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ISOTOPIC UNKNOTTING IN $F \times I$

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ABSTRACT. This paper is essentially a generalization of *Unknotting in* $M^2 \times I$, by E. M. Brown. The major results in this paper concern the existence of ambient isotopies of unknotted arcs (families of arcs) properly embedded in $F \times I$.

I. **Introduction.** This paper is essentially a generalization of [1]. Loosely speaking when Brown shows the existence of a homeomorphism having certain properties, we show the existence of an isotopy having many of the same properties. Our major results concern the existence of an ambient isotopy of unknotted arcs in $F \times I$. The results in this paper come from the author's doctoral thesis at Dartmouth College; however the proofs have been simplified by following some of the techniques used by Waldhausen in [6]. We also adopt much of the terminology used in [6]. The author would like to thank E. M. Brown of Dartmouth College for directing his dissertation and suggesting these problems. The author would also like to thank the referee for a number of helpful suggestions and for pointing out the necessity of an added hypothesis in the statement of Lemma 3.4.

II. **Notation.** In this paper all spaces are simplicial complexes and all maps are piecewise linear. We shall denote the boundary, closure, and interior of a subspace X of a space Y by $\mathrm{bd}(X, Y)$, $\mathrm{cl}(X, Y)$, and $\mathrm{int}(X, Y)$ respectively. Since no confusion can result, we will be able to abbreviate these by $\mathrm{bd}(X)$, $\mathrm{cl}(X)$, and $\mathrm{int}(X)$. The symbol F will denote a compact, connected surface and I the unit interval. A surface F is *properly embedded* in a 3-manifold M if $F \cap \mathrm{bd}(M) = \mathrm{bd}(F)$. A surface F properly embedded in a 3-manifold M is *incompressible in* M if for every disk D embedded in M such that $D \cap F = \mathrm{bd}(D)$, $\mathrm{bd}(D)$ is nullhomotopic in F.

Let F be a two-sided surface properly embedded in the 3-manifold M. Then the manifold M', obtained by splitting M along F, has by definition the properties:

(1) $\mathrm{bd}(M')$ contains surfaces F_1 and F_2 which are copies of F.

(2) Identification of F_1 and F_2 gives a natural projection $(M', F_1 \cup F_2) \to (M, F)$.

We may also speak of splitting M along a one-sided surface properly embedded in M.

An *isotopy of a homeomorphism* $h : X \to Y$ is a map $H : X \times I \to Y$ such that for $h_t = H \mid X \times \{t\}$, we have $h_0 = h$ and h_t is a homeomorphism onto Y. An

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isotopy of subspaces Z_1 and Z_2 in X is an isotopy of the identity map on X such that $h_1(Z_1) = Z_2$.

Let $M = F \times I$ and $p : M \to F$ be the natural projection map onto the factor F. A subspace X of M is called *vertical* if $X = p^{-1}p(X)$. Let α be an arc properly embedded in M such that $p(\mathrm{bd}(\alpha))$ is a point in $\mathrm{int}(F)$. We shall say that α is *homeomorphically unknotted* (or *h-unknotted*) if there is a homeomorphism $h : M \to M$ such that $h \mid \mathrm{bd}(F) \times I \cup F \times \{0\}$ is the identity and $h(\alpha)$ is vertical. We shall say that α is *isotopically unknotted* (or *i-unknotted*) if α is isotopic to a vertical subspace of M leaving $\mathrm{bd}(M)$ fixed. Similarly we can consider collections of arcs.

The major results in this paper concern necessary and sufficient conditions for an arc (collection of arcs) which are h-unknotted to be i-unknotted.

We shall try to follow Waldhausen's principal of "induction on niceness" (see [6, p. 58]). That is, after we convince ourselves that there is no obstruction to achieving some niceness, we take up our problem again assuming that niceness.

Let G_1 and G_2 be incompressible surfaces properly embedded in a 3-manifold M. We suppose that M is irreducible (every embedded 2-sphere bounds a 3-ball). We suppose that there is a simple loop $\lambda \subset G_1 \cap G_2$ such that $\lambda \neq \mathrm{bd}(G_1)$. If λ is nullhomotopic in G_1 (and thus in G_2), λ bounds a disk both in G_1 and in G_2. We may assume that λ bounds an "innermost" disk on G_1 and thus that the union of the disks bounded by λ on G_1 and G_2 is a 2-sphere S^2 embedded in M. Now it is not hard to show that there is an isotopy leaving $\mathrm{bd}(M)$ fixed so that $h_1(G_1) \cap G_2$ is "less complicated" than $G_1 \cap G_2$. The isotopy above is found by pushing G_1 across the ball bounded by S^2. We will use this fact repeatedly in the proofs to follow.

III. **Useful results.** In this section we prove a number of results which will be useful in the proofs of the theorems on isotopic unknotting of arcs.

A restriction of the following lemma has also been proved by W. Heil in his thesis at Rice University.

