomorphic on U^n and $0 \leq r < 1$, let $f_r(z_1, \cdots, z_n) = f(rz_1, \cdots, rz_n)$. $N(U^n)$ consists of those f holomorphic on U^n for which

$$\sup_{0<r<1} \int_{T^n} \log^+ |f_r| \, dm_n < \infty.$$

An $f \in N(U^n)$ is said to be in $N^*(U^n)$ if the family $\{\log^+ |f_r|\}$ is uniformly integrable [2]. An $f \in N(U^n)$ is in $N_k(U^n)$ if

$$\sup_{0<r<1} \int_{T^n} \log^+|f_r|(\log^+ \log^+|f_r|)^k \, dm_n < \infty.$$

Let ϕ be strongly convex [2]. Then an f, holomorphic on U^n, is in $H_\phi(U^n)$ if

$$\sup_{0<r<1} \int_{T^n} \phi(\log^+|f_r|) \, dm_n < \infty.$$

Equivalently, $\phi(\log^+ |f|)$ has an n-harmonic majorant in U^n. Then we have: $N^*(U^n)$ is the union of all the $H_\phi(U^n)$ and $N_k(U^n) \subsetneq N^*(U^n)$, [2]. $A(U^n)$ consists of those f holomorphic on U^n and continuous on the closure of U^n.

For $f \in N(U^n)$ we let $f^*(w) = \lim_{r \to 1} f(rw_1, \cdots, rw_n)$, for $w \in T^n$ (see [2]). Then f^* is measurable on T^n and $\log^+ |f^*| \in L^1(T^n)$. If $f \in N(U^n)$ and $w \in T$, let $f^{w_1}(z_2, \cdots, z_n) = \lim_{r \to 1} f(rw_1, z_2, \cdots, z_n)$, whenever this limit defines a holomorphic function on U^{n-1}. Zygmund in [4] proves:

1.1 **Theorem.** *If $f \in N(U^n)$, then for almost all $w_1 \in T$, $f^{w_1} \in N(U^{n-1})$.*

Similar theorems hold for H_ϕ and N^*. For a $w_1 \in T$ satisfying 1.1, we may then consider

$$f^{w_1 w_2}(z_3, \cdots, z_n) = \lim_{r \to 1} f^{w_1}(rw_2, z_3, \cdots, z_n) \in N(U^{n-2}),$$

for almost all $w_2 \in T$. Continuing in this manner we may then consider

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$$f^{w_1 \cdots w_n} = \lim_{r \to 1} f^{w_1 \cdots w_{n-1}}(rw_n),$$

whenever this limit exists. Zygmund in [4] proves

1.2 **Theorem.** *If* $f \in N_{n-1}(U^n)$, *then* $f^{w_1 \cdots w_n} = f^*(w)$ *for almost all* $w \in T^n$.

Zygmund [4] and Calderón and Zygmund [1] pose the question whether $N_{n-1}(U^n)$ may be replaced by $N(U^n)$ in 1.2. This paper shows that $N_{n-1}(U^n)$ may be replaced by $N^*(U^n)$.

In the next two sections, the following fact will be used repeatedly: If $f \in N(U^n)$ $(H_\phi(U^n), N^*(U^n))$ and $z \in U^{n-k}$ $(1 \leq k \leq n-1)$, then $f(\cdot, z) \in N(U^k)$ $(H_\phi(U^k), N^*(U^k))$. This is seen most easily by using the n-harmonic majorant form of the definitions for N, H_ϕ, and N^*.

2. In this section we show that $f^{w_1 \cdots w_n}$ exists for almost all $w \in T^n$, if $f \in N(U^n)$. This material (in slightly different form) may be found in [4] and [5, Chapter 17]. We include it here for two reasons. First, it makes this paper self-contained. Second, it leads to certain measurability questions which arise in extending 1.2 to $N^*(U^n)$.

We assume 1.1. A proof similar to the proof of 3.4 gives 1.1.

Let $H(k, n)$ and $M(n)$ denote the following:

$H(k, n)$: *If* $f \in N(U^n)$, *then* $f^{w_1 \cdots w_k} \in N(U^{n-k})$ *for almost all* $(w_1, \cdots, w_k) \in T^k$ $(1 \leq k \leq n-1)$.

