TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 179, May 1973

EUCLIDEAN n-SPACE MODULO AN $(n-1)$-CELL[1]

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ABSTRACT. This paper, together with another paper by the author titled similarly, provides a complete answer to a conjecture raised by Andrews and Curtis: *if D is a k-cell topologically embedded in euclidean n-space E^n, then $E^n/D \times E^1$ is homeomorphic to E^{n+1}.* Although there is at present only one case outstanding ($n \geqslant 4$ and $k = n-1$), the proof we give here works whenever $n \geqslant 4$. We resolve this conjecture (for $n \geqslant 4$) by proving a stronger result: *if $Y \times E^1 \approx E^{n+1}$ and if D is a k-cell in Y, then $Y/D \times E^1 \approx E^{n+1}$.* This theorem was proved by Glaser for $k \leqslant n-2$ and has as a corollary: *if K is a collapsible polyhedron topologically embedded in E^n, then $E^n/K \times E^1 \approx E^{n+1}$.* Our method of proof uses radial engulfing and a well-known procedure devised by Bing.

1. **Introduction.** In [1] Andrews and Curtis proved that if A is an arc in euclidean n-space E^n, then $E^n/A \times E^1$ is homeomorphic to E^{n+1}. They conjectured that a similar phenomenon occurs for a k-cell D topologically embedded in E^n. In [4] the author proved that $E^n/D \times E^1 \approx E^{n+1}$ whenever D is flat in E^{n+1}. This condition is known to be satisfied except (possibly) when $n \geqslant 4$ and $k = n-1$. (See [11], [7], [5], and [6].)

The main result of this paper is that $E^n/D \times E^1 \approx E^{n+1}$ in the one situation that remains ($n \geqslant 4$ and $k = n-1$). The proof we give actually works for any $k = 1, 2, \ldots, n$ so long as $n \geqslant 4$. It uses a generous application of the engulfing theorems of Bing [3], Seebeck [13], and Wright [16] and the methods of [1], [2], and [4]. It has the added feature that it does not involve a higher dimensional PL (or locally flat) approximation theorem for cells—either directly or indirectly. Thus, combining [4] and the present paper we obtain that, in general, *if D is a k-cell topologically embedded in E^n, then $E^n/D \times E^1 \approx E^{n+1}$.*

The theorem we shall prove is a generalization of this statement in the case $n \geqslant 4$.

Theorem 1.1. *Suppose that Y is a space with the property that $Y \times E^1 \approx E^{n+1}$ ($n \geqslant 4$) and that D is a k-cell topologically embedded in Y. Then $Y/D \times E^1 \approx E^{n+1}$.*

This generalization has been proved by Glaser in case $n = 3$ or $n \geqslant 4$ and $k \leqslant n-2$ [8]. (For $n = 3$, one must also assume, however, that D is flat in E^4.) Its importance can be seen from the following corollary.

Received by the editors March 17, 1971 and, in revised form, August 4, 1971 and April 21, 1972.
AMS (MOS) subject classifications (1970). Primary 54B15, 57A15, 57A30.
Key words and phrases. Euclidean space, usc decompositions of E^n, cartesian factors of E^n, collapsible polyhedron.
[1] This research was supported in part by NSF Grant GP-19964.

Corollary (Glaser, [8]). *If K is a collapsible polyhedron topologically embedded in E^n, then $E^n/K \times E^1 \approx E^{n+1}$.*

Many thanks go to the referee for discovering a serious error in the engulfing theorem of the original version of this paper. The correcting of this mistake led to a considerable simplification of the original manuscript.

2. **Definitions and notation.** We use "$\sim$", "$\simeq$", and "$\approx$" to mean "is homologous to" (integer coefficients), "is homotopic to," and "is homeomorphic to," respectively. Let G be a subset of a metric space X. The ϵ-neighborhood of a point $x \in X$ is denoted by $N_\epsilon(x)$. We say that G is p-lc (p-LC) at a point $x \in X$ iff for each $\epsilon > 0$ there exists $\delta > 0$ such that every singular p-cycle (p-sphere) in $N_\delta(x) \cap G$ is homologous to zero (homotopic to zero) in $N_\epsilon(x) \cap G$. The set G is lc^p (LC^p) at $x \in X$ iff G is q-lc (q-LC) at x for $0 \leqslant q \leqslant p$. G is lc^∞ (LC^∞) at x iff G is q-lc (q-LC) for all $q \geqslant 0$. The terms p-ulc, p-ULC, ulc^p, and ULC^p are used whenever the δ corresponding to ϵ and x above may be chosen independently of x. Finally, we say that G has property 1-ALG at $x \in X$ [9] iff for each $\epsilon > 0$ there exists $\delta > 0$ such that for each singular 1-sphere Γ in $N_\delta(x) \cap G$, $\Gamma \sim 0$ in $N_\epsilon(x) \cap G$ iff $\Gamma \simeq 0$ in $N_\epsilon(x) \cap G$.

We use I to denote the unit interval $[0, 1]$ and I^k to denote the cartesian product $I \times \ldots \times I$, k factors. Suppose that $Y \times E^1 \approx E^{n+1}$. The metric we shall use on $Y \times E^1$ is the metric d defined by

$$d((y,t),(y',t')) = \max\{\|(y,0) - (y',0)\|, |t - t'|\},$$

where $\|\cdot\|$ is the usual norm on E^{n+1}. The projections of E^{n+1} onto Y and E^1 are denoted by π_1 and π_2, respectively.