Lemma 3.1. *Let F be a surface other than the 2-sphere and $M = F \times I$. In M, let G be a system of properly embedded surfaces, such that each component of G is either a disk which intersects $\mathrm{bd}(F) \times I$ in two vertical arcs or an incompressible annulus which has one boundary curve in $\mathrm{int}(F \times \{0\})$ and the other in $\mathrm{int}(F \times \{1\})$. Then there is an isotopy constant on $F \times \{0\} \cup \mathrm{bd}(F) \times I$ which makes G vertical.*

Proof. In case F is not the projective plane the proof of this lemma is the same as that of Lemma 3.4 in [6] except that we use Theorem 2.1 or Theorem 3.3 in [4] to do the work of Baer's theorem because Baer's theorem applies only to orientable manifolds.

We now assume that F is the projective plane. Let G_1 be a component of G. Since F is without boundary, G_1 is an annulus. Let $\lambda \times \{0\} = G_1 \cap F \times \{0\}$ and $\lambda_1 \times \{1\} = G_1 \cap F \times \{1\}$. We observe that neither λ_1 nor λ separates a regular neighborhood of itself in F. Thus $\lambda_1 \cap \lambda \neq \varnothing$. It is easily seen that after a

deformation constant on $F \times \{0\}$, we may assume that $\lambda_1 \cap \lambda$ is a single point. After a small deformation constant on $\mathrm{bd}(M)$, we may assume that $\lambda \times I \cap G_1$ consists of $\lambda \times \{0\}$, a number of simple closed loops in the interior of $\lambda \times I$, and a number of simple arcs with their endpoints lying in $\lambda \times \{0, 1\}$. Since one of the arcs in $\lambda \times I \cap G_1$ runs from $\lambda \times \{0\}$ to $\lambda \times \{1\}$, every simple loop in $\lambda \times I \cap G_1 \cap \mathrm{int}(M)$ is nullhomotopic in $\lambda \times I$. Since $F \times I$ is irreducible, after a deformation constant on $\mathrm{bd}(M)$ we may assume that $\lambda \times I \cap G_1$ consists of $\lambda \times \{0\}$ together with a number of arcs with their endpoints in $\lambda \times \{0, 1\}$. If such an arc has both its endpoints on $\lambda \times \{0\}$, it together with an arc in $\lambda \times \{0\}$ bounds a disk in $\lambda \times I$. Thus, after a deformation leaving $\mathrm{bd}(M)$ fixed, we may assume that $\lambda \times I \cap G_1$ consists of $\lambda \times \{0\}$ and a single arc running from $\lambda \times \{0\}$ to $\lambda \times \{1\}$.

One observes that the arc in $\lambda \times I \cap G_1$ is isotopic in $\lambda \times I$ to a vertical arc mod $\lambda \times \{0\}$ and that one can cover each stage of the isotopy with a 3-ball meeting $\lambda \times I$ in a disk. After another deformation leaving $F \times \{0\}$ fixed, we may assume that the arc running from $\lambda \times \{0\}$ to $\lambda \times \{1\}$ is vertical. Finally after a deformation leaving $F \times \{0\}$ fixed, we may assume that $G_1 = \lambda \times I$ since one obtains a 3-ball by splitting M along $\lambda \times I$.

Since $M - \lambda \times I$ is simply connected, it can be seen that G_1 was the only component of G and the proof of Lemma 3.1 is complete.

Lemma 3.2. *Let F be a surface other than the 2-sphere. Let $M = F \times I$ and $h : M \to M$ a homeomorphism which is the identity on $F \times \{0\} \cup \mathrm{bd}(F) \times I$. Then there is an isotopy constant on $F \times \{0\} \cup \mathrm{bd}(F) \times I$ of h to the identity.*

Proof. The proof of Lemma 3.2 is essentially the same as that given for Lemma 3.5 in [6] except that we refer to Lemma 3.1 above.

Lemma 3.3. *Let F be a surface other than the 2-sphere. Let $M = F \times I$. Let D_1 and D_2 be disks properly embedded in M such that $\mathrm{bd}(D_1) = \mathrm{bd}(D_2)$. Then there is an isotopy of the subspaces D_1 and D_2 constant on $\mathrm{bd}(M)$.*

Proof. Since every 2-sphere embedded in M bounds a 3-cell in M, this is an easy application of the techniques used to prove Lemmas 3.1 and 3.2 above.

Let λ be a simple loop properly embedded in a surface F. Then λ is *nonorientable* if it fails to separate a regular neighborhood of itself in F.

Lemma 3.4. *We suppose that our surface F is not the 2-sphere or projective plane. Let λ be a simple non-nullhomotopic loop properly embedded in F such that $F - \lambda$ is not the interior of an annulus. Let A be an annulus properly embedded in $M = F \times I$ such that $\mathrm{bd}(A) = \lambda \times \{0, 1\}$. Then the subspaces $\lambda \times I$ and A are isotopic under a deformation constant on $\mathrm{bd}(M)$.*