$M(n)$: *If* $f \in N(U^n)$, *then* $f^{w_1 \cdots w_n}$ *exists (finitely) for almost all* $w \in T^n$, *and the function* $F(w) = f^{w_1 \cdots w_n}$ *is a measurable function on* T^n.

2.1 **Lemma.** *If* $H(n-1, n)$ *and* $M(n-1)$ *hold, then* $M(n)$ *holds.*

Proof. Let $f \in N(U^n)$. $H(n-1, n)$ says that there exists $A \subset T^{n-1}$, $m_{n-1}(A) = 1$, such that $f^{w_1 \cdots w_{n-1}} \in N(U)$ for $w \in A$. Therefore, for $w \in A$, there exists $B(w) \subset T$, $m_1(B(w)) = 1$, such that if $w_n \in B(w)$, then $F(w, w_n)$ exists. It follows that if the set of definition for F is measurable, it must have measure 1. We will show that this set is measurable and that, on this set, F is measurable. The lemma follows.

Write $f = u + iv$. If $\lambda \in U$ is fixed, $f(\cdot, \lambda) \in N(U^{n-1})$. Thus $u^{w_1 \cdots w_{n-1}}(\lambda)$ is a continuous function of λ (for $w \in A$ fixed) and a measurable function on A (and hence on T^{n-1}), for $\lambda \in U$ fixed. It follows that the two functions

$$\overline{U}(w, w_n) = \lim_{k \to \infty} \sup_{r > 1 - 1/k} u^{w_1 \cdots w_{n-1}}(rw_n)$$

and

$$\underline{U}(w, w_n) = \lim_{k \to \infty} \inf_{r > 1 - 1/k} u^{w_1 \cdots w_{n-1}}(rw_n)$$

are measurable on T^n (the supremum and infimum need only be taken over a countable dense set). Similarly for $\overline{V}$ and $\underline{V}$. Hence the sets $\{\overline{U} = \underline{U}\}$, $\{\overline{V} = \underline{V}\}$, $\{\underline{U} = \infty$ or $\overline{U} = -\infty\}$, and $\{\underline{V} = \infty$ or $\overline{V} = -\infty\}$ are measurable. It follows that the set of definition for F is measurable; and since $F = \overline{U} + i\overline{V}$ on this set, F is measurable there.

2.2 **Lemma.** *If* $H(k-1, n)$ *and* $M(k-1)$ *hold for some* k, $2 \le k \le n-1$, *then* $m_k(A) = 1$; *where* $A \subset T^k$ *is defined by*

$$w \in A \text{ if and only if } f^{w_1 \cdots w_{k-1}} \in N(U^{n-k+1})$$
$$\text{and } \{f^{w_1 \cdots w_{k-1}}(rw_k, \cdot)\} \text{ is a normal family}$$
$$\text{on } U^{n-k} \text{ (indexed by } r,\ 0 < r < 1).$$

Proof. For $z \in U^{n-k}$ fixed and $0 \le r < 1$ fixed, it is easy to see that $f^{w_1 \cdots w_{k-1}}(rw_k, z)$ is a measurable function on T^k. Let $\{z_j : j = 1, 2, \cdots\} \subset U^{n-k}$ be dense. Let $\{r_i : i = 1, 2, \cdots\} \subset [0, 1)$ be dense. Let $\{D_l : l = 1, 2, \cdots\}$ be an expanding collection of compact subsets of U^{n-k} such that $U^{n-k} = \bigcup_l D_l$. Define (for $l = 1, 2, \cdots$, and $m = 1, 2, \cdots$):

$$A(l, m) = \left\{ w \in T^k : f^{w_1 \cdots w_{k-1}} \in N(U^{n-k+1}),\ \sup_{i; z_j \in D_l} |f^{w_1 \cdots w_{k-1}}(r_i w_k, z_j)| < m \right\}.$$

Then $A(l, m)$ is measurable, hence

$$A = \bigcap_l \bigcup_m A(l, m)$$

is measurable. Theorem 1.1 and considerations similar to those in 2.1 show that $m_k(A) = 1$ (since $f^{w_1 \cdots w_k} \in N(U^{n-k})$ implies $w \in A$).

2.3 **Lemma.** *If* $H(k-1, n)$ *and* $M(k-1)$ *hold for some* k, $2 \le k \le n-1$, *then* $f^{w_1 \cdots w_k}$ *is holomorphic on* U^{n-k} *for almost all* $w \in T^k$.