3. **$E^{n+1} - D$ is 1-ALG at points of D.** Throughout this section Y is a space such that $Y \times E^1 \approx E^{n+1}$ and D is a k-cell topologically embedded in Y ($= Y \times 0$). In the original version of this paper we showed that the embedding of D into E^{n+1} has a much stronger property than that stated in the title of this section; namely, that D is *locally homotopically unknotted* [6] in E^{n+1}. Our present proof of Theorem 1.1, however, does not require anything this strong.

Theorem 3.1. *$E^{n+1} - D$ is LC^1 at each point of* Bd D.

Proof. Notice that $E^{n+1} - D$ is lc^∞ at each point of Bd D by local duality [15]; hence, we need only show that $E^{n+1} - D$ is 1-LC at all such points. Also, $E^{n+1} - (D \times E^1)$ is lc^∞ at points of Bd $D \times E^1$, which implies that $Y - D$ is lc^∞ at points of Bd D.

Suppose that $y \in \text{Bd } D$ and $\epsilon > 0$. Choose $\gamma > 0$ and $\delta > 0$ so that $N_\gamma(y) \cap Y$ is contractible to a point in $N_\epsilon(y) \cap Y$ and each pair of points in $N_\delta(y) \cap (Y - D)$ can be joined by a path in $N_\gamma(y) \cap (Y - D)$. (Y is locally contractible since it is a retract of E^{n+1}.)

Let $f : S^1 \to (N_\delta(y) - D)$ be given. Choose $\eta > 0$ so that if $g : S^1 \to E^{n+1}$ is a map with $d(f,g) < \eta$, then $f \simeq g$ in $N_\delta(y) - D$. Let T be a subdivision of S^1 with the following properties:

(i) diam $f(A) < \eta/2$ for all $A \in T$,

(ii) if $f(A) \cap Y \neq \varnothing$, then $\pi_1 f(A) \subset N_\delta(y) \cap (Y - D)$,

(iii) if v is a vertex in T, then $f(v) \notin Y$. (We may have to change f by a small homotopy in order to satisfy this condition.)

Let A be a 1-simplex of T such that $f(A) \cap Y \neq \varnothing$. Parameterize A by t, $0 \leqslant t \leqslant 1$, and define $g_A : A \to E^{n+1}$ by

$$g_A(t) = (\pi_1 f(t), (1-t)\pi_2 f(0) + t\pi_2 f(1)).$$

Observe that $g_A(0) = f(0)$, $g_A(1) = f(1)$ and $d(g_A(t), f(t)) \leqslant d(g_A(t), g_A(0)) + d(f(0), f(t)) < \eta$ for each $t \in I$. Define $g : S^1 \to E^{n+1}$ by

$$\begin{aligned} g(x) &= g_A(x), && \text{if } x \in A \text{ and } f(A) \cap Y \neq \varnothing, \\ &= f(x), && \text{otherwise.} \end{aligned}$$

Then $d(f,g) < \eta$ and so $f \simeq g$ in $N_\delta(y) - D$. Observe also that $g(S^1) \cap Y$ is a finite set.

Let α be a subarc of $g(S^1)$ joining successive points a and b of $g(S^1) \cap Y$. By our choice of δ, a and b can be joined by a path β in $N_\gamma(y) \cap (Y - D)$. By our choice of γ, $\beta \cup \pi_1(\alpha)$ bounds a singular disk Δ in $N_\epsilon(y) \cap Y$. Let $v * S^1 = \{tv + (1-t)w \mid w \in S^1, 0 \leqslant t \leqslant 1\}$ be the (abstract) cone over S^1, and let $g' : v * S^1 \to \Delta$ be a map with $g'(S^1) = \beta \cup \pi_1(\alpha)$. Let $h : S^1 \to \alpha \cup \beta$ be a map such that $\pi_1 h = g' \mid S^1$. Extend h to a map of $v * S^1$ into E^{n+1} by the formula

$$h(tv + (1-t)w) = (g'(tv + (1-t)w), t\epsilon' + (1-t)\pi_2 h(w))$$

for $0 \leqslant t \leqslant 1$, $w \in S^1$, where $\epsilon' = \pm\epsilon$ accordingly as Int $\alpha \subset Y \times (0, \infty)$ or Int $\alpha \subset Y \times (-\infty, 0)$.

Then $h(v * S^1) \cap Y = \beta$ yields $h(v * S^1) \subset N_\epsilon(y) - D$. Applying this procedure to each arc α in $g(S^1)$ joining successive points of $g(S^1) \cap Y$, we obtain a homotopy of g in $N_\epsilon(y) - D$ to a map $g_1 : S^1 \to N_\gamma(y) \cap (Y - D)$. Next homotope g_1 to $g_2 : S^1 \to N_\gamma(y) \cap (Y \times \{\epsilon/2\})$ by pushing along the E^1 factor of $Y \times E^1$. Again by our choice of γ, g_2 is null-homotopic in $N_\epsilon(y) \cap (Y \times \{\epsilon/2\})$. Piecing these homotopies together, we get a homotopy of g to 0 in $N_\epsilon(y) - D$; hence, $f \simeq 0$ in $N_\epsilon(y) - D$.

Corollary 3.2. *If $k \neq n-1$, then $E^{n+1} - D$ is 1-LC at each point of D.*

Proof. For $k < n-1$ this follows from the proof of Theorem 3.1, since local duality implies that $Y - D$ is 0-lc at each point of D. If $k = n$, then the fact that Y is locally contractible implies that $E^{n+1} - D$ is 1-LC at points of Int D. (In fact, D is locally flat at points of Int D.)