Proof. *Case* 1. We suppose that λ is orientable. After a small deformation leaving $\mathrm{bd}(M)$ fixed, we may assume that $A \cap \lambda \times I$ consists of the common boundary of A and $\lambda \times I$ together with a number of arcs with their endpoints in $\lambda \times \{0, 1\}$ and a number of simple closed loops properly embedded in $\lambda \times I$. After

a deformation constant on bd(M), we may assume that $A \cap \lambda \times I$ does not contain any simple loops which are nullhomotopic in A. Thus there are no arcs in $A \cap \lambda \times I$ with both endpoints in $\lambda \times \{j\}$ where $j = 1$ or 2. Similarly there are not two disjoint arcs running from $\lambda \times \{0\}$ to $\lambda \times \{1\}$. Since λ separates a regular neighborhood of itself, we may assume that $\lambda \times I \cap A$ is a collection of disjoint, simple, non-nullhomotopic loops. We split M along $\lambda \times I$ to obtain $M_1 = F_1 \times I$. Then there is a subannulus A_1 of A with $\mathrm{bd}(A_1) \subset \mathrm{bd}(F_1) \times I$. Since $F - \lambda$ is not the interior of an annulus, $\mathrm{bd}(A_1)$ lies on a single component of $\mathrm{bd}(F_1) \times I$. Now it is a consequence of the lemma on p. 91 in [6] that there is an annulus A_2 in $\mathrm{bd}(F_1 \times I)$ such that $A_1 \cup A_2$ bounds a solid torus T and that we can push A_1 across the projection of T with a deformation constant on bd(M). Thus we can assume that there are exactly two simple loops in $A \cap \lambda \times I$. Now $A \cup \lambda \times I$ bounds a solid torus embedded in M. Since by the proof above we have a "nice" product structure on this solid torus, there is a deformation, constant on bd(M), which moves A to $\lambda \times I$. This completes the proof of Case 1.

Case 2. λ is nonorientable.

Let λ_1 be a simple loop properly embedded in F such that

(1) λ_1 bounds a möbius band F_1 in F.

(2) $\lambda \subset \mathrm{int}(F_1)$.

After a deformation constant on bd(M) we may assume that $A \cap \lambda_1 \times I$ is a collection of simple loops. Since λ and λ_1 are not freely homotopic in F, every loop in $A \cap \lambda_1 \times I$ is nullhomotopic both in A and in $\lambda_1 \times I$. Thus, after a deformation leaving bd(M) fixed, we may assume that $A \cap \lambda_1 \times I$ is empty. Now we can assume that F is a möbius band.

Let β be an arc properly embedded in F which fails to separate F and which meets λ in a single point. After a deformation constant on bd(M), we may assume that $A \cap \beta \times I$ is a vertical arc. Next we cut M along $\beta \times I$. This allows us to apply Lemma 3.3 to complete the proof of Case 2.

Thus the proof of Lemma 3.4 is complete.

Lemma 3.5. *Let F be a closed surface other than the klein bottle or torus. Let λ be a simple non-nullhomotopic loop embedded in F. We suppose that λ separates F. Let $h : F \times I \to F \times I$ be a homeomorphism such that*

(1) $h \mid \mathrm{bd}(F \times I) = \mathrm{id}$.

(2) $h(\lambda \times I) = \lambda \times I$.

Let x be a point in λ. Then the loop $ph(\{x\} \times I)$ is homotopic to a point in λ.

Proof. Let F_1 and F_2 be the closures of the components of $F - \lambda$. Since F is not the klein bottle either $\pi_1(F_1)$ or $\pi_1(F_2)$ is not abelian. Thus we may assume that $\pi_1(F_1)$ is free on two or more generators and that $[\lambda] = a_1 a_2 a_1^{-1} a_2^{-1} \ldots a_{2n-1} a_{2n} a_{2n-1}^{-1} a_{2n}^{-1}$ where $a_1 \ldots a_{2n}$ are generators of $\pi_1(F_1 \times \{0\}, (x, 0))$ (or $[\lambda] = a_1^2 \ldots a_n^2$ if F_1 is not orientable). Let μ be a loop in F_1.

Observe that $h \mid \mu \times \{0, 1\} = \mathrm{id}$. Consider the loop ν defined by

$$\begin{aligned} \nu(t) &= (\mu(4t), 0) && \text{if } 0 \leqslant t \leqslant \tfrac{1}{4}, \\ &= (x, 4t - 1) && \text{if } \tfrac{1}{4} \leqslant t \leqslant \tfrac{1}{2}, \\ &= (\mu(3 - 4t), 1) && \text{if } \tfrac{1}{2} \leqslant t \leqslant \tfrac{3}{4}, \\ &= (x, 4 - 4t) && \text{if } \tfrac{3}{4} \leqslant t \leqslant 1. \end{aligned}$$

Then ν is homotopic to a point in $F_1 \times I$. Now $h : F_1 \times I \to F_1 \times I$ induces an automorphism of $\pi_1(F_1 \times I, (x, 0))$. We see

$$p_* h_*[\nu] = [ph\nu] = [\mu][ph(\{x\} \times I)][\mu]^{-1}[ph(\{x\} \times I)]^{-1} = 1.$$

Thus $[\mu]$ and $[ph(\{x\} \times I)]$ commute for all $\mu \subset F_1$. Thus the loop $ph(\{x\} \times I)$ is homotopic to a point and Lemma 3.5 follows.