Proof. Let $\{z_j : j = 1, 2, \cdots\} \subset U^{n-k}$ be dense. By 2.1 there exists $C_j \subset T^k$, $m_k(C_j) = 1$, such that $f^{w_1 \cdots w_k}(z_j)$ exists for $w \in C_j$. Let $C = \bigcap_j C_j$ and $D = A \cap C$. Then $m_k(D) = 1$. We claim that $w \in D$ implies that $f^{w_1 \cdots w_k}$ is holomorphic on U^{n-k}. Let $w \in D$, $r_m \to 1$, $z_0 \in U^{n-k}$. There exists $z_{l(j)} \to z_0$. Since $w \in A$, there exists a subsequence $\{r_{p(m)}\}$ such that

$$g(\cdot) = \lim_{m \to \infty} f^{w_1 \cdots w_{k-1}}(r_{p(m)} w_k, \cdot)$$

is holomorphic on U^{n-k}. Let $\eta_j = f^{w_1 \cdots w_k}(z_{l(j)})$ (since $w \in C$) and $\eta_0 = g(z_0)$. Then $\eta_j \to \eta_0$. Since η_j is independent of $\{r_m\}$ (again since $w \in C$), we see that $\lim_{r \to 1} f^{w_1 \cdots w_{k-1}}(rw_k, z_0)$ exists. Since $w \in A$, we see that $f^{w_1 \cdots w_k}$ is holomorphic on U^{n-k}.

2.4 **Lemma.** *If* $f \in N(U^n)$ *and* $f^{w_1 \cdots w_k}$ *is holomorphic on* U^{n-k} *for almost* *all* $w \in T^k$, *then* $f^{w_1 \cdots w_k} \in N(U^{n-k})$ *for almost all* $w \in T^k$.

Proof. If $f \in N(U^n)$, n-harmonic majorant considerations show that there exists $M < \infty$ such that

$$\int_{T^n} \log^+|f(r_1 w_1, \cdots, r_k w_k, rw_{k+1}, \cdots, rw_n)|\, dm_n < M,$$

for $0 \le r_i < 1$ $(1 \le i \le k)$ and $0 \le r < 1$. Fubini's theorem and Fatou's lemma (applied several times) give

$$\int_{T^k} dm_k \int_{T^{n-k}} \log^+|f^{w_1 \cdots w_k}(rw_{k+1}, \cdots, rw_n)|\, dm_{n-k} \le M,$$

for $0 \le r < 1$. Since $f^{w_1 \cdots w_k}$ is holomorphic for almost all $w \in T^k$, the inner integral is a nondecreasing function of r almost everywhere on T^k. Lebesgue's monotone convergence theorem then gives

$$\int_{T^k} \left(\lim_{r \to 1} \int_{T^{n-k}} \log^+|(f^{w_1 \cdots w_k})_r|\, dm_{n-k} \right) dm_k \le M.$$

Therefore

$$\lim_{r \to 1} \int_{T^{n-k}} \log^+|(f^{w_1 \cdots w_k})_r|\, dm_{n-k} < \infty,$$

for almost all $w \in T^k$.

2.5 **Theorem (Zygmund).** *If* $f \in N(U^n)$, *then* $f^{w_1 \cdots w_{n-1}} \in N(U)$ *for almost all* $(w_1, \cdots, w_{n-1}) \in T^{n-1}$, *and* $f^{w_1 \cdots w_n}$ *exists for almost all* $w \in T^n$.

Proof. We will first establish the following statement by induction on k: $H(k, n)$ and $M(k)$ hold for $1 \le k \le n-1$, For $k = 1$, this is 1.1 and the fact that $f \in N(U)$ implies f^* measurable and $\log^+|f^*| \in L^1(T)$. Assume the statement for $k-1$. Clearly if $H(k-1, n)$ holds, then $H(k-1, k)$ holds. Lemma 2.1 then gives that $M(k)$ holds. Lemmas 2.3 and 2.4 give that $H(k, n)$ holds. Q.E.D. Lemma 2.1 and the case $k = n-1$ in the above then give the theorem.

2.6 **Theorem.** *If* $f \in H_\phi(U^n)$, *then* $f^{w_1 \cdots w_{n-1}} \in H_\phi(U)$ *for almost all* $w \in T^{n-1}$

Proof. The proof is an obvious extension of the ideas in 2.4 and 2.5.

