GENERALIZED DEDEKIND ETA-FUNCTIONS
AND GENERALIZED DEDEKIND SUMS(1)

BY
BRUCE C. BERNDT

ABSTRACT. A transformation formula under modular substitutions is derived
for a very large class of generalized Eisenstein series. The result also gives a
transformation formula for generalized Dedekind eta-functions. Various types of
Dedekind sums arise, and reciprocity laws are established.

1. Introduction. In [12], J. Lewittes proved transformation formulae for the
analytic continuation of a very large class of Eisenstein series. From another
viewpoint, these results give transformation formulae for a large class of functions
which generalize the classical Dedekind eta-function $\eta(z)$. However, the formulae
[12, Theorem 3, equation (51)] are so complicated that even in the simplest case
of the Dedekind eta-function it is exceedingly difficult to deduce the usual trans-
formation formulae in terms of Dedekind sums.

Our objective here is to take Lewittes’ proof and give a different account of
the last parts of his proof. Our new proof will yield more elegant transformation
formulae in which Dedekind sums or various generalizations of Dedekind sums
appear. From this new version of Lewittes’ theorem, we shall show that results
of several other authors are special cases. In addition, we shall deduce several
new results as well.

Firstly, we easily deduce the transformation formulae of Dedekind’s eta-func-
tion. Secondly, we derive transformation formulae for a large class of functions
which generalize $\eta(z)$. This class includes those functions studied by C. Meyer
[14], U. Dieter [6], and B. Schoeneberg [17] and which are connected with F. Klein’s
functions [10]. Appearing in our transformation formulae are the generalized Dede-
kind sums $s(h, k; x, y)$ first defined by H. Rademacher in 1964. It is interesting
to observe that in the work of Meyer [14], [15], Dieter [6], and Schoeneberg [17],
only rational values of the parameters $x$ and $y$ appear, while in our generalization

Received by the editors March 8, 1972.


Key words and phrases. Eisenstein series, Dedekind eta-function, generalized Dedekind eta-function, Dedekind sum, generalized Dedekind sum, Bernoulli numbers and polynomials, reciprocity law, Lambert series, modular transformation.

(1) Research partially supported by NSF grant GP21335.
B. C. BERNDT

x and y may assume any real values. We next show how the reciprocity theorem for generalized Dedekind sums \[16\] may be derived from the transformation formulae. Another special case of our general theorem is the result of T. Apostol [2] wherein the transformation formulae of certain Lambert series are derived. S. Iseki [9] considered a more general class of functions than Apostol but derived a transformation formula only in the special case \(V(z) = -1/z\). However, we shall easily derive the transformation formula for any modular substitution. Another generalization of the classical Dedekind sums appears in Apostol's formulae, and L. Carlitz ([4], [5]) derived a reciprocity theorem for these sums. Still another generalization of Dedekind sums appears in our transformation formulae from which we derive a reciprocity theorem which includes Carlitz's as a special case.

2. An improved version of Lewittes' theorem. We review some notation from Lewittes' paper [12]. Put \(z = x + iy\) and \(s = \sigma + it\) with \(x, y, \sigma\) and \(t\) real. For any complex number \(w\), we choose that branch of \(\log w\) with \(-\pi < \arg w < \pi\). Let \(V(z) = Vz = (az + b)/(cz + d)\) be an arbitrary modular transformation. Let \(r_1\) and \(r_2\) be arbitrary real numbers, and define \(R_1\) and \(R_2\) by

\[
R_1 = ar_1 + cr_2 \quad \text{and} \quad R_2 = br_1 + dr_2.
\]

Let \(Z\) denote the ring of rational integers. Let \(\mathbb{H}\) denote the upper half-plane \(\{z: y > 0\}\). For \(z \in \mathbb{H}\) and \(\sigma > 2\), define the Eisenstein series \(G(z, s, r_1, r_2)\) by

\[
G(z, s, r_1, r_2) = \sum_{m, n} ((m + r_1)z + n + r_2)^{-s},
\]

where \(m\) and \(n\) range over all pairs of integers except for the possible pair \(m = -r_1, n = -r_2\). For \(z \in \mathbb{H}\) and arbitrary \(s\), define the following generalization of Dedekind's eta-function by

\[
A(z, s, r_1, r_2) = \sum_{m > -r_1} \sum_{k=1}^{\infty} k^{s-1} e^{2\pi i kr_2 + 2\pi i k(m + r_1)z}.
\]

Put \(H(z, s, r_1, r_2) = A(z, s, r_1, r_2) + e^{\pi is} A(z, s, -r_1, -r_2)\). For \(\alpha\) real and \(\sigma > 1\), Lewittes defined \(\zeta(s, \alpha)\) by

\[
\zeta(s, \alpha) = \sum_{n > -\alpha} (n + \alpha)^{-s}.
\]

Observe that \(\zeta(s, \alpha) = \zeta(s, \lfloor\alpha\rfloor + \chi(\alpha))\), where \(\lfloor\alpha\rfloor\) denotes the fractional part of \(\alpha\), and \(\chi(\alpha)\) denotes the characteristic function of the integers. These two latter notations will be repeatedly used in the sequel. Since \(0 < \lfloor\alpha\rfloor + \chi(\alpha) \leq 1\), \(\zeta(s, \lfloor\alpha\rfloor + \chi(\alpha))\) denotes the classical Hurwitz zeta-function.

