ALGEBRAIC COHOMOLOGY OF TOPOLOGICAL GROUPS

BY

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ABSTRACT. A general cohomology theory for topological groups is described, and shown to coincide with the theories of C. C. Moore [12] and other authors. We also recover some invariants from algebraic topology.

This article contains proofs of results announced in [15]. We consider algebraic cohomology groups of topological groups, which are shown to include the invariants considered by Van Est [6], Hochschild and Mostow [7], C. C. Moore [12], and Tate (see [5]). We identify some of these groups as invariants familiar from algebraic topology.

Let G be a topological group. A topological G-module is an abelian topological group A together with a continuous map $G \times A \to A$ satisfying the usual relations $g(a + a') = ga + ga'$, $(gg')a = g(g'a)$, $1a = a$. The category of topological G-modules and equivariant continuous homomorphisms is a quasi-abelian category in the sense of Yoneda [16], and hence we get Ext functors just as in an abelian category. A proper short exact sequence will be a sequence $0 \to A \to B \to C \to 0$ of topological G-modules which is exact as a sequence of abstract groups and such that $A$ has the subspace topology induced by its embedding in $B$, and such that $u$ be an open map. For any G-module $A$ we define the algebraic cohomology groups $H^i(G, A)$ to be the $i$th Ext group $\text{Ext}^i(Z, A)$, where $Z$ denotes the group of integers with the discrete topology and trivial G-action.

There is another set of short exact sequences we might have chosen which also give the category of topological G-modules the structure of a quasi-abelian S-category in the sense of Yoneda. We might have demanded that in addition to being exact in the previous sense, there be a continuous map $s : C \to B$ such that the composition $u \circ s$ be the identity on $C$. If $G$ is locally compact we recover the "continuous cochains" theory, which is discussed in [5], [6], and [7]. If $G$ is not locally compact it must be shown that continuous cochains are effaceable, i.e. that for any continuous cocycle $c : G^n \to A$ there is a short exact sequence $0 \to A \to B \to C \to 0$ such that $\tau \circ c$ is the coboundary of a
continuous cochain $c': G^{n-1} \to B$. If $G$ has the weak topology with respect to a countable collection of compact sets, this will follow from a lemma of Milnor [11].

In this paper we consider only complete metric $G$-modules. This is made plausible by a theorem of L. Brown, [2] that if $C$ and $A$ are complete metric $G$-modules, then the groups $\text{Ext}^n(C, A)$ do not depend on whether we consider all, all pseudometrizable, or all complete metric $G$-modules, provided that $G$ is weakly separable (i.e. that any uniform cover of $G$ has a countable subcover). Furthermore our arguments also apply to the category of complete separable metric $G$-modules, hence to the functors of [12].

1. Definition of the $H^i(G, A)$. (See [16], also [9, Chapter 12, 5].) Let $M$ be an additive category (with direct sums) and $\phi: A \to B$ be a map in $M$. A map $N \to A$ is called the kernel of $\phi$ if the induced sequence of abelian groups $0 \to \text{Hom}(C, N) \to \text{Hom}(C, A) \to \text{Hom}(C, B)$ is exact for any object $C$ of $M$. Dually a map $B \to L$ is called the cokernel of $\phi$ if the sequence

$$0 \to \text{Hom}(L, C) \to \text{Hom}(B, C) \to \text{Hom}(A, C)$$

is exact for any object $C$ of $M$. This implies that the compositions $N \to A \to B$ and $A \to B \to L$ are 0.

Definition. A sequence $0 \to A \xrightarrow{\sigma} B \xrightarrow{\tau} C \xrightarrow{0}$ of maps in $M$ is called proper exact if $\sigma$ is the kernel of $\tau$ and $\tau$ is the cokernel of $\sigma$. An $n$-term long exact sequence in $M$ is a sequence of short exact sequences

$$0 \rightarrow A_i \rightarrow B_i \rightarrow C_i \rightarrow 0,$$

such that $C_i = A_{i+1}$ for $1 \leq i < n$. It will usually be written

$$0 \rightarrow A_1 \xrightarrow{\sigma_1} B_1 \xrightarrow{\rho_1} B_2 \cdots \xrightarrow{\rho_{n-1}} B_n \xrightarrow{\tau_n} C_n \rightarrow 0$$

where $\rho_i = \sigma_{i+1} \circ \tau_i$. Yoneda defines $\text{EXT}^n(C, A)$ as the set of $n$-term long exact sequences with $A_1 = A$, $C_n = C$.