3. Before establishing the main result of this paper, we must first prove some lemmas. The first two lemmas endow $N^*(U^n)$ with a quasi-norm structure, [3], in such a manner that $A(U^n)$ is dense. This fact is exploited to prove the theorem.

Definition. For $f \in N^*(U^n)$, let

$$d_n(f) = \lim_{r \to 1} \int_{T^n} \log(1 + |f_r|)\, dm_n.$$

d_n is well defined, since $\log(1 + |f|)$ is n-subharmonic.

3.1 **Lemma.** $d_n(f) = \int_{T^n} \log(1 + |f^*|)\, dm_n$.

3.2 **Lemma.** $\lim_{r \to 1} d_n(f_r - f) = 0$.

Lemma 3.1 is a straightforward application of the definition of a uniformly integrable family and Egoroff's theorem (note that $f \in N^*(U^n)$ if and only if $\{\log(1 + |f_r|)\}$ is a uniformly integrable family). Lemma 3.2 uses 3.1, uniform integrability, and Egoroff's theorem. Also note that $d_n(f+g) \le d_n(f) + d_n(g)$, because $\log(1 + x + y) \le \log(1+x) + \log(1+y)$, if $x \ge 0$ and $y \ge 0$.

Lemma 3.2 shows that $A(U^n)$ is d_n-dense in $N^*(U^n)$. In what follows, it is helpful to view $d_n(f)$ in three ways:

(i) $d_n(f) = \sup_{0 < r < 1} \int_{T^n} \log(1 + |f_r|)\, dm_n$.

(ii) $d_n(f)$ is the value at the origin of the least n-harmonic majorant of $\log(1 + |f|)$.

(iii) $d_n(f) = \sup \int_{T^n} \log(1 + |f(r_1 w_1, \cdots, r_n w_n)|)\, dm_n(w)$, where the supremum is taken over all $(r_1, \cdots, r_n)$ for which $0 \le r_i < 1$ $(1 \le i \le n)$. ((iii) follows from (ii).)

3.3 **Lemma.** *If $h_k \in N^*(U^n)$ and $\sum_{k=1}^{\infty} d_n(h_k) = M < \infty$, then*

$$\sum_{k=1}^{\infty} |h_k^*(w)| < \infty \tag{1}$$

for almost all $w \in T^n$. In particular, for almost all $w \in T^n$, $\lim_{k \to \infty} h_k^*(w) = 0$.

Proof. We have

$$\int_{T^n} \log\left(1 + \sum_{k=1}^{m} |h_k^*|\right) dm_n \le \int_{T^n} \sum_{k=1}^{m} \log(1 + |h_k^*|)\, dm_n = \sum_{k=1}^{m} d_n(h_k) \le M.$$

By Lebesgue's monotone convergence theorem,

$$\int_{T^n} \log\left(1 + \sum_{k=1}^{\infty} |h_k^*|\right) dm_n \le M.$$

This last integrand must be finite almost everywhere. Thus (1) holds.

3.4 **Lemma.** *Let $f \in N^*(U^n)$. Then there exists $A \subset T^{n-1}$, $m_{n-1}(A) = 1$, such that the functions f_w defined by $f_w(\lambda) = \lim_{r \to 1} f(rw_1, \cdots, rw_{n-1}, \lambda)$, $\lambda \in U$, are in $N^*(U)$ for $w \in A$.*

Proof. For $w \in T^{n-1}$, let $g(w) = \int_T \log^+ |f^*(w, w_n)|\, dm_1(w_n)$. Let

$$u(z_1, \cdots, z_{n-1}) = P_{n-1}[g](z_1, \cdots, z_{n-1}),$$

where $P_{n-1}[\cdot]$ denotes the Poisson integral (see [2]). Since $g \in L^1(T^{n-1})$, there exists $B \subset T^{n-1}$, $m_{n-1}(B) = 1$, such that if $w \in B$, $\lim_{r \to 1} u(rw)$ exists, [2, p. 24]. Now

