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ON THE CLASSIFICATION OF METABELIAN LIE ALGEBRAS(1)

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ABSTRACT. The classification of 2-step nilpotent Lie algebras is attacked by a generator-relation method. The main results are in low dimensions or a small number of relations.

Introduction. According to a theorem of Levi, in characteristic zero a finite-dimensional Lie algebra can be written as the direct sum of a semisimple subalgebra and its unique maximal solvable ideal. If the field is algebraically closed, all semisimple Lie algebras and their modules are classified [15]. Around 1945, Malcev [20] reduced the classification of complex solvable Lie algebras to several invariants plus the classification of nilpotent Lie algebras. The latter problem is investigated in this paper.

In §1, I introduce a generator-relation method for attacking the classification of nilpotent Lie algebras. Most of the results achieved in the remainder of the paper concern metabelian (i.e. 2-step nilpotent) Lie algebras and are obtained by specializing the results of §1.

If N is a metabelian Lie algebra ($[[N, N], N] = 0$) then $g = \dim N/[N, N]$ is the least number of elements required to generate N. Let V be a g-dimensional vector space. In §2 it is shown that isomorphism classes of g-generator metabelian Lie algebras are in one-to-one correspondence with orbits of subspaces of $\bigwedge^2 V$ acted on by $\mathrm{GL}(V)$. This correspondence lends itself to the rapid and natural development of a duality theory for these algebras having the same fundamental properties as Scheuneman's duality [24]. That is, to an algebra such as N, we associate another such algebra N^0, the dual, satisfying $(N^0)^0 \cong N$, $N_1 \cong N_2$ if and only if $N_1^0 \cong N_2^0$, and if $\dim N = g + \binom{g}{2} - p$, then $\dim N^0 = g + p$. Furthermore, a canonical isomorphism is exhibited between $\bigwedge^2 V$ and $\mathrm{Alt}(V^*)$ (the space of alternating forms on V^*) which induces a bijection between the orbit spaces $\bigwedge^2 V/\mathrm{GL}(V)$ and $\mathrm{Alt}(V^*)/\mathrm{GL}(V^*)$. Thus, the classification problem can be viewed as a problem of obtaining a "simultaneous canonical form" for a space of alternating forms on V^*. In particular, determining orbits of 1-dimensional subspaces is equivalent to computing ranks of alternating forms. Due to classical results of Weierstrass and Kronecker on so-called pencils of matrices, a canonical form is obtained for a 2-dimensional space of alternating

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forms. These results are applied in §7 to give a complete classification of metabelian Lie algebras of dimension ≤ 7 and nearly complete results for dimension 8. By an algebro-geometric argument it is shown that there are infinitely many metabelian Lie algebras for each dimension $n \geq 9$.

Over a field of characteristic zero, Jacobson [17] showed that a Lie algebra having an injective derivation was nilpotent. The converse was later disproven by Dixmier and Lister [10]. However, we do exhibit a large class of nilpotent Lie algebras, including metabelian Lie algebras, which do have injective derivations. As a consequence, these algebras are algebraic and can be faithfully represented on a space whose dimension exceeds that of the algebra itself by one.

The first two chapters of Jacobson [15] provide an excellent reference for the basic facts about Lie algebras, while Chapter 1 of Mumford [23] contains most of the necessary algebraic geometry. Results cited on Grassmann varieties can be found in [14].

Immeasurable thanks are due to Carl Riehm for his inspiration and contributions. In particular, though I was aware of a duality theory, it was he who first suggested the "canonical" approach through the natural duality of $\bigwedge^2 V$ and $\bigwedge^2 (V^*)$. He also provided many calculations of the low-dimensional algebras. The dimension argument of Theorem 7.8 was suggested to me by Mario Borelli.

Finally, it was unforeseeable that the referee would add so much to the exposition and theory itself as he did. In many places he improved results by suggesting different and better references. The proof of Lemma 6.2 is due to him. It is not only simpler and more elegant than my original proof (good only in characteristic zero) but it works in all characteristics other than 2. The profound benefit is that the 2-relation isomorphism Theorem 6.15 is also extended to all characteristics except 2. Using his suggestion to postpone low-dimension calculations until the 2-relation problem was solved, a great deal of tedious, ad hoc calculations were replaced by easy applications of Theorem 6.15. Furthermore he offered a proof of the identity of my own and Scheuneman's duality theories. This proof was identical to one I gave myself in a paper under preparation.

1. **Generators and relations for nilpotent Lie algebras.** Let K be an arbitrary field. With one exception, certain free algebras, all algebras in the following are assumed to be finite-dimensional Lie algebras over K. The notation $A \cong B$ will be reserved for isomorphic algebras A and B, while $V \simeq W$ will be used to indicate vector space isomorphisms.

If $x, y, z, \ldots, w$ are elements of a Lie algebra L, we will usually write $[x, y, z, \ldots, w]$ for the more cumbersome $[[\ldots[[x,y],z]\ldots],w]$. Define L^n to be the subspace generated by all elements of the type $[x_1, \ldots, x_n]$ where the x_i belong to L. Clearly $L^{n+1} \subseteq L^n$, and in fact the subspaces L^n are ideals. The algebra is said to be nilpotent if L^n is zero for some positive integer n. If $L^l \neq 0 = L^{l+1}$ we will say L is l-step nilpotent. The first result gives a good indication of how to generate such an algebra in the most economical fashion.

Lemma 1.1. *Suppose $f : M \to N$ is a homomorphism of Lie algebras. Then $f(M^n) \subseteq N^n$ and if f is surjective the equality obtains.*

Proposition 1.2. *Let N be a nilpotent Lie algebra. A subset S of N generates N if and only if the cosets $\{s + N^2 \mid s \in S\}$ span N/N^2.*

Proof. The result is well known and follows from an easy induction argument.

Corollary 1.3. *Let $g = \dim N/N^2$ and suppose N is nilpotent. A subset $\{y_1, \ldots, y_g\}$ generates N if and only if $\{y_i + N^2\}_{i=1}^g$ is a basis of N/N^2.*

Definition. If N is as in the corollary we will say N has g-generators. Note that N can be generated by g elements, but by no fewer than g.

Fixing l and g, we propose to study the l-step nilpotent Lie algebras with g-generators. Every such algebra can be viewed as a quotient of a certain "universal" algebra $N(l,g)$, and two quotients will be isomorphic when the corresponding defining ideals in $N(l,g)$ are congruent under the action of its automorphism group.

Let $\mathcal{F}$ be the free Lie algebra on g-generators $y_1, \ldots, y_g$ [15, p. 167]. $\mathcal{F}$ is infinite-dimensional. Let $\mathcal{F}_n$ denote the subspace of $\mathcal{F}$ generated by all elements of the type $[y_{i_1}, \ldots, y_{i_n}]$ where $i_j \in \{1, 2, \ldots, g\}$. $\mathcal{F}$ is graded with $\mathcal{F}_n$ as the homogeneous component of degree n, and furthermore $\mathcal{F}^n = \bigoplus \sum_{j \geq n} \mathcal{F}_j$. Let $N(l,g) = \mathcal{F}/\mathcal{F}^{l+1}$ and let x_i denote the image of y_i under the canonical surjection $\mathcal{F} \to N(l,g)$. Then the x_i generate $N(l,g)$. Since $\mathcal{F}^{l+1}$ is homogeneous, $N(l,g)$ inherits a grading from $\mathcal{F}$: $N(l,g) = \bigoplus \sum_{j=1}^{l} N(l,g)_j$ where $N(l,g)_j$ is the subspace spanned by all elements $[x_{i_1}, \ldots, x_{i_j}]$ and $[N(l,g)_i, N(l,g)_j] \subseteq N(l,g)_{i+j}$.

Universal mapping property of $N(l,g)$. For any k-step nilpotent Lie algebra M with $k \leq l$, and any g-elements $m_1, \ldots, m_g$ of M, the correspondence $x_i \to m_i$ extends uniquely to a homomorphism.

Proof. Extend $y_i \to m_i$ to a homomorphism $\theta : \mathcal{F} \to M$. Since $k \leq l$, $\mathcal{F}^{l+1}$ is in the kernel of θ; hence θ factors through $N(l,g)$ by a homomorphism τ taking $x_i \to m_i$. The uniqueness follows since the x_i generate $N(l,g)$.

According to a result of Witt [15, p. 194], if the characteristic of K is zero then

$$\dim \mathcal{F}_n = (1/n) \sum_{d|n} \mu(d) g^{n/d} \tag{1.1}$$

where μ is the Moebius function. Thus, since $N(l,g) = \mathcal{F}/\mathcal{F}^{l+1} \simeq \mathcal{F}_1 \oplus \ldots \oplus \mathcal{F}_l$ we have

$$\dim N(l,g) = \sum_{n=1}^{l} (1/n)\left(\sum_{d|n} \mu(d) g^{n/d}\right). \tag{1.2}$$

Proposition 1.4. *$N(l,g)$ is an l-step nilpotent Lie algebra with g-generators. Any other nilpotent Lie algebra with the invariants l, g is a quotient of $N(l,g)$.*

Proof. By construction $N(l,g) \simeq \mathfrak{I}_1 \oplus \ldots \oplus \mathfrak{I}_l$ so it is finite-dimensional. Furthermore, $N(l,g)^l \simeq \mathfrak{I}_l \neq 0$ while $N(l,g)^{l+1} = 0$. Since $N(l,g)^2 \simeq \mathfrak{I}_2 \oplus \ldots \oplus \mathfrak{I}_l$, $N(l,g)/N(l,g)^2 \simeq \mathfrak{I}_1$ and $\dim \mathfrak{I}_1 = g$.

If M is any other l-step nilpotent Lie algebra with g-generators, pick generators $m_1, \ldots, m_g$ and apply the mapping property of $N(l,g)$ to obtain a homomorphism $N(l,g) \to M$ which takes x_i to m_i. The image of this map contains a set of generators so it is surjective.

Remark. If $N = N(l,g)/I$ is an l-step nilpotent Lie algebra with g-generators, the ideal I can be viewed as the relations among $x_1, \ldots, x_g$ of $N(l,g)$ which define N. The dimension of I will be called the number of relations defining N.