**Theorem 1.** Assume that \(c > 0\). Let \(Q = \{z: x > -d/c\}\) and \(y > 0\). Let \(p = p(R_1, R_2, c, d) = |R_2|c - |R_1|d\). Then \(G(z, s, r_1, r_2)\) can be analytically continued to the entire complex \(s\)-plane, and for \(z \in Q\) and all \(s\),
\[(cz + d)^{-s} G(Vz, s, r_1, r_2)\]

\[= G(z, s, R_1, R_2) - 2i \sin(\pi s) \chi(R_1) \zeta(s, -R_2) + \frac{e^{-\pi is}}{\Gamma(s)} L(z, s, R_1, R_2, c, d),\]

where
\[L(z, s, R_1, R_2, c, d)\]

\[= \sum_{j=1}^{c} \int_0^\infty u^{s-1} \frac{e^{-(cz+d)(j-\lfloor R_1 \rfloor)u/c}}{e^{-(cz+d)u-1}} \frac{e^{(jd+\rho)/c}u}{e^{u-1}} du,\]

where \(C\) is a loop beginning at \(+\infty\), proceeding in the upper half-plane, encircling the origin in the positive direction so that \(u = 0\) is the only zero of \((e^{-(cz+d)u} - 1)\).

\((e^u - 1)\) lying "inside" the loop, and then returning to \(+\infty\) in the lower half-plane. Here we choose the branch of \(u^s\) with \(0 < \arg u < 2\pi\).

**Proof.** From (26) and (29) of [12], we have for \(z \in \mathbb{Q}\) and \(\sigma > 2\),

\[(cz + d)^{-s} G(Vz, s, r_1, r_2)\]

\[= G(z, s, R_1, R_2) - 2i \sin(\pi s) \chi(R_1) \zeta(s, -R_2) + \Gamma(s) K(z, s, R_1, R_2, c, d),\]

where
\[\Gamma(s) K(z, s, R_1, R_2, c, d)\]

\[= \sum_{m > R_1} \sum_{n > R_2 + (m - R_1)d/c} \int_0^\infty u^{s-1} e^{-(m-R_1)zu-(n-R_2)u} du.\]

If we put \(m' = m - \lfloor R_1 \rfloor - 1\) and \(n' = n - \lfloor R_2 + (m - R_1)d/c \rfloor - 1\), then the above becomes

\[\sum_{m' = 0}^\infty \sum_{n' = 0}^\infty \int_0^\infty u^{s-1} \exp\left(- (m' + 1 - \lfloor R_1 \rfloor)zu - (n' + 1 - \lfloor R_2 \rfloor) + [(m'd + d + \rho)/c]u\right) du.\]

If we put \(m' = qc + j, 0 \leq j \leq c - 1, 0 \leq q < \infty\), and replace \(n'\) by \(n\), we get

\[\sum_{j = 0}^{c-1} \sum_{q = 0}^\infty \sum_{n = 0}^\infty \int_0^\infty u^{s-1} \exp\left(- (qc + j + 1 - \lfloor R_1 \rfloor)zu - (n + 1 - \lfloor R_2 \rfloor + qd + [(jd + d + \rho)/c]u\right) du\]

\[= \sum_{j = 0}^{c-1} \int_0^\infty u^{s-1} \exp\left(- (j + 1 - \lfloor R_1 \rfloor)zu - (1 - \lfloor R_2 \rfloor + [(jd + d + \rho)/c]u\right) du\]

\[\sum_{q = 0}^\infty \sum_{n = 0}^\infty e^{-(cz+d)u-qnu} du\]

\[= \sum_{j = 1}^{c} \int_0^\infty u^{s-1} \frac{\exp\left(- (j - \lfloor R_1 \rfloor)zu - (1 - \lfloor R_2 \rfloor + [(jd + \rho)/c]u\right)}{(1 - e^{-(cz+d)u})(1 - e^{-u})} du\]

\[= \sum_{j = 1}^{c} \int_0^\infty u^{s-1} \frac{e^{-(cz+d)(j-\lfloor R_1 \rfloor)u/c}}{1 - e^{-(cz+d)u}} \frac{e^{(jd+\rho)/c}u}{e^u-1} du,\]
where, since \( x > -d/c \), we have interchanged the order of summation and integration by absolute convergence. By a classical method of Riemann [18, pp. 18—19], the integrals above may be transformed into loop integrals. Letting \( C \) be as defined in the statement of the theorem, we find that

\[
\Gamma(s)K(z, s, R_1, R_2, c, d)
\]

\[
= \sum_{j=1}^{c} \frac{1}{e^{2\pi is} - 1} \int_{C} u^{s-1} \frac{e^{-(cz+d)(j-\{R_1\})u/c}}{1 - e^{-(cz+d)u}} \frac{e^{(jd+p)/c}u}{e^{u} - 1} du.
\]

If we now substitute (4) into (3) and simplify slightly, we arrive at (1) for \( z \in \Omega \) and \( \sigma > 2 \). However, these loop integrals converge uniformly on any compact set in the \( s \)-plane and thus represent entire functions. Hence, by analytic continuation, (1) is valid for \( z \in \Omega \) and for all \( s \).