Definition (Yoneda). An additive category is called quasi-abelian if it satisfies the following conditions (Q) and (Q*):

- (Q) Any proper exact sequences $0 \to A \to B' \to C' \to 0$ and $0 \to C \to C'$ can be combined into a commutative diagram with proper exact rows and columns:

$$\begin{array}{ccc}
0 & 0 \\
\downarrow & \downarrow \\
0 & A & B & C & 0 \\
\downarrow & \downarrow & \downarrow \\
D & D & D & D \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0
\end{array}$$

(Diagram Q)
Any proper exact sequences $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ and $A \rightarrow A' \rightarrow 0$
can be combined into a commutative diagram with proper exact rows and columns:

$\begin{array}{c}
0 & 0 \\
\downarrow & \downarrow \\
D = D & \\
\downarrow & \\
0 & 0
\end{array}$

(Diagram Q*)

A quasi-abelian $S$-category is an additive category with a distinguished subset $S$ of proper exact sequences which satisfy $Q$ and $Q^*$. As an example we have the category of all abelian topological groups and all proper maps thereof; in this case a map is proper if and only if it is open with respect to the relative topology of its range. In Diagram Q, $C$ is a closed subgroup of $C'$, $B$ is its inverse image in $B'$ which is again a closed subgroup. Since $B \supset A$ we have $B/B' \cong C/C'$ which is $D$. This verifies (Q). In Diagram $Q^*$, $D$ is the kernel of $A \rightarrow A'$, $B' \cong B/D$, and $A \supset D$, we have $B'/A' \cong B/A \cong C$. Also for any fixed Hausdorff topological group $G$ one can consider the category $\mathbb{M}_G$ of $G$-modules, complete metrizable abelian topological groups $A$ with continuous action $G \times A \rightarrow A$ satisfying $1a = a$, $(g^r g')a = g(g'a)$ and $g(a + a') = ga + ga'$ and continuous equivariant homomorphisms. As with abelian topological groups the totality of all proper maps gives $\mathbb{M}_G$ the structure of a quasi-abelian $S$-category and henceforth $\mathbb{M}_G$ will be assumed to be equipped with this structure. In a quasi-abelian category Yoneda defines functors $\text{Ext}^n(C, A)$ as a certain quotient of $\text{EXT}^n(C, A)$, the set of $n$-term long exact sequences. Let $0 \rightarrow A \rightarrow B_1 \rightarrow \cdots \rightarrow B_n \rightarrow C \rightarrow 0$ and $0 \rightarrow A \rightarrow B_1' \rightarrow \cdots \rightarrow B_n' \rightarrow C \rightarrow 0$
be elements of $\text{EXT}^n(C, A)$. We say there is a map between them if there exists a commutative diagram

$\begin{array}{c}
0 & \rightarrow & A & \rightarrow & B_1 & \rightarrow & \cdots & \rightarrow & B_n & \rightarrow & C & \rightarrow & 0 \\
\| & & \downarrow & & \downarrow & & \| & & \downarrow & & \| \\
0 & \rightarrow & A & \rightarrow & B_1' & \rightarrow & \cdots & \rightarrow & B_n' & \rightarrow & C & \rightarrow & 0
\end{array}$

$\text{Ext}^n(C, A)$ is defined as the quotient of $\text{EXT}^n(C, A)$ under the equivalence relation generated by maps between long exact sequences.

If $A$ is a $G$-module, we define $H^n(G, A)$ to be $\text{Ext}^n_{\mathbb{M}_G}(Z, A)$, where $Z$ is the group of integers with the discrete topology and trivial $G$-action.