Definition. Let $\mathcal{I}(l,g)$ be the set of all ideals I of $N(l,g)$ such that $N(l,g)/I$ is l-step nilpotent with g-generators.

Proposition 1.5. *An ideal I of $N(l,g)$ belongs to $\mathcal{I}(l,g)$ if and only if:*
(i) $N(l,g)^l \not\subseteq I$;
(ii) $I \subset N(l,g)^2$.

Proof. Observe that $(N(l,g)/I)^n \cong (N(l,g)^n + I)/I$. Therefore $(N(l,g)/I)^l \neq 0$ if and only if $N(l,g)^l \not\subseteq I$. Also $(N(l,g)/I)/(N(l,g)/I)^2 \cong N(l,g)/(N(l,g)^2 + I)$. But $\dim N(l,g)/N(l,g)^2 = g$, so the latter space is g-dimensional if and only if $I \subseteq N(l,g)^2$. If $I = N(l,g)^2$, condition (i) is violated; hence $I \subset N(l,g)^2$.

Definition. If I and J are in $\mathcal{I}(l,g)$ we say they are equivalent if $N(l,g)/I \cong N(l,g)/J$.

This equivalence relation on $\mathcal{I}(l,g)$ is completely described by the following propositions. Let $\mathrm{Aut}(A)$ represent the group of automorphisms of an algebra A.

Proposition 1.6. *Suppose I and J belong to $\mathcal{I}(l,g)$. Then I is equivalent to J if and only if there is a θ in* $\mathrm{Aut}(N(l,g))$ *satisfying $\theta(I) = J$.*

Proof. ($\Leftarrow$) Easy.

($\Rightarrow$) Let N_1 and N_2 represent quotients of $N(l,g)$ by I and J respectively and suppose $\tau : N_1 \cong N_2$ is an isomorphism. Consider the diagram

$$\begin{tikzcd} N(l,g) \arrow[d, "\pi"'] & N(l,g) \arrow[d, "\rho"] \\ N_1 \arrow[r, "\tau"'] & N_2 \end{tikzcd}$$

where π, ρ are canonical surjections. Now $\tau \circ \pi$ is surjective so $\{\tau(\pi(x_i))\}_{i=1}^g$ generates N_2. Pick $w_i \in N(l,g)$ such that $\rho(w_i) = \tau(\pi(x_i))$ and observe that the w_i

are independent modulo $N(l,g)^2$, for otherwise the $\rho(w_i)$ would be dependent modulo N_2^2 which is impossible. By the mapping property of $N(l,g)$ there is a unique homomorphism $\theta : N(l,g) \to N(l,g)$ taking x_i to w_i. By Corollary 1.3, the w_i generate $N(l,g)$, and being in the image of θ we have shown θ is surjective, hence an automorphism. Since $\tau(\pi(x_i)) = \rho(w_i) = \rho(\theta(x_i))$ and since the x_i generate $N(l,g)$, the diagram

$$\begin{tikzcd} N(l,g) \arrow[r, "\theta"] \arrow[d, "\pi"'] & N(l,g) \arrow[d, "\rho"] \\ N_1 \arrow[r, "\tau"'] & N_2 \end{tikzcd}$$

commutes. This forces $\theta(I) = J$.

Corollary 1.7. *Suppose $I \in \mathcal{J}(l,g)$ and $N = N(l,g)/I$. Then any automorphism θ of N lifts to an automorphism θ' of $N(l,g)$ (i.e. an automorphism θ' such that $\theta'(I) = I$ and such that θ' induces θ).*

Proposition 1.8. *$\mathcal{J}(l,g)$ is stable under* $\mathrm{Aut}(N(l,g))$.

Proof. The result follows from Proposition 1.5 and Lemma 1.1.

By Propositions 1.4–1.8 there is a one-to-one correspondence between isomorphism classes of l-step nilpotent Lie algebras with g-generators and the orbits of $\mathcal{J}(l,g)$ under the action of $\mathrm{Aut}(N(l,g))$. The problem of computing these orbits will be treated shortly. A description of $\mathrm{Aut}(N(l,g))$ is easy.

Proposition 1.9. *Consider the generators $x_1, \ldots, x_g$ of $N(l,g)$.*

(i) *Every correspondence $x_i \to w_i$ where the w_i are independent modulo $N(l,g)^2$ extends uniquely to an automorphism.*

(ii) *Every automorphism arises in this fashion.*

Proof. The result is easily deduced from Corollary 1.3 and the universal mapping property.

Notation. If G is a group acting on a set S then S/G will denote the orbit space of S by G.

2. **Metabelian Lie algebras.** A two-step nilpotent Lie algebra will be called metabelian. These algebras are the central concern of most of what follows.

In this section we specialize the methods of §1 to deal with metabelian Lie algebras having g-generators. Let $x_1, \ldots, x_g$ be generators of $N(2,g)$ as before

and let $V = \langle \{x_i\}_i \rangle$. We can make the vector space $V \oplus \wedge^2 V$ into a metabelian Lie algebra by linearly extending the rules

$$[x_i, x_j] = x_i \wedge x_j, \tag{2.1}$$

$$[x_i, x_j \wedge x_k] = [x_i \wedge x_j, x_k] = 0,$$

$$[x_i \wedge x_j, x_k \wedge x_l] = 0.$$

It is not difficult to show that the unique homomorphism from $N(2,g)$ to $V \oplus \wedge^2 V$ taking x_i to x_i is an isomorphism. Via this isomorphism we make the identification $N(2,g) = V \oplus \wedge^2 V$.

For any vector space W there is a representation $\wedge^p$ of $\mathrm{GL}(W)$ on $\wedge^p W$ given on decomposable p-vectors $w_1 \wedge \ldots \wedge w_p$ by

$$\wedge^p(\theta)(w_1 \wedge \ldots \wedge w_p) = \theta(w_1) \wedge \ldots \wedge \theta(w_p)$$

for $\theta \in \mathrm{GL}(W)$.

Theorem 2.1. *Every metabelian Lie algebra with g-generators is of the type $N(2,g)/I$ where I ranges over the proper subspaces (ideals) of $\wedge^2 V$ (see the identification $N(2,g) = V \oplus \wedge^2 V$ above). Furthermore, if I and J are proper subspaces of $\wedge^2 V$, then $N(2,g)/I \cong N(2,g)/J$ if and only if there is a $\theta \in \mathrm{GL}(V)$ such that $\wedge^2(\theta)(I) = J$.*

Proof. Since $N(2,g)^2 = \wedge^2 V$, by Proposition 1.5 $\mathcal{I}(2,g)$ consists of the proper subspaces of $\wedge^2 V$ (any such subspace is an ideal since $\wedge^2 V$ is the center). By Proposition 1.9 we can embed $\mathrm{GL}(V)$ in $\mathrm{Aut}(N(2,g))$. This embedding $E : \mathrm{GL}(V) \to \mathrm{Aut}(N(2,g))$ is given by $E(\theta) = \theta \oplus \wedge^2(\theta)$. Any automorphism of $N(2,g)$ stabilizes $N(2,g)^2 = \wedge^2 V$, so let $R : \mathrm{Aut}(N(2,g)) \to \mathrm{GL}(\wedge^2 V)$ be the restriction map. By the results of §1, to establish the last statement it suffices to show

$$R(\mathrm{Aut}(N(2,g))) = R(E(\mathrm{GL}(V))).$$

So suppose $\theta \in \mathrm{Aut}(N(2,g))$ and $\theta(x_i) = (\sum_{j=1}^{g} \alpha_{ji} x_j) + \pi_i$ where $\pi_i \in \wedge^2 V$. Since θ is an automorphism, by Proposition 1.9 we achieve $\det|\alpha_{ij}| \neq 0$. Pick $\tau \in \mathrm{GL}(V)$ such that the matrix of τ with respect to $\{x_i\}_{i=1}^{g}$ is $|\alpha_{ij}|$. As a consequence of the identity $N(2,g)^3 = 0$, it follows that $R(E(\tau)) = R(\theta)$ and the proof is complete.

3. **Duality theory.** Keeping the same notation as before, let $\mathcal{S}$ be the collection of all proper subspaces of $\wedge^2 V$ and partition $\mathcal{S}$ by $\mathcal{S} = \bigcup_{p=0}^{n-1} \mathcal{S}_p$ where $n = \binom{g}{2}$ and $\mathcal{S}_p$ consists of p-dimensional subspaces. Each $\mathcal{S}_p$ is stable under $\mathrm{GL}(V)$ and hence is a union of orbits. The following discussion is aimed at establishing a duality theory for metabelian Lie algebras.

Definition. Let M, N be K-vector spaces and suppose $(\,,\,) : M \times N \to K$ is a nondegenerate pairing. Let $\rho : G \to \mathrm{GL}(M)$ be a representation of the group G on M. We define a new representation ρ^* (contragredient of ρ) of G on N by means of the equation

$$(m, \rho^*(g)(n)) = (\rho(g^{-1})(m), n) \tag{3.1}$$

for $m \in M$, $n \in N$ and $g \in G$. Equivalently,

$$(\rho(g)(m), \rho^*(g)(n)) = (m, n). \tag{3.2}$$

In the customary fashion [2], if S is a subspace of M, we define $S^0 = \{n \in N \mid (S, n) = 0\}$, the right orthogonal complement of S. Similarly, we define the left orthogonal complement of a subspace T of N by ${}^0T = \{m \in M \mid (m, T) = 0\}$. By the nondegeneracy of $(\,,\,)$ we get

$${}^0(S^0) = S, \qquad ({}^0T)^0 = T \tag{3.3}$$

where S and T are as above, and

$$\dim S + \dim S^0 = \dim M = \dim N = \dim T + \dim {}^0T. \tag{3.4}$$

There is a canonical nondegenerate pairing $(\,,\,)$ between $\bigwedge^2 V$ and $\bigwedge^2 (V^*)$ given on decomposable vectors by

$$(v \wedge w, \lambda \wedge \alpha) = \det \begin{vmatrix} \lambda(v) & \alpha(v) \\ \lambda(w) & \alpha(w) \end{vmatrix} \tag{3.5}$$

where $v \wedge w \in \bigwedge^2 V$, $\lambda \wedge \alpha \in \bigwedge^2 (V^*)$ [5, p. 100].