We now restate Theorem 1 in terms of the function \( H(z, s, r_1, r_2) \). The proof is like that of equation (51) in Lewittes' paper [12], and so we omit the proof.

**Theorem 2.** For \( z \in \Omega \) and all \( s \),

\[
(cz + d)^{-s}H(Vz, s, r_1, r_2)
\]

\[
= H(z, s, R_1, R_2) - \chi(r_1) e^{\pi is} (2\pi i)^{-s} \Gamma(s)(\zeta(s, r_2) + e^{\pi is} \zeta(s, -r_2))
\]

\[
+ \chi(R_1) (2\pi i)^{-s} \Gamma(s)(\zeta(s, -R_2) + e^{\pi is} \zeta(s, R_2))
\]

\[
+ (2\pi i)^{-s} L(z, s, R_1, R_2, c, d),
\]

where \( L(z, s, R_1, R_2, c, d) \) is given by (2).

Theorems 1 and 2 may be considerably simplified if \( s \) is an integer. For then,

\[
L(z, s, R_1, R_2, c, d) = 2\pi i R_0,
\]

where \( R_0 \) is the sum of the residues of the integrands at \( u = 0 \). Upon the substitution of (6) into (1) and (5), Theorems 1 and 2 will then be valid for all \( z \in \mathbb{H} \) by analytic continuation. Thus, put \( s = -m \), where \( m \) is an arbitrary nonnegative integer. Now [1, p. 804],

\[
\frac{ue^{ux}}{e^u - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{u^n}{n!}, \quad |u| < 2\pi,
\]

where \( B_n(x) \) denotes the \( n \)th Bernoulli polynomial. Hence,

\[
u^m \sum_{\nu=0}^{\infty} B_{\nu} \left( \frac{j - \{R_1\}}{c} \right) \left( -\frac{(cz+d)u}{c} \right)^{\nu} \sum_{\mu=0}^{\infty} \frac{\nu!}{\mu!} \left( \frac{jd+p}{c} \right)^{\nu}
\]

\[
u \nu
\]
where \( B_n(x) = B_n(|x|) \). Thus,

\[
R_0 = -\frac{1}{cz + d} \sum_{j=1}^{c} \sum_{\mu + \nu = m + 2} B_{\mu} \left( \frac{j - \{R_1\}}{c} \right) B_{\nu} \left( \frac{jd + \rho}{c} \right) (-cz + d)^{\mu + \nu - 1} \]

(7)

\[
= \frac{1}{(m + 2)!} \sum_{j=1}^{c} \sum_{k=0}^{m+2} \left( \frac{m + 2}{k} \right) B_k \left( \frac{j - \{R_1\}}{c} \right) (-cz + d)^{k-1}. 
\]

3. The transformation formulae for \( \log \eta(z) \). Let \( s = r_1 = r_2 = 0 \). Put \( A(z, 0, 0, 0) = A(z) \) and observe that \( H(z, 0, 0, 0) = 2A(z) \). We find then that (5) and (6) yield

\[
A(Vz) = A(z) + \lim_{s \to 0} (2\pi i)^{-s} \Gamma(s) \zeta(s) (1 - e^{\pi is} (cz + d)^{-s}) + \pi R_0 \\
= A(z) + \frac{1}{2} \pi i - \frac{1}{2} \log (cz + d) + \pi R_0.
\]

Using the product definition of \( \eta(z) \), it is easy to show that \( \log \eta(z) = \frac{\pi iz}{12} - A(z) \). Hence,

\[
\log \eta(Vz) = \log \eta(z) + \frac{m(\psi(z) - \zeta)}{12} - \frac{\pi i}{12} \log (cz + d) - mR_0.
\]

From (7),

\[
R_0 = \sum_{j=1}^{c} \left\{ -\frac{1}{2} B_2(jd/c)(cz + d)^{-1} + B_1(j/c)B_1(jd/c) - \frac{1}{2} B_2(j/c)(cz + d) \right\}. 
\]

Since \( [1, p. 804, Equation 23.1.10] \)

\[
B_n(cx) = c^{n-1} \sum_{j=0}^{c-1} B_n(x + j/c), 
\]

we find that

\[
\sum_{j=1}^{c} B_2(j/c) = \sum_{j=0}^{c-1} B_2(j/c) = B_2/c = 1/6c.
\]

Since \((c, d) = 1 \_c \_d \)

\[
\sum_{j=1}^{c} B_2(jd/c) = \sum_{j=1}^{c} B_2(j/c) = \sum_{j=0}^{c-1} B_2(j/c) = 1/6c.
\]

Lastly,

\[
\sum_{j=1}^{c} B_1(j/c)B_1(jd/c) = \sum_{j=1}^{c} B_1(j/c)B_1(jd/c) + B_1(1)B_1(0) \\
= s(d, c) - 1/4,
\]

since \( B_1(x) = x - \frac{1}{2} \) and

\[
s(d, c) = \sum_{j \mod c} ((j/c)((jd/c)),
\]

where \( B_n(x) = B_n(|x|) \). Thus,
where
\[
((x)) = \begin{cases} 
\lfloor x \rfloor - \frac{1}{2} & \text{if } x \text{ is not an integer}, \\
0 & \text{if } x \text{ is an integer}.
\end{cases}
\]