It follows from Yoneda's work that if $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a proper
exact sequence of topological $G$-modules, we have a long exact sequence

$$0 \rightarrow H^0(G, A) \rightarrow H^0(G, B) \rightarrow H^0(G, C) \rightarrow H^1(G, A) \rightarrow H^1(G, B) \rightarrow H^1(G, C) \rightarrow H^2(G, A) \rightarrow \ldots.$$ 

We can then complete a diagram chase to show the $H^i(G, A)$ are universal functors [4] and prove a "Buchsbaum criterion" for the $H^i(G, A)$. Namely an exact connected sequence of functors $\widetilde{H}^i(G, A)$ is naturally isomorphic to the $H^i(G, A)$ if $\widetilde{H}^0(G, A) \cong H^0(G, A)$ and satisfies the following condition:

For $i > 0$ and $X \in \widetilde{H}^i(A)$ there exists a proper monomorphism $\theta: A \rightarrow B$ such that $\theta_* (X) = 0$. It follows immediately from Buchsbaum's criterion and results of C. C. Moore [12] that the functors of [12] coincide with the $H^i(G, A)$ described above.

Henceforward let $G$ be locally compact $\sigma$-compact and let $\mathbb{M}_G$ be the category of complete metric $G$-modules. If $A$ is a $G$-module let $C^n(G, A)$ be the set of continuous maps of the $n$-fold cartesian product $G^n$ into $A$. Let $\delta_n: C^n(G, A) \rightarrow C^{n+1}(G, A)$ be the usual coboundary operator:

$$\delta_n(f(g_0, \ldots, g_n)) = g_0 f(g_1 \ldots g_n) - f(g_0, g_2, \ldots, g_n) + \cdots \pm f(g_0, \ldots, g_{n-1}).$$

Define $\tilde{H}^n(G, A)$ as the $n$th cohomology group of the complex $0 \rightarrow C^0(G, A) \rightarrow C^1(G, A) \rightarrow \cdots \rightarrow C^n(G, A) \rightarrow A$ are the continuous functions from $G^0 = \text{point}$ into $A$. $\delta_0 a = ga - a$ so $\tilde{H}^0(G, A) \cong \text{Hom}_{\mathbb{M}_G}(Z, A) \cong H^0(G, A)$. If $F(G, A) \in \mathbb{M}_G$ is the module of continuous functions from $G$ into $A$ topologized with the compact open topology, the natural map $A \rightarrow F(G, A)$ kills $\tilde{H}^0(G, A)$ (cf. [7]). The $\tilde{H}^i$ form an exact connected sequence of functors if we demand that all short exact sequences $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ have a section, i.e. a continuous map $\rho: C \rightarrow B$ such that $\pi \circ \rho = \text{identity}$. We call this the "continuous cochains" theory.

Now suppose $G$ is zero-dimensional. Then the $\tilde{H}^i(G, A)$ are exact for arbitrary short exact sequences because of the following theorem of Michael:

**Theorem M.** If $\pi: B \rightarrow C$ is an open homomorphism of complete metric topological groups, and if $q: G \rightarrow C$ is a continuous map of a 0-dimensional paracompact space into $C$, then there exists a continuous map $\rho: G \rightarrow B$ such that $\pi \circ \rho = q$.

Hence by Buchsbaum's criterion

**Theorem 1.** If $G$ is locally compact, $\sigma$-compact, zero-dimensional, $H^i(G, A) \cong \tilde{H}^i(G, A)$ defined above.

We now show how to embed an arbitrary complete metric $G$-module in a contractible complete metric $G$-module. Let $A$ be a complete metric $G$-module with a bounded, invariant metric $\rho$. Let $S$ be the topological group of step functions from the unit interval $[0, 1]$ to $A$ which have only finitely many steps with metric obtained from integrating $\rho$ on $[0, 1]$ and natural $G$ action. $G \times S \rightarrow S$ is con-
tinuous since the functions of $S$ assume only finitely many values. Let $\mathcal{G}_A$ be the completion of $S$ which is also a $G$-module by [2] or [12]. $\mathcal{G}_A$ will be the space measurable functions $[0, 1] \to A$ modulo functions almost everywhere 0. Let $C: \mathcal{G}_A \times [0, 1]$ be defined by

$$C(f, \alpha)(x) = 0, \quad \text{if } x < \alpha,$$

$$= f(x), \quad \text{if } x \geq \alpha.$$ 

$C$ is a contraction of $\mathcal{G}_A$ which shrinks all distances; hence $\mathcal{G}_A$ is contractible and locally contractible. In fact any contractible topological group is locally contractible.