Let ρ be the representation $\bigwedge^2$ of $\mathrm{GL}(V)$ and let ρ^* be its contragredient with respect to the pairing (3.5). Let $\mathcal{S}_q^*$ be the q-dimensional subspaces of $\bigwedge^2 (V^*)$ and set $n = \binom{g}{2}$ where $g = \dim V$. Consider the map $S \to S^0$ (right orthogonal complementation with respect to (3.5)) from $\mathcal{S}_p$ into $\mathcal{S}_q^*$ where $q = n - p$. We claim it is an orbit preserving bijection when $\mathrm{GL}(V)$ acts on $\mathcal{S}_p$ and $\mathrm{GL}(V^*)$ on $\mathcal{S}_q^*$. The bijectivity follows from (3.3). Furthermore, by (3.2), since $(S, S^0) = 0$, if $\theta \in \mathrm{GL}(V)$ then $(\rho(\theta)(S), \rho^*(\theta)(S^0)) = (S, S^0) = 0$. Thus $\rho^*(\theta)(S^0) \subseteq (\rho(\theta)(S))^0$. A simple dimension argument using (3.4) forces

$$(\rho(\theta)(S))^0 = \rho^*(\theta)(S^0). \tag{3.6}$$

In addition, it is easily checked that $\rho^*(\theta) = \bigwedge^2 ({}^t\theta^{-1})$ where t denotes transpose. Thus $\rho^*(\mathrm{GL}(V)) = \bigwedge^2 (\mathrm{GL}(V^*))$.

Proposition 3.1. *Let $\mathcal{S}_p$ be the set of p-dimensional subspaces of $\bigwedge^2 V$, $\mathcal{S}_q^*$ the q-dimensional subspaces of $\bigwedge^2 (V^*)$. Set $q = n - p$ where $n = \binom{g}{2}$, $g = \dim V$. If $S \in \mathcal{S}_p$, let S^0 denote its right orthogonal complement with respect to the canonical nondegenerate pairing* (3.5) *between $\bigwedge^2 (V)$ and $\bigwedge^2 (V^*)$. Then the map $(\,)^0 : \mathcal{S}_p \to \mathcal{S}_q^*$ induces a bijection between $\mathcal{S}_p/\mathrm{GL}(V)$ and $\mathcal{S}_q^*/\mathrm{GL}(V^*)$.*

Proof. We have seen the map is a bijection, and by (3.6) together with the identity $\rho^*(\mathrm{GL}(V)) = \wedge^2(\mathrm{GL}(V^*))$ it must also preserve orbits.

Make the vector space $V^* \oplus \wedge^2(V^*)$ into a metabelian Lie algebra isomorphic to $N(2,g)$ using the dual basis $\{x_i^*\}_{i=1}^g$ and the multiplication table (2.1).

Definition. Suppose $N = N(2,g)/S = (V \oplus \wedge^2 V)/S$ where S is a nonzero member of $\mathcal{J}(2,g)$ (i.e. a proper subspace of $\wedge^2 V$). The metabelian Lie algebra $N^0 = (V^* \oplus \wedge^2(V^*))/S^0$ will be called the dual of N.

Theorem 3.2. (i) $N \cong (N^0)^0$.

(ii) $N_1 \cong N_2$ *if and only if* $N_1^0 \cong N_2^0$.

(iii) *If* $\dim N = g + p$, *then* $\dim N^0 = g + \binom{g}{2} - p$ *and* $\dim(N^0)/(N^0)^2 = g$.

Proof. The first statement follows from (3.3) provided one gives the following definition of the dual of an algebra

$$M = (V^* \oplus \wedge^2(V^*))/T; \quad M^0 = (V \oplus \wedge^2 V)/{}^0T.$$

The second result follows from the preceding proposition and Theorem 2.1. The final result follows from (3.4).

Roughly speaking, the duality theory cuts the classification problem in half. In addition, it is quite useful in low dimensions as will be seen in §§5–7. Also, for the sake of consistency, we would have to define $N(2,g)^0$ to be the unique g-dimensional abelian Lie algebra and vice versa. A duality theory whose fundamental properties are the same as those listed in the preceding theorem was developed earlier by Scheuneman [24]. In a later paper [26] I will show the identity of these theories and explore duality theories axiomatically.

4. **The connection with alternating forms.** As in §3, we let $(\,,\,)$ be the canonical nondegenerate pairing (3.5) between $\wedge^2 V$ and $\wedge^2(V^*)$. This pairing induces a canonical isomorphism $\Gamma : \wedge^2 V \to (\wedge^2(V^*))^*$ given by

$$\Gamma(v \wedge w)(\lambda \wedge \alpha) = (v \wedge w, \lambda \wedge \alpha)$$

where $v \wedge w \in \wedge^2 V$, $\lambda \wedge \alpha \in \wedge^2(V^*)$. Let $\rho = \wedge^2$ and let ρ^* be its contragredient with respect to (3.5). Let $\pi : V^* \times V^* \to \wedge^2(V^*)$ denote the natural mapping. Then $\Phi : (\wedge^2(V^*))^* \to \mathrm{Alt}(V^*)$ (the space of alternating forms on V^*) given by $\Phi(f) = f \circ \pi$ is also a canonical isomorphism. Thus $\Psi = \Phi \circ \Gamma : \wedge^2 V \to \mathrm{Alt}(V^*)$ where

$$\Psi(v \wedge w)(\lambda, \alpha) = (v \wedge w, \lambda \wedge \alpha) \tag{4.1}$$

is a canonical isomorphism.

There is a representation $\mathrm{GL}(V) \to \mathrm{GL}(V^*)$ which takes θ to ${}^t\theta^{-1}$ (t denotes transpose) and an action ω of $\mathrm{GL}(V^*)$ on $\mathrm{Alt}(V^*)$ given by

$$(\omega(\tau))f(\ ,\ ) = f(\tau^{-1}(), \tau^{-1}()) \tag{4.2}$$

for $f \in \mathrm{Alt}(V^*)$ and $\tau \in \mathrm{GL}(V^*)$.

Now suppose θ is in $\mathrm{GL}(V)$, $v \wedge w$ in $\bigwedge^2 V$, and $\lambda \wedge \alpha$ in $\bigwedge^2 (V^*)$. Then, using the contragredience identity, (4.1) and (4.2), and the fact that $\rho^*(\theta^{-1}) = \bigwedge^2 ({}^t\theta)$, it follows that

$$\Psi(\rho(\theta)(v \wedge w))(\lambda, \alpha) = (\omega({}^t\theta^{-1})\Psi(v \wedge w))(\lambda, \alpha); \tag{4.3}$$

hence

$$\Psi \circ \rho(\theta) = \omega({}^t\theta^{-1}) \circ \Psi. \tag{4.4}$$

Let $x_1, \ldots, x_g$ be as before and let $x_1^*, \ldots, x_g^*$ be the dual basis of V^*. Suppose y is in $\bigwedge^2 V$ and $\Psi(y)(x_k^*, x_l^*) = a_{kl}$ when $k < l$. Then we claim that

$$y = \sum_{1 \le k < l \le g} a_{kl} x_k \wedge x_l. \tag{4.5}$$

This follows since $\Psi(\sum_{k<l} a_{kl} x_k \wedge x_l)(x_i^*, x_j^*) = a_{ij}$ when $i < j$, plus the fact that Ψ is an isomorphism. We summarize these results in

Proposition 4.1. *There is a canonical isomorphism $\Psi : \bigwedge^2 V \to \mathrm{Alt}(V^*)$ given on decomposable vectors $v \wedge w$ in $\bigwedge^2 V$ by*

$$\Psi(v \wedge w)(\lambda, \alpha) = \det \begin{vmatrix} \lambda(v) & \alpha(v) \\ \lambda(w) & \alpha(w) \end{vmatrix}$$

where $\lambda, \alpha \in V^*$. Under the action $\bigwedge^2$ of $\mathrm{GL}(V)$ on $\bigwedge^2 V$ and the action ω (see (4.2)) of $\mathrm{GL}(V^*)$ on $\mathrm{Alt}(V^*)$ we have

$$\Psi \circ \bigwedge^2 (\theta) = \omega({}^t\theta^{-1}) \circ \Psi$$

for all θ in $\mathrm{GL}(V)$. Thus Ψ induces a bijection between $\bigwedge^2 V / \mathrm{GL}(V)$ and $\mathrm{Alt}(V^*)/\mathrm{GL}(V^*)$.

If $x_1, \ldots, x_g$ is a basis of V, let $x_1^*, \ldots, x_g^*$ be the dual basis. If $y \in \bigwedge^2 V$ and

$$\Psi(y)(x_k^*, x_l^*) = a_{kl} \qquad (k < l),$$

then

$$y = \sum_{i<j} a_{ij} x_i \wedge x_j.$$

Proof. Preceding discussion.

Remark. According to this result and Theorem 2.1, the isomorphism problem for metabelian algebras can be viewed as a problem of obtaining a simultaneous canonical form for a space of alternating forms on V^*—a classical problem. Picking a basis of V^* and identifying a form with its (skew-symmetric) matrix in this basis, the problem becomes one of finding a canonical form for a space S of alternating forms (skew-symmetric matrices) operated on by nonsingular matri-

ces A according to the rule $S \to {}^tASA$. This connection is exploited in the next two sections to classify the one- and two-relation metabelian Lie algebras.

5. **One-relation metabelian Lie algebras.** Recall that a 1-relation algebra is one of the type $N(2,g)/I = V \oplus \wedge^2 V/I$ where $\dim I = 1$. These algebras, and hence their duals, will be completely classified in this section over arbitrary fields. Characteristic 2 causes no problems if one takes $f(x,x) = 0$ for the definition of an alternating form f.

Lemma 5.1. *Let V be a finite-dimensional vector space over an arbitrary field K, and let f_1, f_2 be alternating forms on V. Then f_1 and f_2 are equivalent* ($\operatorname{mat} f_2 = {}^tA(\operatorname{mat} f_1)A$ *for some nonsingular matrix A*) *if and only if they have the same rank. In addition,* $\operatorname{rank}(f_i)$ *is an even number.*

Proof [16, p. 161].