Putting (11)–(13) into (9), we find that
\[
R_0 = -\frac{1}{12c(cz + d)} - \frac{(cz + d)}{12c} + s(d, c) - \frac{1}{4}.
\]
Hence, from (8) we obtain
\[
\log \eta(Vz) = \log \eta(z) - \frac{mi}{4} + \frac{1}{2} \log (cz + d) - \frac{mi}{c} s(d, c)
\]
\[
+ \frac{(mi/12)}{V(z) - z + (cz + d)/c + 1/c(cz + d)}
\]
\[
= \log \eta(z) - \frac{mi}{4} + \frac{1}{2} \log (cz + d) - \frac{mi}{c} s(d, c) + \frac{mi(a + d)}{12c},
\]
upon the simplification of the expression in brackets with the aid of the fact \(ad - bc = 1\).

4. Transformation formulae for generalized Dedekind eta-functions. Put \(s = 0\) and suppose that \(r_1\) and \(r_2\) are arbitrary. From (7),
\[
R_0 = \sum_{j=1}^{c} \left\{ -\frac{1}{2} B_2 \left( \frac{id + \rho}{c} \right) (cz + d)^{-1}
\right. \\
+ B_1 \left( \frac{j - |R_1|}{c} \right) B_2 \left( \frac{j + \rho}{c} \right) - \frac{1}{2} B_2 \left( \frac{j - |R_1|}{c} \right) (cz + d) \left. \right\}.
\]
Using (10) again, we have
\[
\sum_{j=1}^{c} B_2 \left( \frac{j - |R_1|}{c} \right) = \frac{1}{c} B_2 (1 - |R_1|) = \frac{1}{c} B_2 (|R_1|),
\]
since [1, p. 804, Equation 23.1.8]
\[
B_n (1 - x) = (-1)^n B_n (x).
\]
Since \((c, d) = 1\), \(jd + \lfloor \rho \rfloor\) runs through a complete residue system (mod \(c\)) as \(j\) does, and so by (10),
\[
\sum_{j=1}^{c} B_2 \left( \frac{j + \lfloor \rho \rfloor}{c} \right) = \sum_{j=0}^{c-1} B_2 \left( \frac{j + \lfloor \rho \rfloor}{c} \right) = \frac{1}{c} B_2 (|\rho|).
\]
If \((c, d) = 1\), and \(x\) and \(y\) are arbitrary real numbers, the generalized Dedekind sum \(s(d, c; x, y)\) is defined by [16]
\[
s(d, c; x, y) = \sum_{j \mod c} \left( d \frac{j + y}{c} + x \right) \left( \frac{j + y}{c} \right).
\]

Let \(g(d, c; |R_2|, - |R_1|)\) denote the second sum on the right side of (15). We see that \(g(d, c; |R_2|, - |R_1|)\) is very closely related to \(s(d, c; |R_2|, - |R_1|)\).
Various possibilities occur. If neither \( R_1 \) or \( \rho \) are integers, then

\[
g(d, c; \{ R_2 \}, - \{ R_1 \}) = s(d, c; \{ R_2 \}, - \{ R_1 \})
\]

If \( R_1 \in \mathbb{Z} \), but \( \rho \notin \mathbb{Z} \), then \( g(d, c; \{ R_2 \}, - \{ R_1 \}) \) contains an "extra" term corresponding to \( j = \ell \). Suppose that \( \rho \) is an integer. Let \( j' \) be the unique integer such that \( 1 \leq j' \leq c \) and \( j'd + \rho \equiv 0 \pmod{c} \). Then there is an "extra" term corresponding to \( j = j' \). Unless both \( R_1 \) and \( \rho \) are integers and \( j' = c \), we have in general that

\[
g(d, c; \{ R_2 \}, - \{ R_1 \}) = s(d, c; \{ R_2 \}, - \{ R_1 \})
\]

(19)

\[+ \frac{1}{2} \chi(r_1)B_1'(R_2) - \frac{1}{2} \chi(\rho)B_1(\{j' - \{ R_1 \}\}/c).\]

Now if both \( R_1 \) and \( \rho \) are integers and \( j' = c \), the "extra" terms coincide. In such a case \( R_2 \) must be integral. If \( R_1, R_2 \in \mathbb{Z} \), then \( r_1, r_2 \in \mathbb{Z} \), since \( ad - bc = 1 \). Conversely, if \( r_1, r_2 \in \mathbb{Z} \), then \( R_1, \rho \in \mathbb{Z} \). We see that we are then in the case of \( \log \eta(z) \) discussed earlier. Thus, in the remainder of this section, assume that at least one of the pair \( r_1, r_2 \) is not an integer. Observe that if \( \rho \in \mathbb{Z} \), but \( R_1 \notin \mathbb{Z} \), then

\[
B_1'(\{j' - \{ R_1 \}\}/c) = B_1'(\{j' - \{ R_1 \}\}/c).
\]

Also, from the definition of \( \rho \), it is easily seen that \( \rho \in \mathbb{Z} \) if and only if \( r_1 \notin \mathbb{Z} \).