2. Some fibration properties of open homomorphisms.

Lemma 1. Let $0 \to A \to B \xrightarrow{p} C \to 0$ be an exact sequence of complete metric abelian groups with $A$ locally arcwise connected. Let $PB$ (respectively $PC$) denote the space of continuous paths in $B$ (respectively $C$) starting at the identity with the topology of uniform convergence. Then the induced map $p_* : PB \to PC$ is open.

Proof. Since $PB$ and $PC$ are complete metric abelian topological groups, it will be enough to show $p_*$ almost open by the open mapping theorem. Let $d$ be an invariant metric on $B$. $d$ induces an invariant metric $d'$ on $C$ by taking the distance between cosets of $A$. Let $\epsilon > 0$; we must show there exists a $\delta$ such that for any path in $C$, $p: [0, 1] \to C$ such that for all $x \in [0, 1], d(p(x), id) < \delta$ and for all $y > 0$ there is a path in $B$, $q: [0, 1] \to B$ such that for all $y \in [0, 1], d(q(y), id) < \epsilon$ and $d(pq(y), p(y)) < y$. Now $d$ induces a metric on $A$. Pick $\delta < \epsilon/4$ and such that any two points in $A$ at distance $< 4\delta$ of the identity of $A$ can be joined by a path in $A$, all of whose points $s$ satisfy $d(s, id) < \epsilon/4$. Now by a theorem of Michael [11, II, Theorem 1.2], $p$ lifts locally to $q^* : [0, 1] \to \{x \in B | d(x, id) < \delta\} = N$. Since $[0, 1]$ is compact we can assume it covered by a finite number of sub-intervals $I_i = [a_i, b_i], i = 1, \ldots, n$ with $a_1 = 0, b_n = 1, a_i < b_{i-1}, b_i < a_{i+2}$ and $q_i : I_i \to N$ continuous such that $\rho \circ q_i = p | I_i$. Now $d(q_i^*(b_i), q_{i+1}^*(b_i)) < 2\delta$ so there is a path $r_i : [0, 1/2] \to \rho^{-1}(p(b_i))$ with $r_i(0) = q_i^*(b_i), r_i(1/2) = q_{i+1}^*(b_i), d(r_i(x), id) < \epsilon$. Pick $\beta < \min_i (b_i/10, (b_{i+1} - b_i)/10)$ and such that for all $i$ and all $\alpha$ with $0 \leq \alpha \leq \beta, d(p(b_i + \alpha), p(b_i)) < \gamma/2$.

Define $q$ as follows: for

$$0 \leq x \leq b_i, \quad q(x) = q_i^*(x),$$

$$b_i + \beta \leq x \leq b_{i+1}, \quad q(x) = q_{i+1}^*(x),$$

$$b_i \leq x \leq b_i + \frac{1}{2}\beta, \quad q(x) = r_i((x - b_i)/\beta),$$

$$b_i + \frac{1}{2}\beta \leq x \leq b_i + \beta, \quad q(x) = q_{i+1}^*(b_i + (2(x - b_i)/\beta - 1)\beta).$$
It is clear that \( q \) has the required properties. The idea of this construction is to splice the \( q_j \)'s together without going far from the origin. This proves the lemma.

**Definition.** A complete metric abelian topological group \( A \) is said to have property \( F \) if for any short exact sequence of complete metric abelian topological groups \( 0 \to A \overset{\sigma}{\to} B \overset{\tau}{\to} C \to 0 \), \( \tau \) has the homotopy lifting property for finite dimensional (paracompact) spaces. Dimension will be understood in the sense of Lebesgue covering dimension. \( \mathcal{M}_G^F \) will denote the category of complete metric \( G \text{-modules} \) having property \( F \), where a sequence is exact if it is exact in \( \mathcal{M}_G \).