Theorem 3.3. *If* $k = n - 1$, $E^{n+1} - D$ *is* 1-*ALG at points of* Int D.

Proof. We proceed very much the same as in the proof of Theorem 3.1. Suppose that $y \in \operatorname{Int} D$ and $\epsilon > 0$. Choose $\gamma > 0$ so that $N_\gamma(y) \cap Y$ is contractible to a point in $N_\epsilon(y) \cap Y$. Since $D \times E^1$ is locally 2-sided in E^{n+1} at y, D is locally 2-sided in Y at y. Thus there is a neighborhood U of y in E^{n+1} lying in $N_\gamma(y)$ such that $U \cap (Y - D)$ has exactly two components (which are separated in $N_\epsilon(y) \cap (Y - D)$). Choose $\delta > 0$ so that $N_\delta(y) \subset U$. Let Γ' be a simple closed curve in $N_\delta(y) - D$ that is homologous to zero in $N_\epsilon(y) - D$. From the proof of Theorem 3.1, we see that Γ' is homotopic (in $N_\delta(y) - D$) to a simple closed curve Γ such that Γ meets Y "transversally" in a finite number of points —that number necessarily being an even integer.

Write $\Gamma \cap Y = \{x_0, x_1, \ldots, x_{2m-1}\}$, where the x_i's are arranged cyclically on Γ. Let

$$j : H_1(N_\epsilon(y) - D) \to H_0(N_\epsilon(y) \cap (Y - D))$$

be the homomorphism obtained from the Mayer-Vietoris sequence of the triad $(N_\epsilon(y) - D; N_\epsilon(y) \cap ((Y \times [0, \infty)) - D), N_\epsilon(y) \cap ((Y \times (-\infty, 0]) - D))$, and let

$$i : H_1(N_\delta(y) - D) \to H_1(N_\epsilon(y) - D)$$

be the inclusion induced homomorphism. Then

$$ji([\Gamma]) = \sum_{r=0}^{2m-1} (-1)^{r+1}[x_r] = 0.$$

Since $\{x_0, x_1, \ldots, x_{2m-1}\}$ lies in the union of two components of $N_\epsilon(y) \cap (Y - D)$, it must be true that for some $r \pmod{2m}$, x_r and x_{r+1} lie in the same component of $N_\epsilon(y) \cap (Y - D)$; hence, in the same component of $U \cap (Y - D)$. An argument similar to one given in the proof of Theorem 3.1 can now be used to show that Γ is homotopic (in $N_\epsilon(y) - D$) to a simple closed curve Γ_1 in $U - D$ that has two fewer intersections with Y. Applying this process m times, we arrive at a curve Γ_m in $U - D$, homotopic to Γ in $N_\epsilon(y) - D$, such that $\Gamma \cap Y = \varnothing$. Thus, by the choice of γ, $\Gamma \simeq 0$ in $N_\epsilon(y) - D$, and we are through.

4. **Engulfing.** Throughout this section we shall use the following notation:
$f : I^{k-1} \times I \to E^m$ is an embedding ($m \geqslant 5, k < m$),
$D[a, b] = f(I^{k-1} \times [a, b])$,
$D[a] = D[a, a]$, and
$E^m - D[a, b]$ is 1-ALG at each point of $D[a, b]$ for $0 \leqslant a \leqslant b \leqslant 1$.

Proposition 4.1. *Suppose that W is a neighborhood of $D[0, a]$ and $\epsilon > 0$. Then there exist $\delta > 0$ and a neighborhood W' of $D[0, a]$ with the following properties: If P is an $(m - 3)$-polyhedron in $N_\delta(D[a, b])$, then there exists an isotopy h_t $(t \in I)$ of E^m such that*

(i) $h_0 =$ *identity,*

(ii) $h_t =$ *identity on* W' *and outside* $N_\epsilon(D[a,b])$,

(iii) $h_1(W) \supset P$, *and*

(iv) *if* $z \in E^m$, *either* $h_t(z) = z$ *for all* $t \in I$ *or there exists* $x \in I^{k-1}$ *such that* $h_t(z) \in N_\epsilon(f(x \times [a,b]))$ *for all* $t \in I$.

Proof. Suppose that W is a neighborhood of $D[0,a]$ and $\epsilon > 0$. We shall construct the homotopies necessary to apply radial engulfing.

Let $r_t : E^m \to E^m$ be the "straight-line" homotopy between the identity (r_0) and a retraction (r_1) of E^m onto $D[a,b]$. Choose $c > a$ so that $D[0,c] \subset W$. Then there exist neighborhoods U of $D[0,a]$ and V of $D[c,b]$ such that $r_s(\overline{U}) \cap r_t(\overline{V}) = \varnothing$ for all $s, t \in I$. Let V' be a neighborhood of $D[c,b]$ such that $\overline{V'} \subset V$ and let $\alpha : E^m \to I$ be a mapping such that $\alpha(E^m - V) = 0$ and $\alpha(\overline{V'}) = 1$.

Define $\phi_t : E^m \to E^m$ by $\phi_t(y) = r_{t\alpha(y)}(y)$. Then $\phi_0 =$ identity, $\phi_t(y) \in U$ for some $t \in I$ implies $\phi_t(y) = y$ for all $t \in I$, $\phi_1 | V' : V' \to V' \cap D[a,b]$ is a retraction and $\phi_t | D[0,b] =$ identity for all $t \in I$.