Theorem 5.2. *There are exactly* $[g/2]$ $(g+1)$*-dimensional metabelian Lie algebras with g-generators and* $(g + \binom{g}{2} - 1)$*-dimensional metabelian Lie algebras with g-generators. The former are distinguished by the dimensions of their centers while the latter are distinguished by the dimensions of the centers of their duals.*

Proof. By Theorem 2.1 the algebras correspond to orbits of codimension one subspaces of $\wedge^2 V$ under $\mathrm{GL}(V)$, and to orbits of one-dimensional subspaces respectively. By the duality theory, the number of orbits is the same in either case. If $S = \langle v \rangle$ is a one-dimensional subspace of $\wedge^2 V$ and Ψ is as in Proposition 4.1, then by Lemma 5.1 the orbit of $\langle v \rangle$ is completely determined by $\operatorname{rank}(\Psi(v))$ which has exactly $[g/2]$ possibilities.

To establish the second part, we will exhibit $[g/2]$ $(g+1)$-dimensional metabelian Lie algebras with g-generators having centers of pairwise distinct dimensions. Let N be a $(g+1)$-dimensional K-vector space with basis $x_1, \ldots, x_g, z$, and let n be a positive integer less than or equal to $[g/2]$. Make N into a metabelian Lie algebra by linearly extending the rules:

(i) $[x_{2i-1}, x_{2i}] = z = -[x_{2i}, x_{2i-1}]$, $i = 1, \ldots, n$,

(ii) every other product of basis vectors is zero. Call N with this Lie structure N_n. The center of N_n is $\langle x_{2n+1}, \ldots, x_g, z \rangle$, that is, it is $(g + 1 - 2n)$-dimensional. The last statement follows from the duality theory.

Remark. Using Scheuneman's construction of the dual [24, pp. 152–155] one can easily see that the g-generator 1-relation algebras are

$$(N_1)^0 = (V \oplus \wedge^2 V)/I_1, \ldots, (N_{[g/2]})^0 = (V \oplus \wedge^2 V)/I_{[g/2]}$$

where

$$I_n = \left\langle \sum_{j=1}^{n} x_{2j-1} \wedge x_{2j} \right\rangle, \qquad n = 1, \ldots, [g/2]. \tag{5.1}$$

6. **Two-relation metabelian Lie algebras.** For this section we must restrict the field K by requiring it to be algebraically closed and of characteristic different

from 2. Over these fields we will obtain a complete classification of all g-generator, 2-relation metabelian Lie algebras. Due to Theorem 2.1 and the remark following Proposition 4.1, the 2-relation isomorphism problem is equivalent to finding a canonical form (invariants) for a 2-dimensional space $\mathcal{S}$ of $g \times g$ skew-symmetric matrices operated on by nonsingular matrices A according to the rule $\mathcal{S} \to {}^tA\mathcal{S}A$. This problem was basically solved by Weierstrass and Kronecker who obtained a simultaneous canonical form for a pair of matrices. The extension of their results to 2-dimensional spaces of matrices is bridged by my own "canonical form set of a set of elementary divisors". A very readable account of this theory can be found in Gantmacher [11]. Dieudonné [8] also gives a quite unique and elegant treatment of the subject.

Following the ideas of Weierstrass and Kronecker, we begin by studying pairs (A, B) of $m \times n$ matrices.

Definition. Let λ, μ be algebraically independent variables over K. The matrix $\mu A + \lambda B \in \mathrm{Mat}(m, n; K[\lambda, \mu])$ is called an $m \times n$ pencil. Two such pencils $\mu A + \lambda B$, $\mu C + \lambda D$ are said to be strictly equivalent if there are nonsingular matrices $S(m \times m)$ and $T(n \times n)$ with elements in K satisfying $S(\mu A + \lambda B)T = \mu C + \lambda D$; equivalently, $SAT = C$, $SBT = D$. The pencil $\mu A + \lambda B$ is called skew if both A and B are (square) skew-symmetric matrices. Two $m \times m$ skew pencils $\mu A + \lambda B$, $\mu C + \lambda D$ are said to be strictly congruent if there is a nonsingular $m \times m$ matrix S with elements in K satisfying

$$ {}^tS(\mu A + \lambda B)S = \mu C + \lambda D. $$

Proposition 6.1. *Suppose K is algebraically closed and its characteristic is not* 2. *Then skew-symmetric pencils are strictly equivalent if and only if they are strictly congruent.*

Proof. Let $\mu A + \lambda B$, $\mu C + \lambda D$ be strictly equivalent skew pencils, and let P, Q be nonsingular matrices over K satisfying ${}^tPAQ = C$, ${}^tPBQ = D$. Taking transposes and setting $S = QP^{-1}$, we obtain ${}^tSA = AS$, ${}^tSB = BS$. Thus, for any polynomial $p(S)$ in S, we have ${}^t(p(S))A = A(p(S))$, ${}^t(p(S))B = B(p(S))$. There is a polynomial $r(S)$ in S (see Lemma 6.2) such that $(r(S))^2 = S$. Setting $T = r(S)$, we see that T is nonsingular and $TP = T^{-1}Q$, $A = {}^tTAT$, $B = {}^tTBT$. Thus $C = {}^tP{}^tTAT^{-1}Q = {}^t(TP)A(TP)$, $D = {}^tP{}^tTB^{-1}TQ = {}^t(TP)B(TP)$ and the result is established.

Lemma 6.2. *Let K be as before. If S is a nonsingular matrix over K, there is a polynomial $r(S)$ in S satisfying $(r(S))^2 = S$.*

Proof. Let $f(x)$ be the minimum polynomial of S. It is only necessary to find $r(x)$ such that $f(x) \mid (r(x)^2 - x)$. Suppose $f(x) = \prod_{i=1}^{k} (x - a_i)^{n_i}$ where $a_i \neq a_j$ if $i < j$. Note that the a_i are nonzero since S is nonsingular. Suppose $(x - a_i)^{n_i} \mid$

$(r_i(x)^2 - x)$ for all i. Since the polynomials $f(x)/(x - a_1)^{n_1}, \ldots, f(x)/(x - a_k)^{n_k}$ are relatively prime, there are polynomials $q_i(x)$ satisfying

$$1 = \sum_{i=1}^{k} (f(x)/(x - a_i)^{n_i})q_i(x).$$

Set $r(x) = \sum_{i=1}^{k} (f(x)/(x - a_i)^{n_i})q_i(x)r_i(x)$. Squaring these identities, multiplying the former by x, and subtracting, one gets

$$r(x)^2 - x = \sum_{i=1}^{k} (f(x)/(x - a_i)^{n_i})^2 q_i(x)^2(r_i(x)^2 - x)$$
$$+ 2 \sum_{i<j} (f(x)^2/(x - a_i)^{n_i}(x - a_j)^{n_j})q_i(x)q_j(x)(r_i(x)r_j(x) - x)$$

and $f(x)$ divides each term of both sums.

Finding (inductively) the $r_i(x)$ for each i is easy. Notice that $(x - a) \mid (a^{1/2})^2 - x$. Suppose $(x - a)^n \mid (r(x)^2 - x)$ for some $n \geq 1$. Write

$$r(x)^2 - x = h(x)(x - a)^n.$$

Since $a \neq 0$, we must have $(x - a) + r(x)$. Hence there are polynomials $g(x)$, $k(x)$ satisfying $1 = (x - a)g(x) + r(x)k(x)$. Let $l(x)$ be some unknown polynomial and consider

$$(r(x) + (x - a)^n l(x))^2 - x = (x - a)^n(h(x) + 2r(x)l(x)) + (x - a)^{2n} l(x)^2.$$

To complete the proof we need only find an $l(x)$ such that $(x - a) \mid (h(x) + 2r(x)l(x))$. Certainly $h(x) = (x - a)g(x)h(x) + r(x)k(x)h(x)$. Setting $l(x) = -(1/2)k(x)h(x)$ we are done.

With this result in mind we begin the study of strict equivalence of pencils.

Definition. The pencil $\mu A + \lambda B$ is called regular (nonsingular) if it is square and $d(\mu, \lambda) = \det|\mu A + \lambda B|$ is not the zero polynomial. Otherwise the pencil is called singular.

The strict equivalence of regular pencils was solved by Weierstrass in terms of his elementary divisors.

Let $\mu A + \lambda B$ be a square pencil and let $G_m(\mu, \lambda)$ be the greatest common divisor of all its minor determinants of order m. Then $G_m(\lambda, \mu) \mid G_{m+1}(\lambda, \mu)$ for all relevant m. Set $I_m(\lambda, \mu) = G_m(\lambda, \mu)/G_{m-1}(\lambda, \mu)$ for $m > 1$ and $I_1(\lambda, \mu) = G_1(\lambda, \mu)$.

Definition. The homogeneous polynomials $\{I_k(\lambda, \mu)\}_k$ are called the invariant factors of the pencil $\mu A + \lambda B$. Each polynomial $I_k(\lambda, \mu)$ can be written as a product of powers of prime (linear) polynomials. These prime power factors are called the elementary divisors. By the multiplicity of an elementary divisor $(a\lambda + b\mu)^e$, we mean the number of k for which $(a\lambda + b\mu)^e \mid I_k(\lambda, \mu)$ but $(a\lambda + b\mu)^{e+1} \nmid I_k(\lambda, \mu)$.

Remark. The invariant factors (elementary divisors) are unique up to scalar multiples. Hence any elementary divisor is of the type $(\mu + a\lambda)^e$ or $(\lambda)^f$. The latter type are referred to as the infinite elementary divisors in the literature while the former are called finite.