**Proposition 1.** Let \( 0 \to A \to B \to C \to 0 \) be exact in \( \mathcal{M}_G \) where \( A, C \) have property \( F \). Then \( B \) has property \( F \).

**Proof.** Let \( 0 \to A \to B \to C \to 0 \) in \( \mathcal{M}_G \) where \( A \) and \( C \) have property \( F \). Let also \( 0 \to B \to D \overset{\rho}{\to} E \to 0 \) in \( \mathcal{M}_G \). Consider the diagram in \( \mathcal{M}_G \).

\[
\begin{array}{c}
\text{0} \\
\downarrow \\
A \\
\downarrow \\
0 \to B \to D \overset{\rho}{\to} E \to 0 \\
\downarrow \\
0 \to C \to C' \overset{\sigma}{\to} E \to 0 \\
\downarrow \\
\text{0} \\
\end{array}
\]

Let \( b: X \times I \to E \) be a homotopy of which property \( F \) would guarantee a lifting. Since \( C \) has property \( F \), \( h \) can be lifted to \( C \). Since \( A \) has property \( F \), \( b \) can be lifted to \( D \). This proves \( B \) has property \( F \).

**Corollary.** \( \mathcal{M}_G^F \) is a quasi-abelian \( S \)-category.

**Proposition 2.** If \( A \) is a locally compact closed subgroup of a topological group \( G \) the projection \( G \to G/A \) is a fibration.

**Proof.** First suppose \( A \) compact. Let \( h \) be a homotopy of \( X \times I \to G/A \) and \( b_1 \) be a lifting \( X \times I \to G \). Consider the set \( S \) of pairs \( (A_{a}, b_{a}) \) where \( A_{a} \) is closed in \( A \), \( \pi_{a} : G \to G/A_{a}, b_{a} : X \times I \to G/A_{a}, \pi_{a} \circ b_{a} = b, \pi_{a} \circ b_1 = b_{a} \mid X \times I \). We define a partial order on \( S \). If \( A_{a} \subseteq A_{b}, \pi_{b} : G/A_{a} \to G/A_{b} \) and \( \pi_{a} \circ b_{a} = b_{b} \) we say \( (A_{a}, b_{a}) \geq (A_{b}, b_{b}) \). If \( \{ (A_{\gamma}, b_{\gamma}) \}_{\gamma \in I} \) is a linearly ordered subset of \( S \) we obtain

\[
\bar{b} : X \times I \to \lim_{\gamma \in I} A_{\gamma} = \bigcap_{\gamma \in I} A_{\gamma}
\]

and \( \{ (A_{\gamma}, b_{\gamma}) \} \) is an upper bound. Hence Zorn's lemma applies, and \( S \) has
(Aₜ, bₜ) maximal. But if Aₜ ≠ {1}, Aₜ has a proper closed subgroup Aₜ ≠ Aₜ such that
Aₜ/Aₜ is a Lie group. Hence G/Aₜ → G/Aₜ has a local section and is a fibration, hence (Aₜ, bₜ) cannot have been maximal. Hence Aₜ = {1}. This shows
G → G/A has a homotopy lifting property for A compact. But by the structure
theorem any locally compact A has an open subgroup A' such that A' has a
compact normal subgroup A" such that A'/A" is a Lie group. G → G/A" is a fibration. Since A'/A" is a Lie group G/A" → G/A' is a fibration by [14, The-
orem 1]. A/A' is discrete so G/A' → G/A is even a covering space. Since
G → G/A is a composite of fibrations it is a fibration.

Corollary. A locally arcwise compact metric G-module is in Mₚ².

Proposition 3. A locally connected complete metric abelian topological
group has property F.