A short calculation shows that \( j' = ar_1 - \{ R_1 \} \pmod{c} \). Hence, from the definition of \( R_1 \), \( (j' - \{ R_1 \})/c \equiv -r_2 \pmod{1} \). From the above remarks and (17), we conclude that

\[
\chi(\rho)B_1'(\{j' - \{ R_1 \}\}/c) = \chi(r_1)B_1(-r_2) = - \chi(r_1)B_1(r_2).
\]

Finally, putting the above into (19) and then using (16), (18) and (19) in (15), we find that

\[
R_0 = - B_2(\rho)/2c(cz + d) - (cz + d)B_2(R_1)/2c
\]

(20)

\[+ s(d, c; R_2, - R_1) + \frac{1}{2} \chi(r_1)B_1'(R_2) + \frac{1}{2} \chi(r_1)B_1(r_2).
\]

Let

\[
I_m(r_1, r_2, c, d) = \lim_{s \to -m} (2\pi)^{-s}(-\chi(r_1)e^{\pi is}(cz + d)^{-s}\Gamma(s)\zeta(s, r_2) + e^{\pi is}\zeta(s, -r_2))
\]

\[+ \chi(r_1)\Gamma(s)\zeta(s, -r_2) + e^{\pi is}\zeta(s, r_2)).
\]

From (5), we see that we must determine \( I_0(r_1, r_2, c, d) \) for the various possibilities.

Case 1. Let \( r_1 \notin \mathbb{Z}, r_2 \in \mathbb{Z} \). Suppose that \( R_2 \in \mathbb{Z} \). By the definition of \( R_2 \), \( bR_1 \in \mathbb{Z} \). If also \( R_1 \in \mathbb{Z} \), then \( ar_1 \in \mathbb{Z} \). Since \( r_1 \notin \mathbb{Z} \) and \( (a, b) = 1 \), we have a contradiction. Hence, \( R_1 \notin \mathbb{Z} \), and so \( I_0(r_1, r_2, c, d) = 0 \). Assume then that \( R_2 \notin \mathbb{Z} \). Now for \( 0 < \alpha < 1 \) \( \zeta(0, \alpha) = \frac{1}{2} - \alpha \) and \( \zeta'(0, \alpha) = \log \Gamma(\alpha) - \frac{1}{2} \zeta(2, \alpha) \).
\[ f_0(r_1, r_2, c, d) = \lim_{s \to 0} \Gamma(s)(\zeta(s, R_2) + e^{\pi i s} \zeta(s, R_2)) \]
\[ = \lim_{s \to 0} \Gamma(s)(\frac{1}{2} - (1 - \{R_2\}) + (\log \Gamma(1 - \{R_2\}) - \frac{1}{2} \log(2\pi))s + \ldots \]
\[ + \frac{1}{2} - \{R_2\} + (\log \Gamma(\{R_2\}) - \frac{1}{2} \log(2\pi))s + \ldots \]
\[ = \log \Gamma(1 - \{R_2\}) - \log(2\pi) - \frac{1}{2} \log(2\pi) \]
\[ = \log(\pi \sin \pi R_2) - \log(2\pi) \]
\[ = -\log(2\sin \pi R_2 - \pi) \]
\[ = -\log(1 - e^{-2\pi i R_2}) - 2\pi i B_1(R_2). \]

**Case 2.** Let \( r_1 \in \mathbb{Z}, r_2 \notin \mathbb{Z}. \) Suppose that \( R_2 \in \mathbb{Z}. \) By an argument completely analogous to that in Case 1, \( R_1 \notin \mathbb{Z}. \) Proceeding as in Case 1, we find that
\[ f_0(r_1, r_2, c, d) = \log(1 - e^{-2\pi i \{r_2\}}) + 2\pi i B_1(-r_2) \]
\[ - \chi(R_1)(\log(1 - e^{-2\pi i R_2}) + 2\pi i B_1(R_2)) \]
\[ = \log(1 - e^{-2\pi i r_2}) - \chi(R_1)(\log(1 - e^{-2\pi i R_2}) + 2\pi i B_1(R_2)). \]

**Case 3.** Let \( r_1, r_2 \notin \mathbb{Z}. \) Suppose that \( R_2 \in \mathbb{Z}. \) If also \( R_1 \in \mathbb{Z}, \) it follows that \( R_1, R_2 \in \mathbb{Z} \) since \( ad - bc = 1. \) We have a contradiction, and so \( R_1 \notin \mathbb{Z}. \) We see that \( f_0(r_1, r_2, c, d) \) is the same as in Case 1.