Let ψ_t be the natural homotopy of the identity on $D[c,b]$ to the projection of $D[c,b]$ onto $D[c]$.

The homotopy ϕ_t followed by ψ_t will pull all sufficiently small neighborhoods of $D[a,b]$ into W. Moreover, for every $\delta > 0$ there is a neighborhood N of $D[a,b]$ such that $\phi_t \mid N$ is a δ-homotopy. Thus, the engulfing techniques of [3], [13], and [16] can be applied to give the desired isotopies.

An important observation is that the neighborhood W' of $D[0,a]$ depends only upon W and the embedding f (or, more precisely, the deformation retraction r_t).

Lemma 4.2. *Suppose that* K *is a 2-complex in* E^m *such that* $K \cap D[0,a] = \varnothing$. *Then for each* $\epsilon > 0$ *there exists a homotopy* $g_t : K \to E^m$ $(t \in I)$ *such that*

(1) $g_0 =$ *inclusion,*

(2) $g_t \mid K - N_\epsilon(D[a,b]) =$ *inclusion,*

(3) $g_t(K) \cap D[0,a] = \varnothing$ *for each* $t \in I$,

(4) $g_1(K) \cap D[0,b] = \varnothing$, *and*

(5) *for each* $y \in K$ *either* $g_t(y) = y$ *for all* $t \in I$ *or there exists* $x \in I^{k-1}$ *such that* $g_t(y) \in N_\epsilon(f(x \times [a,b]))$.

Proof. *Case* 1. $k \leqslant m - 3$. This situation is easy to handle since $E^m - D$ is 1-ULC.

Case 2. $k = m - 2$. Let T be a fine triangulation of K. Since $m \geqslant 5$ and $k = m - 2$, the 1-skeleton T^1 of T can be moved off of $D[a,b]$ with an arbitrarily small isotopy of E^m that is fixed outside a neighborhood of $D[a,b]$. So we will assume that $T^1 \cap D[a,b] = \varnothing$. Let σ be a 2-simplex of T that meets $D[a,b]$, and let v be a point of $\sigma \cap D[a,b]$ ($v \in \operatorname{Int} \sigma$). Then $\sigma = \{sv + (1-s)z \mid s \in I, z \in \operatorname{Bd} \sigma\}$. Given $0 \leqslant r \leqslant s \leqslant 1$, define

$$C(r) = \{rv + (1-r)z \mid z \in \operatorname{Bd} \sigma\}$$

and

$$C(r,s) = \{tv + (1-t)z \mid r \leqslant t \leqslant s,\ z \in \operatorname{Bd} \sigma\}.$$

Choose $s_0, s_1 \in I$ so that $0 < s_0 < s_1 < 1$ and $C(0, s_1) \cap D[a,b] = \varnothing$. Write $v = f(x,c)$, where $x \in I^{k-1}$ and $a < c \leqslant b$. (See Figure 1.)

Let $\alpha_t : [s_0, 1] \to (0, \infty)$ $(t \in I)$ be the linear map satisfying $\alpha_t(s_0) = s_0$ and $\alpha_t(s_1) = (1-t)s_1 + t$, and let $\beta(s,t) \in [c,b]$ $(s,t \in I)$ satisfy the equation

$$\frac{\beta(s,t) - c}{[(1-t)c + tb] - c} = \frac{s - \alpha_t^{-1}(t)}{1 - \alpha_t^{-1}(1)}$$

whenever $t > 0$. (Note, $\alpha_0^{-1}(1) = 1$.)

Define $g'_t : \sigma \to E^m$ $(t \in I)$ by

$$\begin{aligned} g'_t(sv + (1-s)z) &= sv + (1-s)z && \text{if } 0 \leqslant s \leqslant s_0 \text{ or } t = 0,\\ &= \alpha_t(s)v + (1 - \alpha_t(s))z && \text{if } s_0 \leqslant s \leqslant \alpha_t^{-1}(1)\ (t > 0),\\ &= f(x, \beta(s,t)) && \text{if } \alpha_t^{-1}(1) \leqslant s \leqslant 1\ (t > 0). \end{aligned}$$

Then g'_t has the property that g'_0 = inclusion, $g'_t \mid C(0, s_0)$ = inclusion, $g'_1(C(s_0, s_1)) = C(s_0, 1)$, and $g'_1(C(s)) = f(x, \beta(s, 1))$ for $s_1 \leqslant s \leqslant 1$.