$$\log^+|f(z_1, \cdots, z_n)| \le P_n[\log^+|f^*|](z_1, \cdots, z_n).$$

Therefore if $|z_n| < r_0 < 1$ and $w \in B$,

$$\log^+|f(rw, z_n)| \le K(w)(1/(1 - r_0)),$$

where $K(w)$ is a constant depending on w. Hence $\{f(rw, \cdot)\}$ is a normal family on U (indexed by r, $w \in B$ fixed). Then f_w will be holomorphic if for some dense set $\{\lambda_k : k = 1, 2, \cdots\} \subset U$, $\lim_{r \to 1} f(rw, \lambda_k)$ exists. Now for k fixed, $f(\cdot, \lambda_k) \in N^*(U^{n-1})$. Thus there exists $C_k \subset T^{n-1}$, $m_{n-1}(C_k) = 1$, such that if $w \in C_k$, $\lim_{r \to 1} f(rw, \lambda_k)$ exists. Then if $C = B \cap (\bigcap_k C_k)$, $m_{n-1}(C) = 1$, and $\lim_{r \to 1} f(rw, \lambda_k)$ exists for all k, if $w \in C$. Therefore f_w is holomorphic on U for $w \in C$. Then if $f \in H_\phi(U^n)$, an argument similar to the proof of 2.4 gives $f_w \in H_\phi(U)$ for $w \in A \subset T^{n-1}$, $m_{n-1}(A) = 1$. The lemma follows.

3.5 **Lemma.** *If $f \in N^*(U^n)$, then $G(w, w_n) = \lim_{r \to 1} f_w(rw_n)$ exists (finitely) almost everywhere, and G is a measurable function on T^n.*

(Note that $G(w, w_n) = (f_w)^*(w_n)$.)

Proof. Similar to 2.1.

3.6 **Lemma.** *With A as in 3.4, we have*

$$\int_A d_1(f_w)\, dm_{n-1}(w) \le d_n(f). \tag{2}$$

Proof. We have for $0 \le r < 1$ and $0 \le p < 1$,

$$\int_A dm_{n-1}(w) \int_T \log(1 + |f(rw, pw_n)|)\, dm_1(w_n) \le d_n(f).$$

For fixed (w, w_n) and p, the integrand converges to $\log(1 + |f_w(pw_n)|)$, as $r \to 1$; so that Fatou's lemma gives

$$\int_A dm_{n-1}(w) \int_T \log(1 + |f_w(pw_n)|)\, dm_1(w_n) \le d_n(f).$$

As $p \to 1$, the inner integral converges to $d_1(f_w)$. Another application of Fatou's lemma gives (2).

3.7 **Lemma.** *Suppose that if $f \in N^*(U^{n-1})$, then*

$$f^{\omega_1 \cdots \omega_{n-1}} = f^*(w_1, \cdots, w_{n-1})$$

for almost all $(w_1, \cdots, w_{n-1}) \in T^{n-1}$. Then if $f \in N^*(U^n)$, we have $f_w \in N^*(U)$ and $f_w(\lambda) = f^{w_1 \cdots w_{n-1}}(\lambda)$, $\lambda \in U$, for almost all $w \in T^{n-1}$.

Proof. Let $\{\lambda_k : k = 1, 2, \cdots\} \subset U$ be dense. Then for each k, there exists $A_k \subset T^{n-1}$, $m_{n-1}(A_k) = 1$, such that $f_w(\lambda_k) = f^{w_1 \cdots w_{n-1}}(\lambda_k)$ for $w \in A_k$ (since $f(\cdot, \lambda_k) \in N^*(U^{n-1})$). By 2.6, there exists $B \subset T^{n-1}$, $m_{n-1}(B) = 1$, such that $w \in B$ implies $f^{w_1 \cdots w_{n-1}} \in N^*(U)$. Let A be as in 3.4. Let $C = A \cap B \cap (\bigcap_k A_k)$. Then $m_{n-1}(C) = 1$, and $f_w(\lambda_k) = f^{w_1 \cdots w_{n-1}}(\lambda_k)$ for all k, if $w \in C$.

The continuity of f_w and $f^{w_1 \cdots w_{n-1}}$ for $w \in C$ now gives the lemma.

3.8 **Theorem.** *If $f \in N^*(U^n)$, then for almost all $w \in T^n$, $f^{w_1 \cdots w_n} = f^*(w)$.*

Proof. By induction on n. For $n = 1$ there is nothing to do. Assume the theorem for $N^*(U^{n-1})$. Lemma 3.7 and this assumption give a set $A \subset T^{n-1}$, $m_{n-1}(A) = 1$, such that $f_w \in N^*(U)$ and $f_w = f^{w_1 \cdots w_{n-1}}$, for $w = (w_1, \cdots, w_{n-1}) \in A$.