In summary, in all cases we have
\[ f_0(r_1, r_2, c, d) = \chi(r_1) \log(1 - e^{-2\pi i r_2}) \]
(21)
\[ - \chi(R_1)(\log(1 - e^{-2\pi i R_2}) + 2\pi i B_1(R_2)). \]

Put
\[ \alpha(r_1, r_2) = \begin{cases} 
\frac{e^{\pi i B_1(r_2)}(1 - e^{-2\pi i r_2})}{1 - e^{-2\pi i R_2} + 2\pi i B_1(R_2)}, & r_1 \in \mathbb{Z}, r_2 \notin \mathbb{Z}, \\
1, & \text{otherwise}.
\end{cases} \]

Combining (20) and (21) with (5), we conclude that
\[ H(Vz, 0, r_1, r_2) = H(z, 0, R_1, R_2) \]
(22)
\[ + 2\pi i \left\{ - \frac{1}{2c(cz + d)} B_2(\rho) - \frac{(cz + d)}{2c} B_2(R_1) + s(d, c; R_2, - R_1) \right\} \]
\[ + \log \alpha(r_1, r_2) - \log \alpha(R_1, R_2). \]

The transformation formulae for \( H(z, 0, r_1, r_2) \) given by (22) appear to be new in general. However, Meyer [14], Dieter [6], and Schoeneberg [17] have derived
the result for a subset of these functions, and their formulae were given implicitly by Hecke [8]. Previously, J. Lehner [11] and J. Livingood [13] had achieved results for special cases of the aforementioned results.

We now justify our claim that the results of Meyer, Dieter, and Schoeneberg are special cases of (22). We shall employ some of the notation in [17].

For \( z \in \mathbb{K} \), \( g, h \in \mathbb{Z} \), and a positive integer \( N \), define

\[
\eta_{g, h}(z; N) = \alpha(g/N, h/N) e^{\pi i z \mathcal{B}_2(g/N)} \prod_{m=1; m \neq g(N)}^{\infty} (1 - e^{2\pi i (b + zm)/N})
\]

(23)

\[
= \prod_{m=1; m \neq -g(N)}^{\infty} (1 - e^{2\pi i (b - zm)/N}).
\]

Let \( g' = ag + ch \), \( b' = bg + dh \), and

\[
\begin{cases}
1, & g \equiv b \equiv 0 \pmod{N}, \\
0, & \text{otherwise}.
\end{cases}
\]

Then,

\[
\log \eta_{g, h}(z; N) = \log \eta_{g', b'}(z; N)
\]

(24)

\[
= \left( \frac{\pi a/c}{2} \right) \mathcal{B}_2(g'/N) + \left( \frac{\pi d/c}{2} \right) \mathcal{B}_2(b'/N) - 2\pi i s(d, c; b'/N, -g'/N) + b' g(N) (\log(cz + d) - \frac{1}{2} \pi i).
\]

To prove (24) we may without loss of generality assume that \( 0 \leq g, b < N \). Furthermore, note that \( \eta_{0,0}(z; N) = \eta^2(z) \), and (24) reduces to (14). Hence, we shall assume that at least one of the pair \( g, b \) is not zero. Taking logarithms of both sides of (23) and then putting, respectively, \( m = jN + g, j > -g/N \), and \( m = jN - g, j > g/N \), in the resulting two sums, we have

\[
\log \eta_{g, b}(z; N) = \log \alpha(g/N, h/N) + \pi i \mathcal{B}_2(g/N)
\]

(25)

\[
- \sum_{j=-g/N}^{\infty} \sum_{k=1}^{\infty} \frac{1}{k} e^{2\pi i k b/N + 2\pi i (j + g/N) z}
\]

\[
- \sum_{j=g/N}^{\infty} \sum_{k=1}^{\infty} \frac{1}{k} e^{-2\pi i k b/N + 2\pi i (j - g/N) z}
\]

\[
= \log \alpha(g/N, h/N) + \pi i \mathcal{B}_2(g/N) - H(z, 0, g/N, b/N).
\]

Observe that \( R_1 = g'/N \), \( R_2 = b'/N \), and \( |\rho| = \frac{1}{2} g/N \). Using these facts, substituting (25) into (22), and then simplifying slightly with the use of (17), we arrive at (24).

5. The reciprocity formula for generalized Dedekind sums. We shall assume that at least one of the pair \( r_1, r_2 \) is not an integer for it is well known that the
reciprocity formula for the ordinary Dedekind sums may be derived from (14). Our assumption on the pair \( r_1, r_2 \) implies that at least one of the pair \( R_1, R_2 \) is not an integer.

Let \( Vz = (az + b)/(cz + d), \) \( V^*z = (bz - a)/(dz - c), \) and \( Tz = -1/z, \) where \( c, d > 0. \) Replacing \( z \) by \(-1/z\) in (22), we obtain

\[
H(V^*z, 0, r_1, r_2) - H(Tz, 0, R_1, R_2)
\]

(26) can be derived from (14). Our assumption on the pair \( r_1, r_2 \) implies that at least one of the pair \( R_1, R_2 \) is not an integer.

Next apply (22) with \( V \) replaced by \( V^* \). Observe that \( R_1 \) and \( R_2 \) are replaced by \( R_2 \) and \(-R_1\), respectively. Thus, we get

\[
H(V^*z, 0, r_1, r_2) - H(z, 0, R_2, -R_1)
\]

(27) can be derived from (14). Our assumption on the pair \( r_1, r_2 \) implies that at least one of the pair \( R_1, R_2 \) is not an integer.