Theorem 3.6. *Suppose that the surface F is not the 2-sphere, projective plane, klein bottle, or torus and that $M = F \times I$. Let $h : M \to M$ be a homeomorphism such that $h \mid \mathrm{bd}(M) = \mathrm{id}$. Then there is an isotopy of h to the identity which is constant on $\mathrm{bd}(M)$.*

Proof. We suppose that $\mathrm{bd}(F)$ is not empty. Let β_i, $i = 1, \ldots, n$, be disjoint simple arcs properly embedded in F such that we obtain a disk by splitting F along $\bigcup_{i=1}^n \beta_i$. It is a consequence of Lemma 3.3 that we may assume that $h(\bigcup_{i=1}^n \beta_i \times I) = \bigcup_{i=1}^n \beta_i \times I$. After a deformation leaving $\mathrm{bd}(M)$ fixed, we may assume that $h \mid \beta_i \times I = \mathrm{id}$ for $i = 1, \ldots, n$. The theorem is now an immediate consequence of Alexander's theorem.

We now assume that F is without boundary. Let λ be a simple non-nullhomotopic loop properly embedded in F. We suppose that λ separates F. It is a consequence of Lemma 3.4 that we may assume $h(\lambda \times I) = \lambda \times I$. Let x be a point in λ. Since by Lemma 3.5 the loop $ph(\{x\} \times I)$ is nullhomotopic in λ, after a deformation leaving $\mathrm{bd}(M)$ fixed, we may assume that $h \mid \{x\} \times I = \mathrm{id}$ and $h(\lambda \times I) = \lambda \times I$. After another deformation constant on $\mathrm{bd}(M)$, we may assume that $h \mid \lambda \times I = \mathrm{id}$. We may now split M along $\lambda \times I$ to reduce the case when F is without boundary to the bounded case. The theorem follows.

IV. **Unknotting of arcs.** In this section we investigate the relationship between homeomorphically unknotted arcs and isotopically unknotted arcs. The following result is Theorem 6.1 in [1].

Theorem 4.1. *Let F be a surface and $M = F \times I$. Let α_i, $i = 1, \ldots, n$, be a collection of disjoint simple arcs properly embedded in M such that $p\,\mathrm{bd}(\alpha_i)$ is a single point in $\mathrm{int}(F)$ for $i = 1, \ldots, n$. Then there is homeomorphism $h : M \to M$ such that*

(1) $h \mid \mathrm{bd}(F) \times I \cup F \times \{0\} = \mathrm{id}$.

(2) $h(\bigcup_{i=1}^n \alpha_i)$ *is vertical*

if and only if the natural homomorphism induced by inclusion

$$\pi_1\left(F \times \{0\} - \bigcup_{i=1}^{n} \alpha_i\right) \to \pi_1\left(M - \bigcup_{i=1}^{n} \alpha_i\right)$$

is an epimorphism.

The above theorem gives us a relationship between the property "homeomorphically unknotted" and an algebraic map.

Let α be an arc properly embedded in $F \times I$ and $p\ \mathrm{bd}(\alpha)$ be a single point in $\mathrm{int}(F)$. Then α is embedded *monotonically* in M if α meets $F \times \{t\}$ in a single point for t in I. A *level preserving isotopy* $H : F \times I \times I \to F \times I$ is an isotopy such that $h_t(F \times \{s\}) \subset F \times \{s\}$ for s in I.

Theorem 4.2. *We assume that F is not the 2-sphere. Let α be an arc properly embedded in $M = F \times I$ such that $p(\mathrm{bd}(\alpha))$ is a single point in $\mathrm{int}(F)$. Then there is an isotopy constant on $\mathrm{bd}(M)$ carrying α to a monotonic arc if and only if α is homeomorphically unknotted.*

Proof. Suppose α is homeomorphically unknotted. Let $\{x\} = p\ \mathrm{bd}(\alpha)$. After a small deformation constant on $\mathrm{bd}(M)$, we may assume that $\alpha_2 = \alpha \cap F \times [t_0, 1] = \{x\} \times [t_0, 1]$ where $t_0 < 1$ is sufficiently near 1. Let $\alpha_1 = \mathrm{cl}(\alpha - \alpha_2)$. Consider the diagram of groups promised by van Kampen's theorem and shown in Figure 1.

$$\begin{tikzcd}
 & \pi_1(F \times \{t_0\} - \alpha) \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\
\pi_1(F \times [0, t_0] - \alpha_1 \arrow[dr, "f_3"'] & & \pi_1(F \times [t_0, 1] - \alpha_2) \arrow[dl, "f_4"] \\
 & \pi_1(M - \alpha) &
\end{tikzcd}$$

Figure 1

Now f_1 and f_2 are monomorphisms and f_4 is onto; thus f_1 is onto by 4.2 in [3]. It follows from 4.1 above that there is a homeomorphism $k : F \times [0, t_0] \to F \times [0, t_0]$ such that

$$k(\alpha_1) = \{x\} \times [0, t_0] \quad \text{and} \quad k \mid \mathrm{bd}(F) \times [0, t_0] \cup F \times \{0\} = \mathrm{id}\,.$$

It is an immediate consequence of Lemma 3.2 that there is an isotopy, constant on $\mathrm{bd}(F) \times [0, t_0] \cup F \times \{0\}$ of k to the identity. It is not hard to use this isotopy together with a level preserving isotopy constant on $\mathrm{bd}(F) \times [t_0, 1] \cup F \times \{1\}$ which "pays for the motion on $F \times \{t_0\}$" to construct the isotopy required in the theorem. We may omit the details of this construction.