Choose $g_k \in A(U^n)$ such that

$$d_n(g_k - f) < 2^{-k}, \qquad k = 1, 2, 3, \cdots. \tag{3}$$

Lemma 3.3 (with $g_k - f$ in place of h_k) and (3) imply that

$$\lim_{k\to\infty} g_k(w') = f^*(w'), \tag{4}$$

for almost all $w' \in T^n$. Lemma 3.6 and (3) give

$$\int_A d_1((g_k)_w - f_w)\, dm_{n-1}(w) < 2^{-k}, \qquad k = 1, 2, 3, \cdots.$$

Hence there is a set $A' \subset A$, $m_{n-1}(A') = 1$, such that

$$\sum_{k=1}^{\infty} d_1((g_k)_w - f_w) < \infty, \tag{5}$$

for $w \in A'$. Now apply 3.3 again (with $n = 1$) to conclude from (5) that

$$\lim_{k\to\infty} g_k(w, w_n) = (f_w)^*(w_n), \tag{6}$$

if $w \in A'$ and $w_n \in B(w) \subset T$, $m_1(B(w)) = 1$. Since $A' \subset A$, (6) and the definition of $(f_w)^*$ say that

$$\lim_{k\to\infty} g_k(w, w_n) = f^{w_1 \cdots w_n}, \tag{7}$$

for $w = (w_1, \cdots, w_{n-1}) \in A'$, $w_n \in B(w)$. Statements (4) and (7) thus give the existence of a set $A_0 \subset T^{n-1}$, $m_{n-1}(A_0) = 1$, and a set $B_0(w)$, $m_1(B_0(w)) = 1$, for each $w \in A_0$, such that

$$f^{w_1 \cdots w_n} = f^*(w, w_n) \tag{8}$$

for $w = (w_1, \cdots, w_{n-1}) \in A_0$ and $w_n \in B_0(w)$. (We may assume $A_0 \subset A'$ and $B_0(w) \subset B(w)$.)

Lemma 3.5 and the induction hypothesis show that $(w_1, \cdots, w_n) \to f^{w_1 \cdots w_n}$ is a measurable function on T^n. Hence the set $C = \{(w, w_n) \in T^n : f^{w_1 \cdots w_n} = f^*(w, w_n)\}$ is measurable. Since, from (8),

$$\bigcup_{w \in A_0} \{(w, w_n) : w_n \in B_0(w)\} \subset C,$$

$m_n(C) = 1$. This is the theorem.

4. If z_1 approaches w_1 nontangentially, [1] and [4] show that if $f \in N(U^n)$, then

$$\lim_{z_1 \to w_1} f(z_1, \cdots, z_n) = f^{w_1}(z_2, \cdots, z_n),$$

for almost all $w_1 \in T$. Similarly

$$\lim_{z_n \to w_n} \cdots \lim_{z_1 \to w_1} f(z_1, \cdots, z_n) = f^{w_1 \cdots w_n},$$

for almost all $w \in T^n$, where $z_j \to w_j$ nontangentially $(1 \leq j \leq n)$. Therefore we have

4.1 Corollary. *If $f \in N^*(U^n)$,*

$$\lim_{z_n \to w_n} \cdots \lim_{z_1 \to w_1} f(z_1, \cdots, z_n) = f^*(w_1, \cdots, w_n)$$

for almost all $(w_1, \cdots, w_n) \in T^n$, when $z_j \to w_j$ nontangentially $(1 \leq j \leq n)$.

The following is an immediate consequence of 3.8.

4.2 Corollary. *The value of the iterated limit of a function in $N^*(U^n)$ is almost everywhere independent of the order of iteration.*

d_n is actually a quasi-norm on $N^*(U^n)$, and $N^*(U^n)$ is complete under d_n. This topological algebra is the subject of the author's thesis. The author appreciates the encouragement of his thesis advisor, Walter Rudin.

d_n is not a quasi-norm on $N(U^n)$, for scalar multiplication fails to be continuous (even when $n = 1$, as may be seen by considering the function $\exp((1 + z)/(1-z)))$. The above proof is not applicable to $N(U^n)$, for both 3.1 and 3.2 fail to hold.

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