Proof. Let PX denote the space of base-pointed paths of X. Consider the
diagram

```
0 → PA → PΣ_A^φ → PΣ_A/A → 0
  ↓   ↓ψ   ↓χ
0 → A → Σ_A^r → Σ_A/A → 0
```

The top row is exact by Lemma 1 and φ has the homotopy lifting property for
finite dimensional spaces since PA is locally contractible by Michael [10, The-
orem 3.4, Proposition 4.1 and Corollary 4.2]. Let Z be finite dimensional, b:
Z × I → Σ_A/A, b': Z → Σ_A with r ∘ b' = h|Z × 0. ψ is a fibration with con-
tractible base so it has a section s: Σ_A → PΣ_A. X ∘ φ has the HLP for Z
since both X and φ do, hence there exists g: Z × I → PΣ_A with g|Z × 0 =
s ∘ b', and X ∘ φ ∘ g = h, X ∘ g is a lifting of h to Σ_A by the commutativity
of the diagram. This shows that r has the HLP for Z.

We form the diagram

```
0 → 0
↓   ↓
0 → A σ→ B φ→ C → 0
|φ| |φ'|
0 → Σ_A σ'→ P φ'→ C → 0
|r| 1rr'
Σ_A/A = Σ_A/A
↓   0
0 → 0
```

Let b: X × I → C, b': X → B with b' = h|X × 0 and X finite dimensional.
Since $\mathcal{E}_A$ is locally contractible, $\rho'$ has the homotopy lifting property for finite dimensional spaces again by Theorem 3.4 of [10] so there exists $g: X \times I \to P$ with $\rho' \circ g = b$ and $g|X \times 0 = \phi' \circ b'$. Since $r' \circ \phi' \circ b' = 0$ there exists $f: X \times I \to \mathcal{E}_A$ with $r \circ f = g$ and $f|X \times 0 = 0$. Since $r$ has the HLP for $X$, $r' \circ (g - \sigma' \circ f) = 0$ so the range of $g - \sigma' \circ f$ lies entirely in $B$. Hence $\phi'^{-1} \circ (g - \sigma' \circ f)$ is defined and lifts $b$ as required. This proves the proposition.

**Proposition 4.** If $A, C$ are in $\mathcal{M}_G^F$, $\text{Ext}_{\mathcal{M}_G^F}(C, A) \cong \text{Ext}_{\mathcal{M}_G^C}(C, A)$.

**Proof.** Consider

$$
\begin{array}{c}
0 \to A \to B \\
\| \\
0 \to A \to \mathcal{E}_B \to \mathcal{E}_B/A \to 0
\end{array}
$$

with $A \in \mathcal{M}_G^F$ and $B \in \mathcal{M}_G^C$. $\mathcal{E}_B$ is locally arcwise connected, hence $\mathcal{E}_B/A$ is locally arcwise connected and in $\mathcal{M}_G^C$. Hence anything which is effaceable in $\mathcal{M}_G$ is effaceable in $\mathcal{M}_G^F$ and Buchsbaum's criterion is verified.

3. **Double complex.** We now assign to the topological group $G$ a semisimplicial $G$-space $S(G)$. $S(G)$ is a semisimplicial object in the category of topological spaces with jointly continuous action of the group $G$ and equivariant maps. The $n$-simplex $S_n$ of this semisimplicial complex was the $(n + 1)$-fold cartesian power $G^{n+1}$ of the space underlying the group $G$, and the faces and degeneracies were as follows:

- $d_0 g(g_1, g_2, \ldots, g_n) = gg_1(g_2, \ldots, g_n)$,
- $d_i g(g_1, \ldots, g_n) = g(g_1, \ldots, g_{i-1}, g_i, \ldots, g_n)$ for $0 < i < n$,
- $d_n g(g_1, \ldots, g_n) = g(g_1, \ldots, g_{n-1})$,
- $s_i g(g_1, \ldots, g_n) = g(g_1, \ldots, g_{i-1}, 1, g_i, \ldots, g_n)$.

$G$ acts by left multiplication on the argument outside the parenthesis.