Suppose α is monotonically embedded in M. Let $\alpha(t) = \alpha \cap F \times \{t\}$. If F is a disk, it is clear that there exists a deformation retraction of $F \times I - \alpha$ onto $F \times \{0\} - \alpha(0)$ which slides the boundary vertically downward and lets the missing point follow the track of α. It follows that the natural map from $\pi_1(F \times \{0\} - \alpha(0))$ into $\pi_1(M - \alpha)$ induced by inclusion is an epimorphism and α is h-unknotted.

In case F is not a disk, we can find a finite sequence of disks $D_1, \ldots, D_n$ covering the projection of α to F. We may assume that $\alpha(t)$ is in $\mathrm{int}(D_i) \times [(i-1)/n, i/n]$ for t in $[(i-1)/n, i/n]$, $i = 1, \ldots, n$, by using the Lebesgue number. We define a deformation retraction of $F \times [0, i/n] - \alpha$ to $F \times [0, (i-1)/n] - \alpha$ on $D_i \times [(i-1)/n, i/n] - \alpha$ as the one above and sliding $(F - D_i) \times [(i-1)/n, i/n]$ vertically downward.

We compose finitely many of these deformation retractions to deformation retract $F \times I - \alpha$ to $F \times \{0\} - \alpha(0)$. It is now clear that the natural map $\pi_1(F \times \{0\} - \alpha(0)) \to \pi_1(F \times I - \alpha)$ is onto and α is h-unknotted. This completes the proof of Theorem 4.2.

Theorem 4.3. *Let F be a surface not the 2-sphere. Let α be an arc properly embedded in $M = F \times I$ such that $p\ \mathrm{bd}(\alpha)$ contains a single point in $\mathrm{int}(F)$. Then α is isotopically unknotted if and only if α is homeomorphically unknotted and the loop $p(\alpha)$ is homotopic to a point.*

Proof. Clearly we need only prove that α is isotopically unknotted in case α is h-unknotted and the loop $p(\alpha)$ is nullhomotopic. We will divide the proof of Theorem 4.3 into six lemmas.

Lemma 4.4. *Let F be a disk. Let α be embedded in $F \times I$ as above. Let α be homeomorphically unknotted in $F \times I$. Then α is isotopically unknotted.*

Proof. Let $h : F \times I \to F \times I$ be a homeomorphism such that $h(\alpha)$ is vertical and $h \mid \mathrm{bd}(F) \times I \cup F \times \{0\} = \mathrm{id}$. Then $k(m) = ph(m, 1)$ defines a homeomorphism from F onto F. By Theorem 5.2 in [4], there is an isotopy of k to the identity constant on $\mathrm{bd}(F) \cup p\ \mathrm{bd}(\alpha)$. Thus after a deformation, we may assume that $h \mid \mathrm{bd}(F \times I) = \mathrm{id}$. The desired result is now a consequence of Alexander's theorem or Theorem 3.6 above.

Lemma 4.5. *Let F be an annulus or a möbius band. Let α be embedded in $F \times I$ as in Theorem 4.3. Let the loop $p(\alpha)$ be nullhomotopic in F. Let α be h-unknotted in $F \times I$. Then α is isotopically unknotted in $F \times I$.*

Proof. It is a consequence of Theorem 4.2 that we may assume that α is embedded monotonically in $F \times I$. Let β_1 and β_2 be disjoint simple arcs properly embedded in F such that $F - \beta_j$ is connected for $j = 1, 2$; and $p\ \mathrm{bd}(\alpha)$ is not contained in $\beta_1 \cup \beta_2$. After a deformation constant on $\mathrm{bd}(F \times I)$, we may assume that $J = \alpha \cap (\beta_1 \cup \beta_2) \times I$ is a finite collection of points and that α crosses $(\beta_1 \cup \beta_2) \times I$ at each point in J.

It can be checked that the monotonic arc constructed in 4.2 projects to a trivial loop if α does. We assume that J is not empty. Since the loop $p(\alpha)$ is nullhomotopic, there must be an arc $\alpha_1 \subset \alpha$ with both its endpoints on one component of $(\beta_1 \cup \beta_2) \times I$ such that $\alpha_1 \cap (\beta_1 \cup \beta_2) \times I = \mathrm{bd}(\alpha_1)$. Since α is monotonic in $F \times I$, it is possible to find a level preserving deformation constant on $\mathrm{bd}(F \times I)$ carrying α to α' so that $\alpha' \cap (\beta_1 \cup \beta_2) \times I = J - \mathrm{bd}(\alpha_1)$. It follows that after a deformation constant on $\mathrm{bd}(F \times I)$, we may assume that J is empty. Thus we may split $F \times I$ along $\beta_1 \times I$. Lemma 4.5 is now a consequence of Lemma 4.4.