Theorem 6.3 (Weierstrass). *Two regular pencils are strictly equivalent if and only if they have the same elementary divisors (counting multiplicities). If the elementary divisors of a regular pencil* $\mu A + \lambda B$ *are* $(\mu + a_1\lambda)^{e_1}, \ldots, (\mu + a_n\lambda)^{e_n}, \lambda^{f_1}, \ldots, \lambda^{f_m}$, *then it is strictly equivalent to a pencil*

$$\begin{vmatrix} E_1 & & & & & & & \\ & \cdot & & & & & & \\ & & \cdot & & & & & \\ & & & \cdot & & & & \\ & & & & E_n & & & \\ & & & & & F_1 & & \\ & & & & & & \cdot & \\ & & & & & & & \cdot \\ & & & & & & & & \cdot \\ & & & & & & & & & F_m \end{vmatrix}$$

where E_i *is the* $e_i \times e_i$ *pencil*

$$E_i = \begin{vmatrix} \mu + a_i\lambda & & & & & & \\ \lambda & \mu + a_i\lambda & & & & & \\ & & \cdot & & & & \\ & & & \lambda & & & \\ & & & & \cdot & & \\ & & & & & \mu + a_i\lambda & \\ & & & & & \lambda & \mu + a_i\lambda \end{vmatrix} \tag{6.1}$$

and F_j *is the* $f_j \times f_j$ *pencil*

$$F_j = \begin{vmatrix} \lambda & & & & & \\ \mu & \lambda & & & & \\ & \mu & \cdot & & & \\ & & & \cdot & & \\ & & & & \lambda & \\ & & & & \mu & \lambda \end{vmatrix} \tag{6.2}$$

Proof [11, Chapter 12], [8, pp. 138–140], or [1].

Now suppose $P = \mu A + \lambda B$ is an $r \times s$ singular pencil with more rows than rank $(\mu A + \lambda B)$. If so, there is a vector U in $(K[\lambda, \mu])^r$ such that $UP = 0$. Since the entries of P are homogeneous, we need only consider the case when U is

homogeneous, that is, all its entries are homogeneous of some fixed degree d which we call the degree of U.

Definition. If $r > \operatorname{rank}(P)$, the smallest degree m of a homogeneous vector U in $(K[\lambda, \mu])^r$ satisfying $UP = 0$ is called the minimal row index of P. Similarly, if $s > \operatorname{rank}(P)$, the smallest degree m' of a homogeneous vector V in $(K[\lambda, \mu])^s$ satisfying $P({}^tV) = 0$ is called the minimal column index of P.

Exhausting the row and column dependences of a singular pencil using Theorem 4 [11, p. 30], one can show

Theorem 6.4 (Kronecker). *Let $\mu A + \lambda B$ be a singular pencil. Then there are unique sequences of nonnegative integers $m_1 \leq m_2 \leq \ldots \leq m_r$, $n_1 \leq n_2 \leq \ldots \leq n_s$ such that the given pencil is strictly equivalent to a pencil of the type*

$$\begin{vmatrix} L_{m_1} & & & & & & & & \\ & \ddots & & & & & & & \\ & & L_{m_r} & & & & & & \\ & & & {}^tL_{n_1} & & & & & \\ & & & & \ddots & & & & \\ & & & & & {}^tL_{n_s} & & & \\ & & & & & & P_1 \end{vmatrix}$$

where L_0 is the 1×0 zero matrix, L_m is the $(m+1) \times m$ matrix

$$L_m = \begin{vmatrix} \lambda & & & & & \\ \mu & \lambda & & & & \\ & \mu & \ddots & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \lambda & \\ & & & & \lambda & \\ & & & & \mu & \\ & & & & & \mu \end{vmatrix} \qquad (m+1) \times m \tag{6.3}$$

and P_1 is a square regular pencil.

Proof [11, Chapter 12] or [1, pp. 121–125].

In saying L_0 is a 1×0 zero matrix we mean: if A is an $m \times n$ pencil with a zero row index then A is strictly equivalent to a pencil

$$\begin{vmatrix} 0 & 0 & \dots & 0 \\ & B & & \end{vmatrix}$$

where B is an $(m-1) \times n$ pencil, that is, L_0 consumes one row but no columns.

Definition. The nonnegative integers $m_1 \leq m_2 \leq \dots \leq m_r$, $n_1 \leq n_2 \leq \dots \leq n_s$ are called the minimal row and column indices respectively.

From Theorems 6.3 and 6.4 follows the

Theorem 6.5. *Two pencils (of the same dimensions) are strictly equivalent if and only if they have the same elementary divisors and minimal indices. If the pencil $\mu A + \lambda B$ has elementary divisors $(\mu + a_1\lambda)^{e_1}, \dots, (\mu + a_r\lambda)^{e_r}$, $\lambda^{f_1}, \dots, \lambda^{f_s}$ and minimal indices $m_1 \leq \dots \leq m_p$, $n_1 \leq \dots \leq n_q$, then it is strictly equivalent to the pencil*

$$(6.4) \qquad \begin{vmatrix} L_{m_1} & & & & & & & & & & & & & & & \\ & \cdot & & & & & & & & & & & & & & \\ & & \cdot & & & & & & & & & & & & & \\ & & & \cdot & & & & & & & & & & & & \\ & & & & L_{m_p} & & & & & & & & & & & \\ & & & & & {}^tL_{n_1} & & & & & & & & & & \\ & & & & & & \cdot & & & & & & & & & \\ & & & & & & & \cdot & & & & & & & & \\ & & & & & & & & \cdot & & & & & & & \\ & & & & & & & & & {}^tL_{n_q} & & & & & & \\ & & & & & & & & & & E_1 & & & & & \\ & & & & & & & & & & & \cdot & & & & \\ & & & & & & & & & & & & \cdot & & & \\ & & & & & & & & & & & & & \cdot & & \\ & & & & & & & & & & & & & & E_r & \\ & & & & & & & & & & & & & & & F_1 \\ & & & & & & & & & & & & & & & & \cdot \\ & & & & & & & & & & & & & & & & & \cdot \\ & & & & & & & & & & & & & & & & & & \cdot \\ & & & & & & & & & & & & & & & & & & & F_s \end{vmatrix}$$

where L_m is given in (6.3), *E_i in* (6.1), *and F_j in* (6.2).

Proof [11, pp. 37–40].

Corollary 6.6. *Two skew pencils are strictly congruent if and only if they have the same elementary divisors and minimal indices.*

Proof. Proposition 6.1 and Theorem 6.5.

In fact, much more can be said. It is possible to obtain a skew canonical form for a skew pencil. This is mainly due to

Proposition 6.7. *Let $P = \mu A + \lambda B$ be a skew pencil.*

(i) *If P is singular with minimal indices $m_1 \leq \ldots \leq m_p$, $n_1 \leq \ldots \leq n_q$, then $p = q$ and $m_i = n_i$.*

(ii) *The elementary divisors of P occur in pairs.*

Proof. (i) follows from a simple transpose argument [1, p. 134] while (ii) can be found on page 51 of [18].

Let J_m be the $m \times m$ matrix obtained by reversing the order of the rows of the $m \times m$ identity matrix. The matrices

$$\mathcal{L}_m = \begin{vmatrix} & & & & & & \lambda \\ & & & & & \lambda & \mu \\ & & & & \cdot & & \\ & & & \cdot & & \mu & \\ & & \cdot & & \cdot & & \\ & \cdot & & & & & \\ & \lambda & \cdot & & & & \\ \lambda & \mu & & & & & \\ \mu & & & & & & \end{vmatrix} \qquad (m+1) \times m, \tag{6.5}$$

$$E_n(a) = \begin{vmatrix} & & & & & & \mu + a\lambda \\ & & & & & \cdot & \lambda \\ & & & & \cdot & & \\ & & & \cdot & & \cdot & \\ & & & & \cdot & & \\ & & \mu + a\lambda & \cdot & & & \\ \mu + a\lambda & & \lambda & & & & \end{vmatrix} \qquad (n \times n), \tag{6.6}$$

and

$$G_n = \begin{vmatrix} & & & & & & \lambda \\ & & & & & \cdot & \mu \\ & & & & \cdot & & \\ & & & \cdot & & \cdot & \\ & & & & \cdot & & \\ & & \lambda & \cdot & & & \\ \lambda & \mu & & & & & \end{vmatrix} \qquad (n \times n) \tag{6.7}$$

can be obtained from the $L_{()}$, $E_{()}$, and $F_{()}$ blocks of the canonical form (6.4) by multiplying by a suitable $J_{()}$.

In addition, let

$$\mathcal{M}_m = \begin{vmatrix} 0 & \mathcal{L}_m \\ -{}^t\mathcal{L}_m & 0 \end{vmatrix} \qquad (2m+1)\times(2m+1),\ \mathcal{M}_0 = (0), \tag{6.8}$$

$$\mathcal{E}_n(a) = \begin{vmatrix} 0 & E_n(a) \\ -E_n(a) & 0 \end{vmatrix} \qquad (2n)\times(2n), \tag{6.9}$$

$$\mathcal{G}_n = \begin{vmatrix} 0 & G_n \\ -G_n & 0 \end{vmatrix} \qquad (2n)\times(2n) \tag{6.10}$$

where $\mathcal{L}_*$, $E_*(\cdot)$, G_* are as above.

Definition. The minimal row indices of a skew pencil will be called its minimal indices (see Proposition 6.7).

Theorem 6.8 (Canonical form for skew pencils). *Let $P = \mu A + \lambda B$ be a skew pencil and suppose it has minimal indices $m_1 \le m_2 \le \ldots \le m_p$ and elementary divisors $(\mu + a_1\lambda)^{e_1}$, $(\mu + a_1\lambda)^{e_1}, \ldots, (\mu + a_q\lambda)^{e_q}$, $(\mu + a_q\lambda)^{e_q}$, $(\lambda)^{f_1}$, $(\lambda)^{f_1}, \ldots,$ $(\lambda)^{f_s}$, $(\lambda)^{f_s}$. Then P is strictly congruent to the skew pencil*:

$$Q = \begin{vmatrix} \mathcal{M}_{m_1} & & & & & & & & & & \\ & \ddots & & & & & & & & & \\ & & \mathcal{M}_{m_p} & & & & & & & & \\ & & & \mathcal{E}_{e_1}(a_1) & & & & & & & \\ & & & & \ddots & & & & & & \\ & & & & & \mathcal{E}_{e_q}(a_q) & & & & & \\ & & & & & & \mathcal{G}_{f_1} & & & & \\ & & & & & & & \ddots & & & \\ & & & & & & & & \mathcal{G}_{f_s} & & \end{vmatrix}$$

Proof. P is strictly equivalent to a pencil of the type (6.4). Shuffle the blocks of (6.4) so that L_{m_i} and ${}^tL_{m_i}$ are adjacent and so that the blocks corresponding to paired elementary divisors are adjacent. Multiply this pencil by suitable blocks of $\pm J_*$ to get P strictly equivalent to the skew pencil Q. By Proposition 6.1 the strict equivalence is a strict congruence.