Let $A$ be a $G$-module. Using the action of $G$ on $S_n$ and $A$ we form the space $S_n \times_G A$ and consider the natural projections $p_n: S_n \times_G A \to S_n/G$. The faces and degeneracies of $S(G)$ induce faces and degeneracies on the $S_n \times_G A$ and on the $S_n/G$ making them into semisimplicial spaces and these faces and degeneracies commute with the natural projections $p_n$. Let $T_n$ be the sheaf of germs of continuous sections of $p_n$. Since the identity of $A$ is fixed by $G$, there is an isomorphism of $T_n$ with the sheaf of germs of continuous $A$-valued functions on $S_n/G$. The $T_n$ have faces and degeneracies induced by the faces and degeneracies of $S(G)$. The $T_n$ thus form a semisimplicial sheaf $T(G, A)$ over the $S_n/G$, i.e. a semisimplicial object in the category of spaces with sheaves and...
We apply the canonical semisimplicial resolution functor \([1, \text{Chapter II}]\) to the semisimplicial sheaf \(T(G, A)\). We then get a double complex of abelian groups, \(D^{p,q}(G, A) = \mathcal{J}^p(S_q/G, T_q)\) the \(p\)th stage of the canonical semisimplicial resolution of the sheaf \(T_q\) over \(S_q/G\). We denote the \(p\)th cohomology group of this double complex by \(H^{p,q}(G, A)\).

Associated to \(D^{p,q}\) is a spectral sequence with \(E_1\) term \(E_1^{p,q} = H^p(S_q/G, T_q)\), the sheaf cohomology of \(S_q/G\) with coefficient sheaf \(T_q\). Since \(S_0/G\) is a point, \(E_1^{0,0}\) is the abstract group underlying \(A\). If \(z \in A\), \(d_1(a) \in H^0(S_1/G, T_1)\) is a continuous function from \(S_1/G \cong G\) into \(A\). In fact \(d_1(a)\) maps \(g\) into \(ga - a\), hence we see that \(H^0(G, A) \cong A^G \cong \hat{H}^0(G, A)\) where \(A^G\) is the abstract group of points of \(A\) fixed by \(G\).

Now suppose \(G\) is finite dimensional. \(G\) is then locally \(Z \times N\) where \(Z\) is a simplex and \(N\) is 0-dimensional. Now let \(0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0\) be a short exact sequence in \(\mathcal{M}_G^F\). We will show \(r_+ : D^{p,q}(G, B) \rightarrow D^{p,q}(G, C)\) is surjective. If \(q = 1\) and \(I\) is a germ of a continuous map of \(G\) into \(C\), \(I\) can be represented by a continuous map \(l : Z \times N \rightarrow C\) where \(N\) is 0-dimensional and \(Z\) is a simplex. If \(z \in Z\), \(l_1 z \times N\) can be lifted by Theorem \(M\). But \(Z\) is contractible hence the lifting \(\tilde{t}\) such that \(r \circ \tilde{t} = \tilde{t}\) is guaranteed by property \(F\). Now \(D^{p,q}(G, \cdot)\) is easily seen to be left exact on \(\mathcal{J}^G\) hence exact on \(\mathcal{M}_G\). We conclude that \(H(G, \cdot)\) is an exact connected sequence of functors on \(\mathcal{M}_G^F\).

To prove effaceability we first consider the proper injection \(A \rightarrow \mathcal{E}_A\). Since \(\mathcal{E}_A\) is contractible we have by \([4, \text{Lemma 4}]\) that \(E_1^{p,q}(G, \mathcal{E}_A) = 0\) for \(p > 0\). Hence \(\hat{H}^*(G, \mathcal{E}_A)\) is given by the complex of continuous cochains. Since \(G\) is locally compact continuous cochains are effaceable, and it follows that continuous cochains are effaceable in \(\mathcal{M}_G^F\). We have verified Buchsbaum's criterion for the \(\hat{H}^*(G, A)\). Therefore:

**Theorem 2.** If \(G\) is locally compact, \(\sigma\)-compact, finite dimensional and \(A\) has property \(F\), \(H^*(G, A) \cong \hat{H}^*(G, A)\) described above.

4. **Spectral sequence.** In this section all groups will be finite dimensional, locally compact, \(\sigma\)-compact and all modules will be in \(\mathcal{M}_G^F\).