Lemma 4.6. *Let F be a torus. Let α be embedded in $F \times I$ as in Theorem* 4.3. *Let α be h-unknotted in $F \times I$. Let the loop $p(\alpha)$ be nullhomotopic. Then α is isotopically unknotted in $F \times I$.*

Proof. Let λ_1 and λ_2 be disjoint, non-nullhomotopic simple loops properly embedded in F such that $p\ \mathrm{bd}(\alpha)$ is not contained in $\lambda_1 \cup \lambda_2$. As in the proof of Lemma 4.5 above we can arrange α so that it is embedded monotonically in $F \times I$ and so that it meets $(\lambda_1 \cup \lambda_2) \times I$ in a finite collection of points. We can find an arc $\alpha_1 \subset \alpha$ such that $\alpha_1 \cap (\lambda_1 \cup \lambda_2) \times I = \mathrm{bd}(\alpha_1)$ and $\mathrm{bd}(\alpha_1)$ is contained in a single component of $(\lambda_1 \cup \lambda_2) \times I$. It is now easy to find a level preserving deformation constant on $\mathrm{bd}(F \times I)$ which carries α to a monotonic arc α' such that $\alpha' \cap (\lambda_1 \cup \lambda_2) \times I = \alpha \cap (\lambda_1 \cup \lambda_2) \times I - \mathrm{bd}(\alpha_1)$. Thus after a deformation constant on $\mathrm{bd}(F \times I)$, we can assume that $\alpha \cap (\lambda_1 \cup \lambda_2) \times I$ is empty. By cutting $F \times I$ along $\lambda_1 \times I$, we reduce the proof of Lemma 4.6 to an application of Lemma 4.5.

Lemma 4.7. *Let F be the projective plane. Let α be embedded in $F \times I$ as in Theorem* 4.3. *Suppose the loop $p(\alpha)$ is nullhomotopic and α is h-unknotted in $F \times I$. Then α is isotopically unknotted in $F \times I$.*

Proof. The proof of Lemma 4.7 is similar to that of Lemmas 4.4, 4.5, and 4.6. One may again assume that α is embedded monotonically in $F \times I$. Let λ be a simple non-nullhomotopic loop properly embedded in F which does not contain $p\ \mathrm{bd}(\alpha)$. Then we can find a level preserving isotopy constant on $\mathrm{bd}(F \times I)$ carrying α to α' such that $\alpha' \cap \lambda \times I$ is a finite collection of points and α' crosses $\lambda \times I$ at each point in $\alpha' \cap \lambda \times I$. After another level preserving isotopy constant on $\mathrm{bd}(F \times I)$, we may assume that $p(\alpha \cap \lambda \times I)$ is a single point. We will denote this point by x_0. We cut $F \times I$ along $\lambda \times I$ to obtain $D \times I$ where D is a disk. Let

$P : D \times I \to F \times I$ be the natural projection map. Since the loop $p(\alpha)$ is nullhomotopic, we can find an arc $\alpha_1 \subset \alpha$ such that $\alpha_1 \cap \lambda \times I = \mathrm{bd}(\alpha_1)$ and so that both endpoints of the arc in $p^{-1}(\alpha_1)$ lie in the same component of $P^{-1}(\{x_0\} \times I)$. It is not hard to find a level preserving deformation of $F \times I$ constant on $\mathrm{bd}(F \times I)$ carrying α to α' such that $\alpha' \cap \lambda \times I = \alpha \cap \lambda \times I - \mathrm{bd}(\alpha_1)$. Thus we may assume that $\alpha \cap \lambda \times I$ is empty. Lemma 4.7 now follows from an application of Lemma 4.4 to the space obtained by cutting $F \times I$ along $\lambda \times I$.

Lemma 4.8. *Let F be a surface other than the 2-sphere. Let $F = F_1 \cup F_2$ and $F_1 \cap F_2 = c$ be a component of* $\mathrm{bd}(F_1)$. *We assume that F_1 and F_2 are closed in F and that F_2 is not a disk. Let α be an arc properly embedded in* $\mathrm{int}(F_1) \times I$ *such that p* $\mathrm{bd}(\alpha)$ *contains a single point. Then α is h-unknotted in $F_1 \times I$ if and only if α is h-unknotted in $F \times I$.*

Proof. We need only show that if α is h-unknotted in $F \times I$ it is also h-unknotted in $F_1 \times I$. Since α is h-unknotted in $F \times I$, the natural map

$$\phi : \pi_1(F \times \{0\} - \alpha) \to \pi_1(F \times I - \alpha)$$

induced by inclusion is onto by 4.1. Consider the diagram of groups promised by van Kampen's theorem and shown in Figure 2 where all maps are induced by inclusion. Observe that

$$\begin{tikzcd}
 & \pi_1(c \times \{0\}) \arrow[dl, "g_1"'] \arrow[d, "f_0"] \arrow[dr, "g_2"] & \\
\pi_1(F_1 \times \{0\} - \alpha) \arrow[d, "f_1"'] & \pi_1(c \times I) \arrow[dl, "h_1"] \arrow[dr, "h_2"'] & \pi_1(F_2 \times \{0\}) \arrow[d, "f_2"] \\
\pi_1(F_1 \times I - \alpha) & & \pi_1(F_2 \times I)
\end{tikzcd}$$

Figure 2

h_1 and h_2 are monomorphisms and f_0 is an epimorphism. Since the natural map

$$\phi : \pi_1(F_1 \times \{0\} - \alpha) \underset{\pi_1(c \times \{0\})}{*} \pi_1(F_2 \times \{0\}) \to \pi_1(F_1 \times I - \alpha) \underset{\pi_1(c \times I)}{*} \pi_1(F_2 \times I)$$

is an epimorphism, it follows from Proposition 2.4 in [2] that f_1 is an epimorphism. The lemma follows immediately from Theorem 4.1.