In getting down to the 2-relation isomorphism problem, let $\mathcal{S} = \langle A, B\rangle$ be a 2-dimensional space of $g \times g$ skew-symmetric matrices. One could associate to this space the pencil $\mu A + \lambda B$ and compute elementary divisors and minimal indices which, of course, depend on the particular basis $\{A, B\}$ of $\mathcal{S}$. Suppose $\{C, D\}$ is another basis. The pencil $\mu C + \lambda D$ can be gotten from $\mu A + \lambda B$ by a nonsingular linear substitution $\mu \to a_{11}\mu + a_{12}\lambda$, $\lambda \to a_{21}\mu + a_{22}\lambda$. The invariants of the second pencil are closely related to those of the first.

Lemma 6.9. *Let P_1, P_2 be 2 pencils related by a nonsingular linear substitution. Then the minimal indices of P_1 and P_2 are identical while the elementary divisors of P_2 can be obtained by performing the same linear substitution on those of P_1.*

Proof [1, pp. 117–120].

Thus, to complete the solution of the strict congruence of 2-dimensional spaces of skew-symmetric matrices we must investigate the behavior of elementary divisors under linear substitutions.

Let $S = \{p(\mu,\lambda)^a, \ldots, q(\mu,\lambda)^c\}$ be a set of elementary divisors, that is (since K is algebraically closed), $p, \ldots, q$ are homogeneous linear polynomials in μ, λ. We consider S to be unchanged if any polynomial $p, \ldots, q$ is replaced by an associate in $K[\mu, \lambda]$.

Definition. (i) Let S, T be sets of elementary divisors. We say S is equivalent to T if there is a linear substitution θ taking $\mu \to a\mu + b\lambda$, $\lambda \to c\mu + d\lambda$ with $ad - bc \neq 0$ and $T = S^\theta \equiv \{(p(\mu,\lambda)^\theta)^a, \ldots, (q(\mu,\lambda)^\theta)^c\}$.

(ii) A line $l = \langle a\mu + b\lambda\rangle$ is said to occur in S if $a\mu + b\lambda$ is an associate of one of $p(\mu,\lambda), \ldots, q(\mu,\lambda)$. Let $\mathcal{L}(S)$ be the set of distinct lines $l_1, \ldots, l_p$ occurring in S; p is called the order of S.

The following lemma is the fundamental tool for finding a "canonical form" for a set of elementary divisors.

Lemma 6.10. *Let l_1, l_2, l_3 be any ordered triple of distinct lines in K^2. Then, up to scalars, there is a unique linear automorphism θ of K^2 satisfying $\theta(l_1) = \langle(1,0)\rangle$, $\theta(l_2) = \langle(0,1)\rangle$, $\theta(l_3) = \langle(1,1)\rangle$.*

Proof. It suffices to show that if θ stabilizes the lines $\langle(1,0)\rangle$, $\langle(0,1)\rangle$, and $\langle(1,1)\rangle$ then it is a scalar. This is a routine calculation.

The point of the lemma is this: if we transform the lines $l_1, \ldots, l_r$ $(r \geq 4)$ to the lines $\langle(1,0)\rangle$, $\langle(0,1)\rangle$, $\langle(1,1)\rangle$, $l'_4, \ldots, l'_r$ (in the given order), then the lines $l'_4, \ldots, l'_r$ are uniquely determined.

Definition. Let S be a set of elementary divisors and let $\mathcal{C}(S)$, the canonical form set of S (consisting of a finite list of sets of elementary divisors equivalent to S), be defined by the following:

I. Order of $S = 1$. If $S = \{p^a, \ldots, p^c\}$, define $\mathcal{C}(S) = \{\mu^a, \ldots, \mu^c\}$.

II. Order of $S = 2$. If $S = \{p^a, \ldots, p^c, q^d, \ldots q^f\}$, define $\mathcal{C}(S) = \{S_1, S_2\}$ where $S_1 = \{\lambda^a, \ldots, \lambda^c, \mu^d, \ldots, \mu^f\}$ and $S_2 = \{\mu^a, \ldots, \mu^c, \lambda^d, \ldots \lambda^f\}$.

III. Order of $S = 3$. If $S = \{p^a, \ldots, p^c, q^d, \ldots, q^f, r^g, \ldots, r^h\}$, define $\mathcal{C}(S) = \{S_1, \ldots, S_6\}$ where

$$S_1 = \{\lambda^a, \ldots, \lambda^c, \mu^d, \ldots, \mu^f, (\mu + \lambda)^g, \ldots, (\mu + \lambda)^h\},$$

$$S_2 = \{\mu^a, \ldots, \mu^c, \lambda^d, \ldots, \lambda^f, (\mu + \lambda)^g, \ldots, (\mu + \lambda)^h\},$$

$$S_3 = \{\lambda^a, \ldots, \lambda^c, (\mu + \lambda)^d, \ldots, (\mu + \lambda)^f, \mu^g, \ldots, \mu^h\},$$

$$S_4 = \{(\mu + \lambda)^a, \ldots, (\mu + \lambda)^c, \lambda^d, \ldots, \lambda^f, \mu^g, \ldots, \mu^h\},$$

$$S_5 = \{\mu^a, \ldots, \mu^c, (\mu + \lambda)^d, \ldots, (\mu + \lambda)^f, \lambda^g, \ldots, \lambda^h\},$$

$$S_6 = \{(\mu + \lambda)^a, \ldots, (\mu + \lambda)^c, \mu^d, \ldots, \mu^f, \lambda^g, \ldots, \lambda^h\}.$$

IV. Order of $S > 3$. Let $\mathcal{L}(S) = \{l_1, \ldots, l_r\}$ $(r > 3)$. For any ordered triple (l_i, l_j, l_k) of distinct lines in $\mathcal{L}(S)$, let θ_{ijk} be the unique (up to scalars) linear substitution satisfying $\theta_{ijk}(l_i) = \langle\lambda\rangle$, $\theta_{ijk}(l_j) = \langle\mu\rangle$, $\theta_{ijk}(l_k) = \langle\mu + \lambda\rangle$. Now define $\mathcal{C}(S) = \{\theta_{ijk}(S) \mid (i,j,k)$ an ordered triple of distinct integers between 1 and $r\}$.

Due to Lemma 6.10 and the construction of $\mathcal{C}(S)$, we have

Proposition 6.11. *Let S and T be sets of elementary divisors. S is equivalent to T if and only if they have the same canonical form sets.*

Example 6.12. Suppose $S_a = \{\lambda, \mu, \lambda + \mu, a\lambda + \mu\}$ where $a \neq 0, 1$ (thus S_a has order 4). Then $\mathcal{C}(S_a)$ has 24 members and among these it is easily checked that the following sets appear 4 times each:

$$S_1 = \{\lambda, \mu, \lambda + \mu, a\lambda + \mu\}, \qquad S_2 = \{\lambda, \mu, \lambda + \mu, (1 - a)\lambda + \mu\},$$

$$S_3 = \{\lambda, \mu, \lambda + \mu, a^{-1}\lambda + \mu\}, \qquad S_4 = \{\lambda, \mu, \lambda + \mu, ((a - 1)/a)\lambda + \mu\},$$

$$S_5 = \{\lambda, \mu, \lambda + \mu, (a/(a - 1))\lambda + \mu\}, \qquad S_6 = \{\lambda, \mu, \lambda + \mu, (1 - a)^{-1}\lambda + \mu\}.$$

Proposition 6.13. *Let S_a, S_b be sets of elementary divisors as in the example. Then S_a is equivalent to S_b if and only if $b \in \{a, a^{-1}, (1 - a), (1 - a)^{-1}, a/(a - 1), (a - 1)/a\}$.*

Proof. The preceding example and Proposition 6.11 are sufficient.

Due to Lemma 6.9 and Proposition 6.11, the following definition makes sense.

Definition. Let $\mathcal{S} = \langle A, B\rangle$ be a 2-dimensional space of $g \times g$ skew-symmetric matrices. The minimal indices of the pencil $\mu A + \lambda B$ will be called the minimal indices of $\mathcal{S}$. If E is the set of elementary divisors of the given pencil, $\mathcal{C}(E)$ will be called the elementary divisors of $\mathcal{S}$. A member of $\mathcal{C}(E)$ will be called a set of canonical elementary divisors.

Theorem 6.14. *Let $\mathcal{S}_1$, $\mathcal{S}_2$ be 2-dimensional spaces of $g \times g$ skew-symmetric matrices over an algebraically closed field of characteristic $\neq 2$. There is a*

nonsingular matrix P satisfying ${}^tP\mathcal{S}_1 P = \mathcal{S}_2$ if and only if $\mathcal{S}_1$ and $\mathcal{S}_2$ have the same minimal indices and elementary divisors.

Proof. Let $\mathcal{S}_1 = \langle A, B\rangle$, $\mathcal{S}_2 = \langle C, D\rangle$. If there is such a matrix P, there is a basis A', B' of $\mathcal{S}_1$ such that the pencils $\mu A' + \lambda B'$ and $\mu C + \lambda D$ are strictly congruent. Apply Theorem 6.5, Lemma 6.9 and Proposition 6.11.

If $\mathcal{S}_1$ and $\mathcal{S}_2$ have the same elementary divisors, the elementary divisors of $\mu A + \lambda B$ are equivalent to those of $\mu C + \lambda D$. Hence there is a new basis A', B' of $\mathcal{S}_1$ for which $\mu A' + \lambda B'$, $\mu C + \lambda D$ have identical sets of elementary divisors. Since minimal indices are unaffected by changing bases, the pencils $\mu A' + \lambda B'$, $\mu C + \lambda D$ are strictly congruent. Hence there is a nonsingular matrix P with the required property ${}^tP\mathcal{S}_1 P = \mathcal{S}_2$.