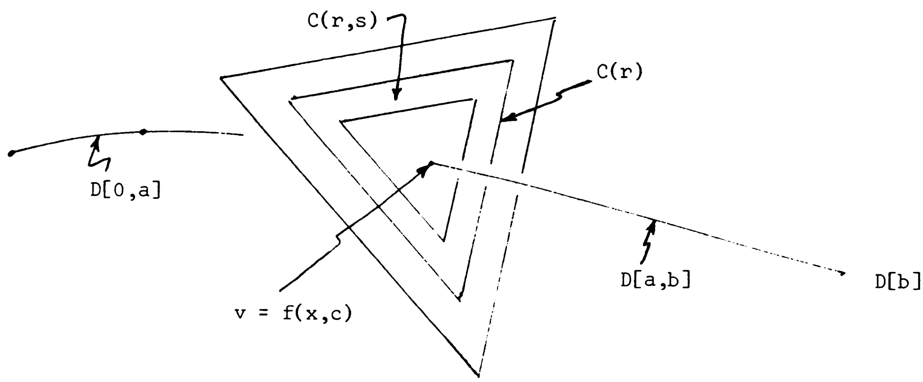

Figure 1

From local duality [15] we know that if $a < d \leqslant b$ and U is a neighborhood of $f(x,d)$ in E^m, then there exists a neighborhood V of $f(x,d)$ in U such that the image of the inclusion $i_* : H_1(V - D[a,b]) \to H_1(U - D[a,b])$ is either zero (if $x \in \operatorname{Bd} I^{k-1}$ or $d = b$) or possibly the integers (if $(x,d) \in \operatorname{Int} D[a,b]$). Hence, if y is a point of $V \cap f(x \times [a,b])$ and if $z \in \operatorname{im} i_*$, then arbitrarily close to y there is a simple closed curve Γ in $V - D[a,b]$ such that $i_*([\Gamma]) = z$. Therefore, assuming that the diameter of σ is sufficiently small, we can use standard compactness arguments to find numbers $s_2, \ldots, s_n \in [s_1, 1]$ with $s_1 < s_2 < \ldots < s_n < 1$ and simple closed curves $\Gamma_1, \ldots, \Gamma_n$ such that (taking $\Gamma_0 = C(s_0)$)

$$\Gamma_i \subset (\text{neighborhood of } g_1'(C(s_i))) - D[a,b],$$

$$\Gamma_i \sim \Gamma_{i-1} \quad \text{in (larger neighborhood of } g_1'(C(s_i))) - D[a,b], \text{ and}$$

$$\Gamma_n \sim 0 \quad \text{in (neighborhood of } f(x,b)) - D[a,b]. \text{ (See Figure 2.)}$$

Using the 1-ALG property of $E^m - D[a,b]$, we see that if $\Gamma_1, \ldots, \Gamma_n$ are suitably chosen, then $\Gamma_i \cup \Gamma_{i-1}$ $(i = 1, \ldots, n)$ bounds a singular annulus in $N_{\epsilon/2}(g_1'(C(s_i))) - D[a,b]$ and Γ_n bounds a singular disk in $N_{\epsilon/2}(f(x,b)) - D[a,b]$. Thus, we can find a map $g : \sigma \to N_{\epsilon/2}(f(x \times [a,b])) - D[a,b]$ such that $g(\sigma) \cap D[0,a] = \varnothing$, $g \mid \text{Bd } \sigma =$ inclusion, and g is $(\epsilon/2)$-homotopic (rel Bd σ) to g_1'. Piecing these homotopies together as σ ranges over the simplexes of K that meet $D[a,b]$ gives the desired homotopy g_t $(t \in I)$ of K in E^m.

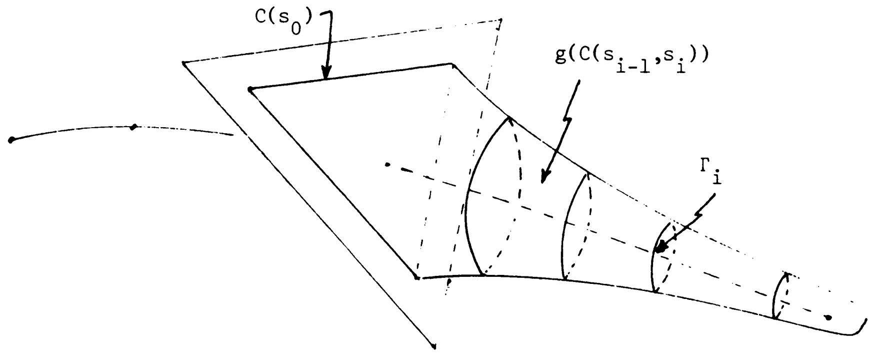

Figure 2

Case 3. $k = m - 1$. Again let T be a fine triangulation of K. We shall assume that no vertex of T lies in $D[a,b]$ and that no 1-simplex of T meets Bd $D[a,b]$. We shall also assume that no 1-simplex of T that meets $D[a,b]$ has both of its vertices "on the same side" of $D[a,b]$. Thus, every 2-simplex of T meets $D[a,b]$ in essentially one of two ways as illustrated in Figure 3.

We proceed in much the same way as in Case 2. Let e be a 1-simplex of T that meets $D[a,b]$, and let v be a point of intersection. Write $v = f(x,c)$, where $x \in I^{k-1}$ and $a < c < b$. Given $0 \leqslant r \leqslant s \leqslant 1$, define

$$A(r) = \{rv + (1-r)z \mid z \in \text{Bd } e\}$$

and

$$A(r,s) = \{tv + (1-t)z \mid z \in \text{Bd } e, r \leqslant t \leqslant s\}.$$

Given a 2-simplex σ of T that meets $D[a,b]$ and $0 \leqslant r \leqslant s \leqslant 1$, define subsets $C(r)$ and $C(r,s)$ of σ as in Figure 3. The specific formula for $C(r)$ (and $C(r,s)$) will, of course, depend on whether one or two of the edges of σ meet $D[a,b]$. Choose s_0 and s_1 so that $0 < s_0 < s_1 < 1$ and $C(0,s_1) \cap D[a,b] = \varnothing$ for the set $C(0,s_1)$ corresponding to each 2-simplex σ of T that meets $D[a,b]$. If a 2-simplex σ of T has two of its edges intersecting $D[a,b]$, we homotope σ (rel Bd σ) so that the segment $C(1)$ is carried "linearly" onto the image under f of the segment in I^k joining (x_1,c_1) and (x_2,c_2), where $v_1 = f(x_1,c_1)$ and $v_2 = f(x_2,c_2)$ are the distinquished points Bd $\sigma \cap D[a,b]$.