Lemma 4.9. *Let F be a surface other than the 2-sphere, projective plane, disk, or torus. Let α be embedded in $F \times I$ as in Theorem 4.3. Let α be h-unknotted and the loop $p(\alpha)$ be homotopic to a point. Then α is isotopically unknotted.*

Proof. By assumption there exists a homeomorphism $k : F \times I \to F \times I$ such that $k \mid \mathrm{bd}(F) \times I \cup F \times \{0\} = \mathrm{id}$ and $k(\alpha)$ is vertical. Define $\hat{k} : F \to F$ by $\hat{k}(m) = pk^{-1}(m, 1)$. Let x be the point in $p(\mathrm{bd}(\alpha))$. Now $\hat{k}$ is homotopic to the identity homeomorphism via the homotopy $H_t(m) = pk^{-1}(m, t)$. We claim that $\hat{k}$ is homotopic to the identity under a homotopy constant on $\{x\} \cup \mathrm{bd}(F)$. Let $K : I \times I \to F$ be an extension of the map defined on $\mathrm{bd}(I \times I)$ by $K(t, 0) = p\alpha(t)$, $K(t, s) = x$ for $s \neq 0$ where $\alpha(t) = k^{-1}(x, t)$. Such an extension exists since the loop $p(\alpha)$ is homotopic to a point. Consider the space $F \times I \times I$. Define a map $\overline{H}$ on the subspace

$$F \times I \times \{0\} \cup \{x\} \times I \times I \cup F \times \{0, 1\} \times I \cup \mathrm{bd}(F) \times I \times I$$

onto F by

(1) $\overline{H}(m, t, 0) = H_t(m)$ for m in F and t in I,
(2) $\overline{H}(x, t, s) = K(t, s)$ for t in I and s in I,
(3) $\overline{H}(m, 0, s) = m$ for m in F and s in I,
(4) $\overline{H}(m, 1, s) = \hat{k}(m)$ for m in F and s in I,
(5) $\overline{H}(m, s, t) = m$ for m in $\mathrm{bd}(F)$, t in I and s in I.

If we let $A = \mathrm{bd}(F) \times I \cup F \times \{0, 1\} \cup \{x\} \times I$ and $X = F \times I$, we see that $\overline{H}$ is defined on $(X \times \{0\} \cup A \times I) \subset X \times I$. It follows from the homotopy extension theorem that $\overline{H}$ can be extended to $X \times I$. We assume that $\overline{H}$ is defined on $X \times I$. Define $L_t : F \to F$ by $L_t(m) = \overline{H}(m, t, 1)$. Then L_t is a homotopy of $\hat{k}$ to the identity which is constant on $\mathrm{bd}(F) \cup \{x\}$ and our claim is established.

Let λ be a simple non-nullhomotopic loop properly embedded in F. Assume that x is in λ and that λ does not bound a möbius band in F. Now $L_t : \lambda \to F$ is a homotopy of λ to $\hat{k}(\lambda)$ leaving x fixed. Then by 4.1 in [4] there is an isotopy of F carrying λ to $\hat{k}(\lambda)$ which is constant on $\{x\} \cup \mathrm{bd}(F)$. Thus we may assume that $k(\lambda) = \lambda$ and $k(\alpha) = \{x\} \times I$. Note that $k^{-1}(\lambda \times I)$ is an annulus A properly embedded in $F \times I$ such that $p\,\mathrm{bd}(A)$ is a simple non-nullhomotopic loop properly embedded in F and that $\alpha \subset A$. By Lemma 3.4 we may assume that $A = \lambda \times I$. Let $R \subset F$ be a möbius band or annulus embedded in $\mathrm{int}(F)$ which contains λ in its interior. We claim that (1) α is h-unknotted in $R \times I$ and (2) the loop $p(\alpha) \subset R$ is nullhomotopic in R. It is easy to see that (1) is a consequence of Lemma 4.8. Since the loop $p(\alpha)$ is nullhomotopic in F and $\mathrm{bd}(R)$ is not nullhomotopic in F, the proof of (2) is a triviality. Thus the proof of Lemma 4.9 can be completed by applying Lemma 4.5. This completes the proof of Theorem 4.3.

Theorem 4.10. *Let λ_1 and λ_2 be simple loops properly embedded in the surface F. Let λ_1 be non-nullhomotopic and freely homotopic to λ_2. Then $\lambda_1 \times \{0\} \cup \lambda_2 \times \{1\}$ bounds an annulus properly embedded in $F \times I$.*

Proof. Theorem 2.1 or Theorem 3.3 in [4] guarantees that there exists an isotopy $H : F \times I \to F$ such that

(1) $H_0 = \mathrm{id}$.

(2) $H_1(\lambda_1) = \lambda_2$.

(3) $H_t \mid \mathrm{bd}(F) = \mathrm{id}$.

The desired annulus is the image of $\lambda_1 \times I$ under the map f defined by $f(m,t) = (H(m,t),t)$ for m in F and t in I.

We will denote the *Euler characteristic* of a manifold F by $\chi(F)$.