Definition. Let $N = N(2,g)/I = V \oplus \wedge^2 V/I$ where I is a 2-dimensional subspace of $\wedge^2 V$. Let $\mathcal{S}$ be the 2-dimensional space of skew-symmetric matrices of $\Psi(I)$ (Ψ as in Proposition 4.1) with respect to the basis $x_1^*, \ldots, x_g^*$ of V^*. The minimal indices and elementary divisors of $\Psi(I) \equiv \mathcal{S}$ will be called the minimal indices and elementary divisors of N.

Theorem 6.15 (Isomorphism Theorem). *If N_1 and N_2 are g-generator, 2-relation metabelian Lie algebras over a closed field K whose characteristic is not 2, then $N_1 \cong N_2$ if and only if they have the same minimal indices and elementary divisors.*

Proof. Let $N_i = V \oplus \wedge^2 V/I_i$ where $\dim I_i = 2$. Then $N_1 \cong N_2$ if and only if there is a nonsingular matrix P such that ${}^tP(\Psi(I_1))P = \Psi(I_2)$ (Proposition 4.1 and Theorem 2.1). Apply Theorem 6.14.

Remarks. (i) The application of the methods developed here to metabelian Lie algebras with more than 2 relations is complicated by several difficult problems. According to Theorem 2.1 and Proposition 4.1, we could view the problem as one of finding a canonical form for a p-dimensional space ($p > 2$) of skew-symmetric matrices $\langle A_1, \ldots, A_p\rangle$. The notion of minimal row and column index are easily extended to the skew pencil $\lambda_1 A_1 + \ldots + \lambda_p A_p$, but what should be the analogue of the "reduction theorem" [11, p. 30, Theorem 4] is unclear. One could define invariant factors as quotients of greatest common divisors of minor determinants of fixed order, but with $p > 2$ they need not factor into powers of linear homogeneous polynomials in $\lambda_1, \ldots, \lambda_p$ (homogeneous polynomials in more than 2 variables need not factor linearly). Furthermore, there is no straightforward extension of Lemma 6.10 to higher-dimensional affine spaces. That is, there is no lemma of the type: given 2 ordered sets $\{l_1, \ldots, l_r\}$, $\{l'_1, \ldots, l'_r\}$ of distinct lines in K^p, there is a unique (up to scalars) $\theta \in \mathrm{GL}(p, K)$ satisfying $\theta(l_i) = l'_i$.

(ii) Let $N = N(2,g)/I$ be a two-relation metabelian Lie algebra. We can obtain a "near" canonical form for the relations I in the following way. Let $\mathcal{S}$ be the 2-dimensional space of $g \times g$ skew-symmetric matrices of the elements of $\Psi(I)$ (Ψ as in Proposition 4.1) with respect to the basis $x_1^*, \ldots, x_g^*$ of V^*. Pick a basis A, B of $\mathcal{S}$ such that the elementary divisors of the pencil $\mu A + \lambda B$ belong to the

canonical form set of the elementary divisors of $\mathcal{S}$. Then, using Theorem 6.8 (canonical form for skew pencils), it can be assumed the pencil $\mu A + \lambda B$ is in canonical form. Pick $u, v \in \wedge^2 V$ such that $\Psi(u) = A$, $\Psi(v) = B$ using Proposition 4.1, equation (4.5). Then u, v can be taken as canonical relations defining $N \cong N(2,g)/\langle u, v\rangle$. This method will be employed in the next section to classify low-dimensional metabelian Lie algebras.

7. **Low dimensions and forms.** In this section we will be concerned with classifying the low-dimensional metabelian Lie algebras and finding the first dimension at which one gets an infinite family. Due to the algebraic geometry involved we must require the field K to be algebraically closed.

When $2 < p < \binom{g}{2} - 2$, it is in general much more difficult to compute isomorphism classes of g-generator, p-relation metabelian Lie algebras, or equivalently, to compute $\mathcal{S}_p/\mathrm{GL}(V)$ where $\dim V = g$ and $\mathcal{S}_p$ is the set of p-dimensional subspaces of $\wedge^2 V$. In most cases the number of isomorphism classes is infinite. Before proceeding, we wish to view the orbit problem differently. With a p-dimensional subspace $T = \langle t_1, \ldots, t_p\rangle$ of $\wedge^2 V$ we associate a decomposable p-vector $t_1 \wedge \ldots \wedge t_p$ in $\wedge^p(\wedge^2 V)$.

Lemma 7.1. *Two linearly independent subsets $\{x_1, \ldots, x_p\}$ and $\{y_1, \ldots, y_p\}$ of a vector space W span the same subspace if and only if $x_1 \wedge \ldots \wedge x_p = \alpha y_1 \wedge \ldots \wedge y_p$ for some $\alpha \in K^{\cdot}$. If $\{w_1, \ldots, w_p\}$ is any p-element subset of W it is independent if and only if $w_1 \wedge \ldots \wedge w_p \neq 0$.*

Proof. [19, p. 566].

The set of all lines of $\wedge^p(\wedge^2 V)$ through nonzero decomposable p-vectors is a projective subvariety of $\mathcal{P}\wedge^p(\wedge^2 V)$ known as the Grassmann variety and will be denoted by G_p [14]. From the preceding discussion, we have a bijective map Φ from $\mathcal{S}_p$ onto G_p given by $\Phi(S) = K^{\cdot}(s_1 \wedge \ldots \wedge s_p)$ if $S = \langle s_1, \ldots, s_p\rangle$. Since $\wedge^p(\wedge^2(\mathrm{GL}(V)))$ stabilizes G_p we have an induced action and it is straightforward to show

Lemma 7.2. *The map $\Phi : \mathcal{S}_p \to G_p$ given by $\Phi(S) = K^{\cdot}(s_1 \wedge \ldots \wedge s_p)$, if $S = \langle s_1, \ldots, s_p\rangle$, is a $\mathrm{GL}(V)$-bijection.*

We will write D_p for the set of decomposable p-vectors in $\wedge^p(\wedge^2 V)$. It is a homogeneous affine subvariety (irreducible Zariski-closed subset) of $\wedge^p(\wedge^2 V)$ and is stable under $\wedge^p(\wedge^2(\mathrm{GL}(V)))$. We claim that the natural projection $\pi : D_p - 0 \to G_p$, which sends a nonzero vector v to the line $K^{\cdot}v$, induces a bijection between the orbit spaces $(D_p - 0)/\mathrm{GL}(V)$ and $G_p/\mathrm{GL}(V)$. For this it suffices to show that if an orbit in $D_p - 0$ contains a vector v it contains $K^{\cdot}v$. This is an easy computation using the formula $\wedge^p(\wedge^2(\alpha I)) = \alpha^{2p} I$ and extracting $2p$th roots in K. Thus $D_p/\mathrm{GL}(V)$ is finite if and only if $G_p/\mathrm{GL}(V)$ is finite. It is well known [14] that

$$\dim G_p = (\tbinom{g}{2} - p)p; \tag{7.1}$$

hence

$$\dim D_p = (\tbinom{g}{2} - p)p + 1. \tag{7.2}$$

Our aim now is to characterize those g and p such that $G_p/\mathrm{GL}(V)$ is infinite, that is, such that there are infinitely many g-generator, p-relation algebras. This is accomplished by comparing the dimension of an orbit in G_p to the dimension of G_p itself. First, however, we need to know the orbits are subvarieties. This requires a look at the evaluation map $\mathrm{GL}(V) \times D_p \to D_p$.

We will show that the evaluation map $\alpha : \mathrm{GL}(V) \times D_p \to D_p$ given by

$$\alpha(\theta, w) = \wedge^p (\wedge^2 \theta)(w) \tag{7.3}$$

is a morphism of affine varieties. We proceed by showing that the representation $\wedge^p : \mathrm{GL}(W) \to \mathrm{GL}(\wedge^p W)$ is a morphism for any vector space W.

Fix a basis $w_1, \ldots, w_n$ of W. Then all elements of the type $w_{i_1} \wedge \ldots \wedge w_{i_p}$ where $1 \leq i_1 < i_2 < \ldots < i_p \leq n$ form a basis of $\wedge^p W$. Let I be the index set consisting of ordered p-tuples of increasing integers between 1 and n. We take g_{ij}, $1 \leq i, j \leq n$, as coordinate functions in $\mathrm{GL}(W)$ and $h_{\alpha\beta}$, $\alpha, \beta \in I$ as coordinate functions in $\mathrm{GL}(\wedge^p W)$ determined by the respective bases chosen above. If α, $\beta \in I$, the (α, β)th coordinate function of $\wedge^p$ is the minor of $|g_{ij}|$ determined by the rows $\alpha(1), \ldots, \alpha(p)$ and the columns $\beta(1), \ldots, \beta(p)$. The coordinate ring of $\mathrm{GL}(\wedge^p W)$ is generated by the $h_{\alpha\beta}$ together with $1/\det|h_{\alpha\beta}|$, while that of $\mathrm{GL}(W)$ is generated by the various g_{ij} together with $1/\det|g_{ij}|$. The coordinate functions of $\wedge^p$ are polynomials in the g_{ij}, so to show $\wedge^p$ is a morphism it suffices to show that $1/\det|\wedge^p (g_{ij})|$ is a polynomial in g_{ij} and $1/\det|g_{ij}|$. For this, the following lemma is useful.

Lemma 7.3. *Every rational character of* $\mathrm{GL}(W)$ *is an integral power of the determinant.*

Proof. By rational is meant that the coordinate function is a sum of quotients of polynomials in the coordinates of W. (See [6, p. 22].)