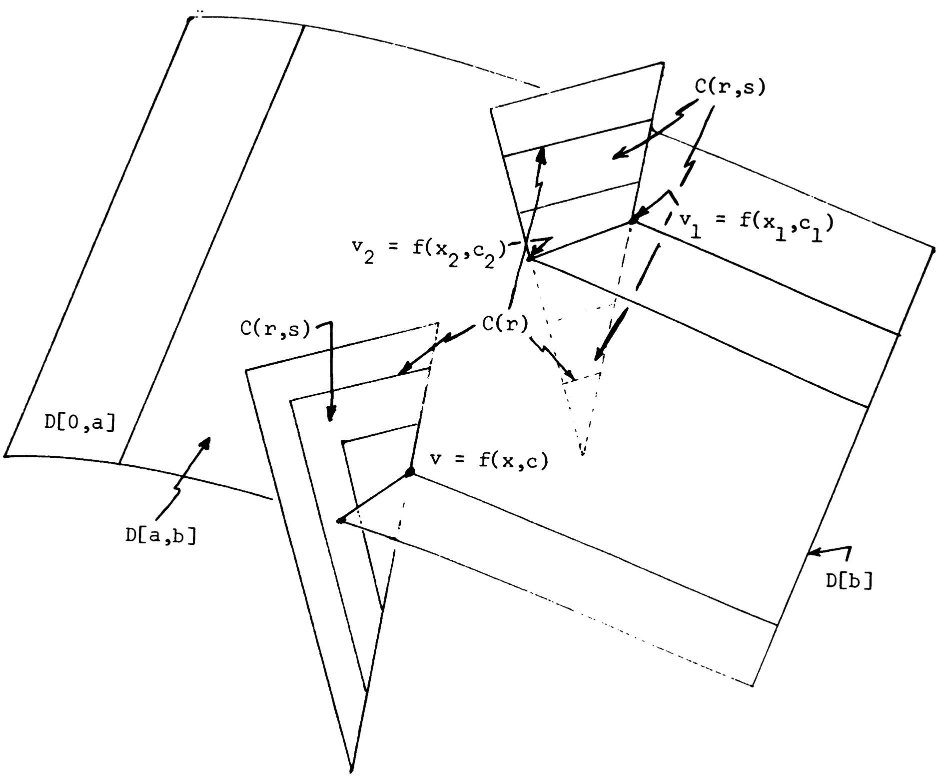

Figure 3

Next, construct $g'_t : T \to E^n$ $(t \in I)$ using the same type of formulas as in Case 2 on each 2-simplex of T that meets $D[a,b]$. Since $E^m - D[a,b]$ is 1-ULC, the map $g'_1 : T \to E^m$ will be homotopic, via a small homotopy, to a map $g : T \to E^m - D[a,b]$. (See Figure 4.) The combination of the three homotopies is the desired homotopy f_t $(t \in I)$.

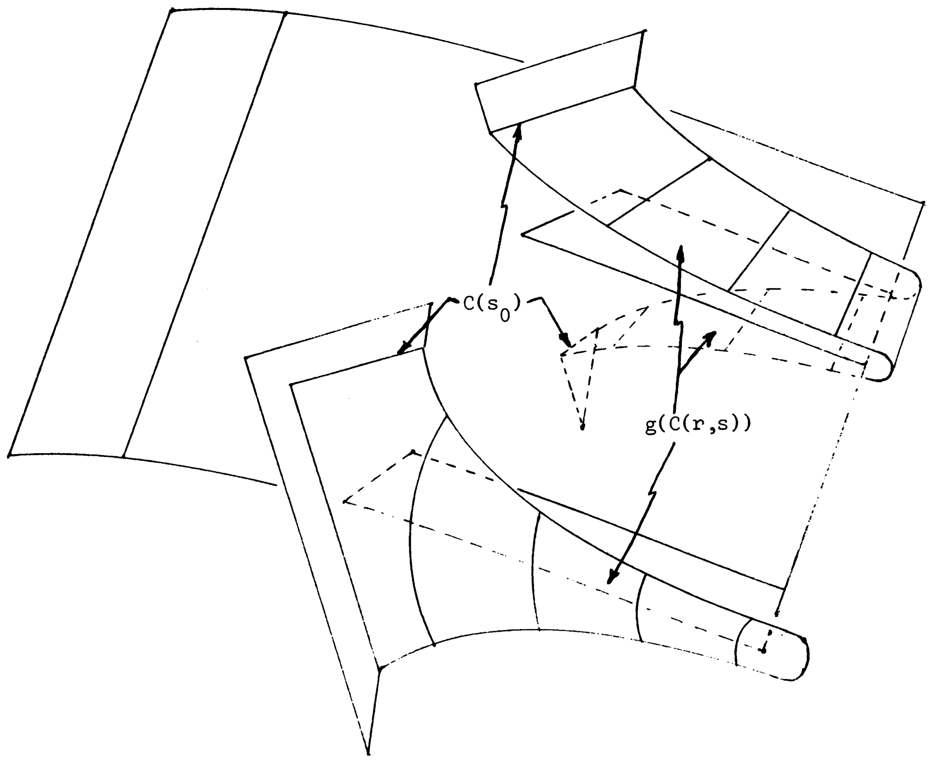

Figure 4

Proposition 4.3. *Suppose that K is a 2-complex in E^m such that $K \cap D[0,a] = \varnothing$. Then there is a neighborhood W of $D[0,a]$ such that for every $\epsilon > 0$ there exists an isotopy g_t $(t \in I)$ of E^m satisfying*:

(i) $g_0 =$ *identity*,

(ii) $g_t \mid W \cup (E^m - N_\epsilon(D[a,b])) =$ *identity for all* $t \in I$,

(iii) $g_1(K) \cap D[0,b] = \varnothing$, *and*

(iv) *for each* $y \in E^m$ *either* $g_t(y) = y$ *for all* $t \in I$ *or there exists* $x \in I^{k-1}$ *such that* $g_t(y) \in N_\epsilon(f(x \times [a,b]))$ *for all* $t \in I$.

Proof. The proof of this lemma uses the homotopy provided by Lemma 4.2 together with the radial engulfing technique of [3], [13], and [16]. In particular, one must use Zeeman's piping lemma [17] in case $m = 5$ as in [16]. To get the neighborhood W of $D[0,a]$, first select $c > a$ $(c \leqslant b)$ so that $K \cap D[0,c] = \varnothing$, and then simply choose W so that $\overline{W} \cap D[0,b] \subset D[0,c]$.

Theorem 4.4. *Suppose that W is a neighborhood of $D[0,a]$ and $\epsilon > 0$. Then there exist a neighborhood W' of $D[0,a]$ and an isotopy h_t $(t \in I)$ of E^m such that*

(i) $h_0 =$ *identity*,

(ii) $h_t \mid W' \cup (E^m - N_\epsilon(D[a,b])) =$ *identity*,

(iii) $h_1(W) \supset D[0,b]$, *and*

(iv) *if $y \in E^m$, either $h_t(y) = y$ for all $t \in I$ or there exists $x \in I^{k-1}$ such that $g_t(y) \in N_\epsilon(f(x \times [a,b]))$ for all $t \in I$.*

Proof. Suppose $\delta > 0$. Choose W'' and $\gamma > 0$ corresponding to W and δ as in Proposition 4.1. Let M be a PL neighborhood of $D[a,b]$ lying in $N_\gamma(D[a,b])$, and let T be a fine triangulation of M with the property that no simplex of T can intersect both $D[0,a]$ and $E^m - W''$. Let K be the subcomplex of T^2, the 2-skeleton of T, obtained by taking all $\sigma \in T^2$ such that $\sigma \cap D[0,a] = \varnothing$, and let L be the dual of K in T (i.e., L consists of all simplexes σ in T', the first barycentric subdivision of T, such that $\sigma \cap K = \varnothing$). Then $\dim(L - W'') \leqslant m - 3$, and hence $L - W''$ lies in a subcomplex L_1 of L such that $\dim L_1 \leqslant m - 3$. Let W' be a neighborhood of $D[0,a]$ corresponding to K as in Proposition 4.3 and having the additional property that $W' \cap M \subset \operatorname{Int} L$ (interior relative to M) and $W' \subset W''$.

Let h'_t $(t \in I)$ be the isotopy of E^m satisfying (i)–(iv) of Proposition 4.1 with $\{\delta, L_1, W''\}$ replacing $\{\epsilon, P, W'\}$. Since $h'_t \mid W'' =$ identity for all $t \in I$, $h'_1(W) \supset L$.

Now let g_t $(t \in I)$ be an isotopy of E^m satisfying (i)–(iv) of Proposition 4.3 with $\{\lambda, K, W'\}$ replacing $\{\epsilon, K, W\}$, where $\lambda > 0$ is small enough so that $\lambda \leqslant \delta$ and $N_\lambda(D[a,b]) \subset M$. Then $g_t \mid W' \cup (E^m - M) =$ identity for all $t \in I$.

We now have $h'_t(W) \supset L$ and $g_1^{-1}(E^m - D[0,b]) \supset K$, where K' and L are dual subcomplexes of T'. Hence, there is an isotopy ϕ_t $(t \in I)$ of E^m that is fixed outside a neighborhood of M and on W' such that $\phi_0 =$ identity and $\phi_1 h'_1(W) \cup g_1^{-1}(E^m - D[0,b]) \supset M$. (This is Stallings' isotopy [14].) Moreover, the distance a point moves under ϕ_t is no greater than the mesh of T (hence, arbitrarily small). Observe that

$$g_1 \phi_1 h'_1(W) \cup (E^m - D[0,b]) \supset g_1(M) = M$$

and of each g_t, ϕ_t, and h'_t $(t \in I)$ is fixed on W' so that

$$g_1 \phi_1 h'_1(W) \supset D[0,b].$$

If δ is sufficiently small, then the desired isotopy h_t $(t \in I)$ of E^m is obtained by "stacking" the isotopies h'_t, ϕ_t, and g_t (in that order).

5. **The proof of Theorem 1.1.** In this section we shall set up the machinery so that we can appeal to the methods of [2] and [4]. As before, Y denotes a space with the property that $Y \times E^1 \simeq E^{n+1}$.