Theorem 4.11. *Let F be a surface such that $\chi(F) < 0$. Let α_i for $i = 1, \ldots, m$ be a nonempty collection of disjoint simple arcs properly embedded in $F \times I$ such that $p\,\mathrm{bd}(\alpha_i)$ contains a single point for $i = 1, \ldots, n$. We assume that the collection α_i, $i = 1, \ldots, n$, is h-unknotted. We assume that for each $\lambda \subset F - \bigcup_{i=1}^n p\,\mathrm{bd}(\alpha_i)$, $\lambda \times \{0\}$ is freely homotopic to $\lambda \times \{1\}$ in $F \times I - \bigcup_{i=1}^n \alpha_i$. Then there is an isotopy constant on $\mathrm{bd}(F \times I)$ moving $\bigcup_{i-1}^n \alpha_i$ to a vertical set, i.e. the collection α_i, $i = 1, \ldots, n$, is i-unknotted.*

Proof. Let $h : F \times I \to F \times I$ be a homeomorphism such that $h(\alpha_i)$ is vertical for $i = 1, \ldots, n$. Now $h(\lambda \times \{0\})$ is freely homotopic to $h(\lambda \times \{1\})$. It is a consequence of Lemma 4.10 that $h(\lambda \times \{0,1\})$ bounds an annulus properly embedded in $F \times I - \bigcup_{i=1}^n h(\alpha_i)$ and thus $\lambda \times \{0,1\}$ bounds an annulus properly embedded in $F \times I - \bigcup_{i=1}^n \alpha_i$.

It is easily shown using standard arguments that if λ_1 and λ_2 are disjoint simple loops properly embedded in $F - \bigcup_{i=1}^n p\,\mathrm{bd}(\alpha_i)$, $\lambda_1 \times \{0,1\}$ and $\lambda_2 \times \{0,1\}$ bound disjoint annuli properly embedded in $F \times I - \bigcup_{i=1}^n \alpha_i$.

We claim that the loop $p(\alpha_i)$ is nullhomotopic for $i = 1, \ldots, n$. Let $\{x_i\} = p\,\mathrm{bd}(\alpha_i)$ for $i = 1, \ldots, n$. Let λ_1 and λ_2 be simple non-nullhomotopic loops such that $\lambda_1 \cap \lambda_2 = \{x_1\}$ and if $[\lambda_1]^m = [\lambda_2]^n$ in $\pi_1(F, x_1)$, $m = n = 0$. Let R_j be a möbius band or annulus such that $\lambda_j \subset R_j \subset F$ and $R_j \cap \{x_i, i = 1, \ldots, n\} = \{x_1\}$ for $j = 1, 2$. We assume that R_1 is an annulus. Then by the argument above $\mathrm{bd}(R_1) \times \{0,1\}$ bounds a pair of disjoint annuli A_1, A_2 properly embedded in $F \times I$. It is a consequence of 3.4 that we may assume that A_1 and A_2 are vertical sets. Thus the loop $p(\alpha_i)$ is a representative for $[\lambda_1]^m$ in $\pi_1(F, x_1)$ for some m. It is easily seen that we would have reached the same conclusion if R_1 had been a möbius band.

Similarly the loop $p(\alpha_1)$ is a representative of $[\lambda_2]^n$ in $\pi_1(F, x_1)$ for some n. It follows that the loop $p(\alpha_1)$ is nullhomotopic in F. Repetition of this argument establishes our claim.

We now pick disjoint möbius bands or annuli $R_1, \ldots, R_n$ such that $\{x_i\} \subset R_1$ and $\mathrm{bd}(R_i)$ is not nullhomotopic for $i = 1, \ldots, n$. Then after an argument similar

to the one given above, we see that we may assume that $\alpha_i \subset \operatorname{int}(R_i) \times I$ for $i = 1, \ldots, n$. We may apply Lemma 4.8 to show that α_i is h-unknotted in $R_i \times I$ for $i = 1, \ldots, n$. We see that the loop $p(\alpha_i)$ is nullhomotopic in R_i since $p(\alpha_i)$ is nullhomotopic in F and $\operatorname{bd}(R_i)$ is not nullhomotopic in F. Theorem 4.11 now follows from Theorem 4.3.

Theorem 4.12. *Let F be a surface such that $\chi(F) = 0$. Let α_i, $i = 1, \ldots, n$, be a collection of disjoint arcs properly embedded in $F \times I$ such that $p \operatorname{bd}(\alpha_i) \subset \operatorname{int}(F)$ contains a single point. Suppose that the collection α_i, $i = 1, \ldots, n$, is h-unknotted. Suppose also for every loop $\lambda \subset F - \bigcup_{i=1}^n p \operatorname{bd}(\alpha_i)$, $\lambda \times \{0\}$ is freely homotopic to $\lambda \times \{1\}$ in $F \times I - \bigcup_{i=1}^n \alpha_i$. Then if the loop $p(\alpha_i)$ is nullhomotopic, the collection α_i, $i = 1, \ldots, n$, is i-unknotted.*

Proof. It is a consequence of Theorem 4.3 that α_1 can be assumed to be a vertical arc in $F \times I$. Thus Theorem 4.12 becomes a consequence of 4.11.

There are a number of theorems which the reader may formulate which we could state in case F is a disk relating h-unknottedness and i-unknottedness. One simply finds hypotheses of the sort used in 4.11 and 4.12 which are sufficient to confine the unknotted arcs to product cells in $F \times I$.

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