Lastly, apply (22) to the transformation \( T \), and set \( r_1 = R_1 \) and \( r_2 = R_2 \). Accordingly, we obtain

\[
H(Tz, 0, R_1, R_2) - H(z, 0, R_2, -R_1)
\]

(28) can be derived from (14). Our assumption on the pair \( r_1, r_2 \) implies that at least one of the pair \( R_1, R_2 \) is not an integer.

Since \( \bar{B}_2(x) = \bar{B}_2(-x) \) from (17), we have upon combining (26)-(28),

\[
\left( -\frac{z}{2c(dz - c)} + \frac{1}{2d(dz - c)} \right) \bar{B}_2(\rho) - \left( \frac{dz - c}{2cz} + \frac{1}{2z} \right) \bar{B}_2(R_1) + \left( \frac{dz - c}{2d} - \frac{z}{2} \right) \bar{B}_2(R_2)
\]

\[
+ \left( (R_1) (R_2) \right) + s(d, c; R_2, -R_1) - s(-c, d; -R_1, -R_2) - (29)
\]

Since \( s(-c, d; -R_1, -R_2) = -s(c, d; -R_1, R_2) \), the above reduces to

\[
s(d, c; R_2, -R_1) + s(c, d; -R_1, R_2)
\]

(29) can be derived from (14). Our assumption on the pair \( r_1, r_2 \) implies that at least one of the pair \( R_1, R_2 \) is not an integer.

6. Transformation formulae of some Lambert series. In Theorem 2, put \( r_1 = r_2 = 0 \) and \( s = -m \), where \( m > 0 \) is even. Here,
GENERALIZED DEDEKIND ETA-FUNCTIONS AND SUMS

\[ H(z, -m, 0, 0) = 2A(z, -m, 0, 0) = 2A(z, -m) \]
\[
= 2 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} k^{-m-1} e^{2\pi i k n z} = 2 \sum_{k=1}^{\infty} k^{-m-1} \frac{e^{2\pi i k z}}{1 - e^{2\pi i k z}},
\]
which is a Lambert series in the variable \( e^{2\pi i z} \). From the functional equation of \( \zeta(s) \) [18, p. 24],
\[
\zeta(0, 0, c, d) = 2(2\pi i)^m (1 - (cz + d)^m) \lim_{s \to -m} \Gamma(s) \zeta(s)
\]
\[
= 2(2\pi i)^m (1 - (cz + d)^m) \lim_{s \to -m} 2^{-s} (m - 1 - m^s \sec(\frac{\pi}{2} ms)) \zeta(1 - s)
\]
\[
= (1 - (cz + d)^m) \zeta(m + 1).
\]

Hence, from (5)-(7),
\[
(cz + d)^m A(Vz, -m)
\]
\[
= A(z, -m) + \frac{1}{2} (1 - (cz + d)^m) \zeta(m + 1)
\]
\[
= \frac{(2\pi i)^{m+1}}{2(m+2)!} \sum_{j=1}^{m+2} \sum_{k=0}^{m+2} \left( \begin{array}{c} m+2 \\ k \end{array} \right) B_k(j/c) B_{m+2-k}(jd/c) (- (cz + d))^{k-1}.
\]

The transformation formulae (30) were first proved by Apostol [2]. However, due to a miscalculation of residues, the term \( \frac{1}{2} (1 - (cz + d)^m) \zeta(m + 1) \) was omitted. See also [9, p. 661]. Consequently, the result is also misstated by Carlitz ([4], [5]), but the other results in [4], [5] are unaffected by this and remain correct. In the notation of Carlitz ([4], [5]), the double sum on the right side of (30) is \(-f(d, c, z)\). To show this, all one needs is (17) and the observation that \( B_k(j/c) \) may be replaced by \( \overline{B}_k(j/c) \). For \( k \neq 1 \), this is clear. For \( k = 1 \), this is also clear for \( 1 \leq j \leq c - 1 \), but for \( j = c \), \( B_1(1) \neq \overline{B}_1(1) \). However, in the latter case \( \overline{B}_{m+1}(d) = B_{m+1}(0) \), since \( m > 0 \) is even. Hence, in all cases \( B_k(j/c) \) may be replaced by \( \overline{B}_k(j/c) \) in (30).

7. Additional new transformation formulae. Let \( s = -m \), where \( m > 0 \) is even, and let \( r_1 \) and \( r_2 \) be arbitrary. Now, for \( 0 < \alpha < 1 \) and \( \sigma < 0 \) [18, p. 37],
\[
\Gamma(s) \zeta(s, \alpha) = \frac{(2\pi)^s}{\sin(\pi s)} \sum_{n=1}^{\infty} \sin(2\pi n\alpha) \zeta(n + \frac{s}{2}).
\]
It follows that for \( \alpha \) real and \( \sigma < 0 \),
\[
\Gamma(s) \left( \zeta(s, |\alpha| + \chi(\alpha)) + e^{\pi is} \zeta(s, |\alpha| + \chi(\alpha)) \right)
\]
\[
= \frac{(2\pi)^s}{2 i \sin(\pi s)} \sum_{n=1}^{\infty} \frac{e^{-2\pi i n \alpha + 3\pi is/2} - e^{-2\pi i n \alpha - \pi is/2}}{n^{1-s}}
\]
\[
= (2\pi)^s e^{\pi is/2} \phi(1 - s, - \alpha),
\]
where for \( \alpha \) real and \( \sigma > 1 \),
In fact, \( \phi(s, a) \) is a special case of Lerch's zeta-function. It follows from (31) that

\[
f_m(r_1, r_2, c, d) = -\chi(r_1)(cz + d)^m\phi(m + 1, -r_2) + \chi(R_1)\phi(m + 1, R_2).
\]