If \(A\) is a vector space the spectral sequence collapses from \(E_2\) onward and we get:

**Theorem 3.** \(H^*(G, A)\) is given by the complex of continuous cochains if \(A\) is a vector group.

**Corollary.** If \(G\) is a connected Lie group \(H^*(G, A) \cong H^*(G, K, A)\) the Lie algebra cohomology of \(G\) modulo the Lie algebra of a maximal compact subgroup, if \(A\) is a finite dimensional vector space on which \(G\) acts linearly and differentiably.
Proof. Hochchild and Mostow [7] have shown $H^*(G, A)$ is given by continuous cochains in this case.

Now let $A$ be a discrete $G$-module. We will see that the algebraic cohomology $H^*(G, A)$ coincides with the sheaf cohomology of the classifying space. Let $\pi: E_G \rightarrow B_G$ be a principal universal $G$-bundle with paracompact base. There is a semisimplicial $G$-space whose $n$-simplex is the $(n+1)$-fold fiber product $F_n$ of $E_G$ over $B_G$, by regarding the $(n+1)$-fold fiber product as the set of maps of $[0, 1, \ldots, n]$ into $E_G$ whose range is contained in a single $G$-orbit, $G$ acts on $E_G \times_{B_G} E_G \times_{B_G} \cdots \times_{B_G} E_G$ by the diagonal action. Consider the sheaves of germs of continuous sections of the associated bundles $F_n \times_G A \rightarrow F_n / G$. They form a semisimplicial sheaf and by applying the canonical semisimplicial resolution functor we get a double complex which we denote by $R^{p,q}$. The injection of $G$ into the fiber of $\pi$ induces a homomorphism $R^{p,q} \rightarrow D^{p,q}(G, A)$. This induces a map from the first spectral sequence of the double complex $R^{p,q}$ into the spectral sequence described in the last section. On the $E_1$ terms we get the map:

$$0 \rightarrow H^*(E_G, A) \rightarrow H^*(E_G \times_{B_G} E_G, A) \rightarrow \cdots$$

But

$$0 \rightarrow H^*(\text{point}, A) \rightarrow H^*(G, A) \rightarrow \cdots$$

is homeomorphic to $E_G \times_G \cdots \times_G G$ which is homotopy equivalent to $G \times \cdots \times G$. Therefore by the homotopy axiom for sheaf cohomology with constant coefficients [2] we have an isomorphism of $E_1$ terms. Hence the $E_{\infty}$ terms coincide.

Now for each point $x$ of $B_G$ pick a section $s_x: B_G \rightarrow E_G$ which is continuous in some neighborhood of $x$. For an $n$-tuple $(e_1, \ldots, e_n)$ in $E_G \times_{B_G} \cdots \times_{B_G} E_G$ with $\pi(e_i) = b$ define $k_x: F_n \rightarrow F_{n+1}$ by $k_x(e_1, \ldots, e_n) = (s_x(b), e_1, \ldots, e_n)$. Now an element of $R^{p,q}$ is represented by a function $f$: $(F^q_q)^{p+1} \rightarrow A$ so define $b: R^{p,q} \rightarrow R^{p,q-1}$ by $b_f(X_0, \ldots, X_p) = f(k_b(X_0), k_b(X_1), \ldots, k_b(X_p))$ where $b = \pi(X_0)$. $b$ is well-defined since $s_b$ is continuous in a neighborhood of $b$. Let $d: R^{p,q} \rightarrow R^{p,q+1}$ be induced by the space map. $d$ is then the 0th differential of the second spectral sequence of the double complex $R^{p,q}$. $db + bd = \text{id}$ unless $q = 0$. The kernel of $d$ on $R^{p,0}$ consists just of functions constant on the $G$-orbits of $E_G$. Hence the $E_1$ term of the second spectral sequence of $R^{p,q}$ is the canonical resolution of the locally constant sheaf $A$ on $B_G$. Therefore
Theorem 4. $H^*(G, A)$ is the sheaf cohomology of the classifying space $B_G$ with coefficients in the locally constant sheaf $A$, if $A$ is a discrete $G$-module.

BIBLIOGRAPHY


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