Theorem 5.1. *If $f: I^{k-1} \times I \to Y$ $(n \geqslant 4)$ is an embedding with $D = f(I^k)$, then for each $\epsilon > 0$ there exists an isotopy h_t $(t \in I)$ of E^{n+1} satisfying*:

(1) $h_0 =$ *identity*,

(2) $h_t =$ *identity outside* $N_\epsilon(D \times E^1)$,

(3) h_t *is uniformly continuous*,

(4) *for each $z \in E^{n+1}$, either $h_t(z) = z$ for all $t \in I$ or there exist $x \in I^{k-1}$ and $w \in E^1$ such that $h_t(z) \in N_\epsilon(f(x \times I) \times w)$ for all $t \in I$, and*

(5) *for all $w \in E^1$ there exists $y \in I$ such that for all $x \in I^{k-1}$,*

$$h_1(f(x \times I) \times w) \subset N_\epsilon(f(x,y) \times w).$$

Theorem 5.1 is Statement $H(n,k,1)$ of [4] with $n \geqslant 4$. Once we have proved this theorem, we will be through because we proved in [4] that $H(n,k,1)$ implies Theorem 1.1. (See Lemmas 2.1 and 2.2 of [4].)

Proof of Theorem 5.1. Suppose we are given $f: I^{k-1} \times I \to Y$ $(n \geqslant 4)$ and $\epsilon > 0$. As usual we shall set $D = f(I^k)$ and $D[a,b] = f(I^{k-1} \times [a,b])$.

Let $a_0 = 0 < a_1 < a_2 < \ldots < a_m = 1$ be numbers in I, and choose δ $(0 < \delta < \epsilon)$ so that $N_\delta(D[0,a_{i-1}] \times E^1) \cap N_\delta(D[a_i,1]) = \varnothing$ for $i = 1, \ldots, m-1$.

Let ϵ_1 be a positive number and let N_1 be a neighborhood of $D[0,a_{m-1}]$ such that $\overline{N}_1 \subset N_\epsilon(D) \cap (Y \times (-\epsilon_1, \epsilon_1))$. Apply Theorem 4.4 and get an isotopy h_t^1 $(t \in I)$ such that

$h_0^1 =$ identity,

$h_t^1 =$ identity outside $N_\delta(D[a_{m-1}, 1] \cap (Y \times (-\epsilon_1, \epsilon_1))$,

$h_1^1(N_1) \supset D$, and

h_t^1 satisfies condition (iv) of Theorem 4.4 with $\{a_{m-1}\, 1, \epsilon/2\}$ replacing $\{a, b, \epsilon\}$.

There exists ϵ_2 $(0 < \epsilon_2 < \epsilon_1)$ such that $h_1^1(N_1) \supset D \times [-\epsilon_2, \epsilon_2]$. Let N_2 be a neighborhood of $D[0, a_{m-2}]$ such that

$$\overline{N}_2 \subset N_\delta(D[0,a_{m-2}]) \cap (Y \times (-\epsilon_2, \epsilon_2)).$$

Let λ_2 be a positive number, and apply Theorem 4.4 to get an isotopy h_t^2 $(t \in I)$ of E^{n+1} such that

$h_0^2 =$ identity,

$h_t^2 =$ identity outside $N_\delta(D[a_{m-2}, 1]) \cap (Y \times (-\epsilon_2, \epsilon_2))$,

$h_1^2(N_2) \supset D$, and

h_t^2 satisfies (iv) of Theorem 4.4 with $\{a_{m-2}, 1, \lambda_2\}$ replacing $\{a, b, \epsilon\}$.

We continue in this manner, obtaining numbers $\epsilon_1 > \epsilon_2 > \ldots > \epsilon_m > 0$, neighborhoods N_i of $D[0, a_{m-i}]$ $(i = 1, 2, \ldots, m-1)$, positive numbers $\lambda_2, \lambda_3, \ldots, \lambda_{m-1}$ and isotopies h_t^i $(t \in I)$ $(i = 1, 2, \ldots, m-1)$ that satisfy

$h_0^i =$ identity,

$h_t^i =$ identity outside $N_\delta(D[a_{m-i}, 1]) \cap (Y \times (-\epsilon_i, \epsilon_i))$,

$h_1^i(N_i) \supset D \times [-\epsilon_{i+1}, \epsilon_{i+1}]$,

h_t^i satisfies (iv) of Theorem 4.4 with $\{a_{m-i}, 1, \lambda_i\}$ replacing $\{a, b, \epsilon\}$,

where $\lambda_1 = \epsilon/2$.

Observe that $h_t^j \mid N_i =$ identity if $i > j$. If the numbers $\lambda_2, \lambda_3, \ldots, \lambda_{m-1}$ are chosen properly, then the homeomorphism $h = (h_1^1)^{-1}(h_1^2)^{-1} \ldots (h_1^{m-1})^{-1}$ will be the 1-level of an isotopy h_t $(t \in I)$ having the following properties:

$h(D \times [-\epsilon_{i+1}, \epsilon_{i+1}]) \subset N_i$ $(i = 1, \ldots, m-1)$,

$h_t =$ identity outside $N_\delta(D[a_{m-i}, 1]) \cap (Y \times (-\epsilon_i, \epsilon_i))$, and

h_t satisfies (iv) of Theorem 4.4 with $\{0, 1, \epsilon\}$ replacing $\{a, b, \epsilon\}$.

We may now appeal to the technique of proof in Lemma 2 of [2] to complete the proof of Theorem 5.1. (See also Theorems 4.2 and 4.3 of [4].)

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