Hence, from (5)–(7),

\[
(cz + d)^mH(Vz, -m, r_1, r_2) = H(z, -m, R_1, R_2) - \chi(r_1)(cz + d)^m\phi(m + 1, -r_2)
\]

(32)

\[
+ \chi(R_1)\phi(m + 1, R_2) + (2m)^{m+1}g(d, c; z; R_1, R_2)/(m + 2)!,
\]

where

\[g(d, c; z; R_1, R_2) = \sum_{j=1}^{c} \sum_{k=0}^{m+2} \binom{m+2}{k} B_k \left( \frac{i \rho - R_1}{c} \right) B_{m+2-k} \left( \frac{jd + \rho}{c} \right) (- (cz + d))^{k-1}.
\]

Iseki [9] has proved (32) in the special case \( V(z) = -1/z \). Another proof of Iseki's result has been given by Apóstol [3].

8. The reciprocity formula for \( g(d, c; z; R_1, R_2) \). Let \( V, V^* \), and \( T \) be as in §5. Put \( c_m = (2\pi i)^{m+1}/(m + 2)! \). Replacing \( z \) by \(-1/z\) in (32), we obtain

\[
(-c/z + d)^mH(V^*z, -m, r_1, r_2) - H(Tz, -m, R_1, R_2)
\]

(33)

\[
= -\chi(r_1)(-c/z + d)^m\phi(m + 1, -r_2)
\]

\[
+ \chi(R_1)\phi(m + 1, R_2) + c_m g(d, c; z; R_1, R_2).
\]

Apply (32) to \( V^* \) to get

\[
(dz - c)^mH(V^*z, -m, r_1, r_2) - H(z, -m, R_2, -R_1)
\]

(34)

\[
= -\chi(r_1)(dz - c)^m\phi(m + 1, -r_2)
\]

\[
+ \chi(R_2)\phi(m + 1, -R_1) + c_m g(-c, d; z; R_2, -R_1).
\]

Lastly, let \( r_1 = R_1, r_2 = R_2 \), and apply (32) to the transformation \( T \). Then,

\[
z^mH(Tz, -m, R_1, R_2) - H(z, -m, R_2, -R_1)
\]

(35)

\[
= -\chi(R_1)z^m\phi(m + 1, -R_2) + \chi(R_2)\phi(m + 1, -R_1) + c_m g(0, 1; z; R_2, -R_1).
\]

Now,

\[
g(0, 1; z; R_2, -R_1) = \sum_{k=0}^{m+2} \binom{m+2}{k} B_k (1 - \{R_2\}) B_{m+2-k}(-R_1)(-z)^{k-1}
\]

(36)

\[
= (-1/z)(B(-R_1) - zB(1 - \{R_2\}))^{m+2},
\]

upon the use of a standard symbolic notation for Bernoulli polynomials. Combining (33)–(36), we find that
\[ c_m z^m g(d, c; Tz; R_1, R_2) - c_m g(-c, d; z; R_2, -R_1) \]

\[ -\left(\frac{c_m}{z}\right)(\bar{B}(-R_1) - \bar{B}(1 - |R_2|))^m + 2 \]

\[ + \chi(R_1)z^m (\phi(m + 1, R_2) - \phi(m + 1, -R_2)) = 0. \]

From the Fourier series \([1, p. 805]\),

\[ B_{2n+1}(x) = \frac{2(-1)^n(2n + 1)!}{(2\pi)^{2n+1}} \sum_{k=1}^{\infty} \frac{\sin(2\pi kx)}{k^{2n+1}}, \]

where \( n \geq 1 \) and \( 0 < x < 1 \), we find that

\[ \phi(m + 1, R_2) - \phi(m + 1, -R_2) = -\frac{(2\pi)^{m+1} B_{m+1}(R_2)}{m+1}. \]

Hence, (37) reduces to

\[ z^m g(d, c; Tz; R_1, R_2) = g(-c, d; z; R_2, -R_1) \]

\[ + \left(\frac{1}{z}\right) (\bar{B}(-R_1) - \bar{B}(1 - |R_2|))^m + 2 \]

\[ + \chi(R_1)z^m (m + 2) \bar{B}_{m+1}(R_2). \]

In the case when \( r_1 = r_2 = 0 \), (38) reduces to a reciprocity theorem proved by Carlitz ([4], [5]) for the function \( f(d, c, z) \) mentioned in §6.

REFERENCES


6. U. Dieter, Das Verhalten der Kleinschen Funktionen \( \log \sigma_{g,b}(\omega_1, \omega_2) \) gegenüber Modultransformationen und verallgemeinerte Dedekindsche Summen, J. Reine Angew. Math. 201 (1959), 37–70. MR 21 #3397.