Now $\det \circ \wedge^p : \mathrm{GL}(W) \to K^{\cdot}$ is a homogeneous polynomial character, thus $\det|\wedge^p (g_{ij})| = (\det|g_{ij}|)^a$ for some positive integer a, and we have proven

Lemma 7.4. *Let W be a K-vector space. The representation* $\wedge^p : \mathrm{GL}(W) \to \mathrm{GL}(\wedge^p W)$ *is a morphism of affine varieties.*

We mention without proof the following

Lemma 7.5. *Let W be a K-vector space.*

(i) *Suppose $\mathcal{W}$ is an affine subvariety of W. Then the evaluation map* $\mathrm{GL}(W) \times \mathcal{W} \to W$ *is a morphism.*

(ii) *If $f : X \to Y$ is a morphism, then so is the restriction $f : X \to \mathrm{Cl}(f(X))$ where* $\mathrm{Cl}(\,)$ *means Zariski-closure.*

Now consider the commuting diagram

$$\begin{tikzcd}
GL(V) \times D_p \arrow[r, "(\wedge^2, I)"] \arrow[d, "\alpha"'] & GL(\wedge^2 V) \times D_p \arrow[r, "(\wedge^p, I)"] & GL(\wedge^p(\wedge^2 V)) \times D_p \arrow[d] \\
D_p \arrow[rr, "i"] & & \wedge^p(\wedge^2 V)
\end{tikzcd}$$

where the vertical maps are evaluations and i is the inclusion map.

Lemma 7.6. *The evaluation map $\alpha : \mathrm{GL}(V) \times D_p \to D_p$ is a morphism of affine varieties.*

Proof. The proof follows from the preceding lemmas, the commutativity of the diagram and the well-known fact that D_p is closed in $\wedge^p(\wedge^2 V)$.

Definition. A subset of a variety is locally closed if it is the intersection of an open subset with an irreducible closed subset.

Lemma 7.7. *Every orbit of* $\mathrm{GL}(V)$ *in D_p is a locally closed subvariety.*

Proof. The proof follows from the proposition or p. 98 of [3] due to Lemma 7.6.

The point of the lemmas is the following

Theorem 7.8. *Suppose for some $p < \binom{g}{2}$ that*

$$g^2 < \left(\binom{g}{2} - p\right)p + 1. \tag{7.4}$$

Then there are infinitely many g-generator, p-relation metabelian Lie algebras. When $g \geq 6$ and $3 \leq p \leq \binom{g}{2} - 3$ the inequality holds.

Proof. Suppose there were only finitely many. Then the orbit space $D_p/\mathrm{GL}(V)$ is also finite. Now D_p is a finite union of orbits, and hence of their closures as well. Since D_p is irreducible it must be equal to one of these closures, say $\mathrm{Cl}(\mathcal{O})$. Hence $\left(\binom{g}{2} - p\right)p + 1 = \dim D_p = \dim \mathcal{O} \leq g^2$, since $\mathcal{O}$ is the image of the g^2-

dimensional variety $\mathrm{GL}(V)$. This contradiction establishes the result.

The expression $(\binom{g}{2} - p)p + 1$, as a polynomial in p, is symmetric about $\binom{g}{2}/2$ at which it attains a maximum. Since $g \geq 6$ forces $\binom{g}{2}/2 > 3$, it suffices to check that (7.4) holds for g-arbitrary and $p = 3$ which is a routine calculation.

We now look for the first dimension at which (7.4) determines an infinite number of isomorphism classes. If for some g there is a p satisfying (7.4), the largest such p-value gives the lowest dimensional infinite family. Consider the following table where $d(g) = \dim N(2,g) - \max\{p \mid g^2 < \dim D_p\}$ is the smallest dimension of an infinite family of g-generator algebras arising from (7.4).

$g = \dim V$	$\max\{p \mid g^2 < \dim D_p\}$	$d(g)$
3	set is empty	—
4	set is empty	—
5	5	10
6	12	9
7	18	10

If N is a metabelian Lie algebra with 8 or more generators then $\dim N \geq 9$. Hence the dimension argument can give no information on algebras of dimension less than nine.

Corollary 7.9. *There are infinitely many isomorphism classes of n-dimensional metabelian Lie algebras for each $n \geq 9$.*

For $p = 1, \binom{g}{2} - 1$ we have seen that $D_p/\mathrm{GL}(V)$ is always finite. For $g \geq 6$, $3 \leq p \leq \binom{g}{2} - 3$ it is infinite. Also, if $p = 2$, $\dim D_2 = g^2 - g - 3 < g^2$, so the dimension argument gives no information about orbits of 2-dimensional subspaces. This can be determined however from results of the last section. One might expect from the dimension argument that $D_2/\mathrm{GL}(V)$ is finite. In general however it is not.

Theorem 7.10. *Let K be algebraically closed and suppose $\chi(K) \neq 2$.*

(i) *Let $g \geq 8$ and set $N_a(g) = N(2,g)/I_a$ where $I_a = \langle x_1 \wedge x_2 + x_5 \wedge x_6 + ax_7 \wedge x_8, x_3 \wedge x_4 + x_5 \wedge x_6 + x_7 \wedge x_8 \rangle$ and where $a \neq 0, 1$. Then $N_a(g) \cong N_b(g)$ if and only if $b \in \{a, a^{-1}, (1-a), (1-a)^{-1}, a/(a-1), (a-1)/a\}$.*

(ii) *There are infinitely many isomorphism classes of g-generator, 2-relation algebras if and only if $g \geq 8$.*

Proof. Let the given basis vectors of I_a be denoted by u_a, v_a. Let $\mathcal{S}_a = \Psi(I_a)$ (Ψ as in Proposition 4.1) and let A_a, B_a be the matrices of $\Psi(u_a)$, $\Psi(v_a)$ in the basis $x_1^*, \ldots, x_g^*$ of V^*. Then the pencil $\mu A_a + \lambda B_a$ is given by

$$\begin{array}{|cc|cc|cc|cc|c|}
\hline
0 & \mu & & & & & & & \\
-\mu & 0 & & & & & & & \\
\hline
 & & 0 & \lambda & & & & & \\
 & & -\lambda & 0 & & & & & \\
\hline
 & & & & 0 & \lambda+\mu & & & \\
 & & & & -(\lambda+\mu) & 0 & & & \\
\hline
 & & & & & & 0 & \lambda+a\mu & \\
 & & & & & & -(\lambda+a\mu) & 0 & \\
\hline
 & & & & & & & & 0 \\
\hline
\end{array}$$

where the lower right 0-block is $(g-8)\times(g-8)$. The minimal indices are all zero while the elementary divisors are $E_a = \{\mu, \mu, \lambda, \lambda, (\lambda+\mu), (\lambda+\mu), (a\mu+\lambda), (a\mu+\lambda)\}$. By Theorem 6.15, $N_a(g) \cong N_b(g)$ if and only if $\mathcal{C}(E_a) = \mathcal{C}(E_b)$. Now apply Proposition 6.13 and Example 6.12.

The second part of the theorem follows since if $g \leq 7$, a g-generator algebra can have at most 3-distinct elementary divisors (see the canonical form for skew pencils—Theorem 6.8) which can be chosen from λ, μ, $\lambda+\mu$ or a 2nd or 3rd power of one of these. Also the minimal indices have only a finite number of possibilities. Apply the isomorphism theorem.

Quite a bit of work has been done over the past 25 years on the classification of metabelian Lie algebras and the listing of low-dimensional nilpotent Lie algebras. Another interesting question for nilpotent Lie algebras is, over what subfields do they have forms?

Definition. Let k be a subfield of K and suppose L is a K-algebra having a basis $x_1, \ldots, x_n$ whose multiplication table is $x_i x_j = \sum_k \alpha_{ijk} x_k$, where the α_{ijk} belong to k. Then L is said to have a k-form or it is sometimes called a k-algebra.

The following appear to be the most significant results on these questions.

Up to and including dimension 5, there are 16 nilpotent Lie algebras over any field [9], and in fact, all these algebras have forms over their respective prime fields.

At dimension 6 over algebraically closed fields, Morozov [22] showed there are finitely many nilpotent Lie algebras all having forms over their prime fields. Over $\mathbf{Q}$, however, Scheuneman [24] constructed an infinite family of metabelian Lie algebras.

For any dimension $n \geq 7$ over an infinite field, there are infinitely many $(n-1)$-step nilpotent Lie algebras, and for any subfield k there is such an algebra having no k-form. This result appears as an exercise [4, p. 122] for the case $n = 7$, and is easily generalized.

At dimension 8 we have the

Theorem 7.11. *Let K be algebraically closed and $\chi(K) = 0$. There are infinitely many 8-dimensional, 3-step nilpotent Lie algebras over K.*

Proof. Consider algebras of the type $N(3,g)/I$ where I is a subspace of $N(3,g)^3$ (any such subspace is an ideal). If V is the subspace of $N(3,g)$ spanned by $x_1, \ldots, x_g$, we can embed $\mathrm{GL}(V)$ in $\mathrm{Aut}\, N(3,g)$, and it is then straightforward to check that $\mathcal{S}_p/\mathrm{Aut}\, N(3,g) = \mathcal{S}_p/\mathrm{GL}(V)$ where $\mathcal{S}_p$ consists of p-dimensional subspaces of $N(3,g)^3$.

Using the same techniques developed to prove Theorem 7.8 and Witt's dimension formula (1.1), we find that $\mathcal{S}_p/\mathrm{GL}(V)$ is infinite whenever

$$g^2 < \dim D_p = (((g^3 - g)/3) - p)p + 1$$

(D_p being the decomposable p-vectors in $\bigwedge^p (N(3,g)^3)$). Setting $g = 3, p = 6$, the inequality is satisfied. The corresponding algebras are 8-dimensional.

At any dimension $n \geq 10$, Chao [7] has succeeded in showing the existence of infinitely many n-dimensional, **R**-metabelian Lie algebras having no **Q**-forms. It had already been shown by Malcev [21] that there was at least one such algebra when $n \geq 16$.

Gurevič [13] studied metabelian Lie algebras over algebraically closed fields. He considered the weights of such an algebra viewed as a module for some Cartan subalgebra of its derivation algebra. He gave an abstract classification for those algebras whose weights are simple (1-dimensional weight spaces), though not all metabelian Lie algebras enjoy this property.

It remains possible there are only finitely many metabelian Lie algebras of dimensions 7 and 8 over algebraically closed fields. We can now give an exact number for dimension 7 when $\chi(K) \neq 2$ and nearly complete results for dimension 8. We begin by classifying the 4-, 5- and 6-generator, 2-relation algebras. By the duality theory these correspond to 6-, 7- and 8-dimensional algebras. Due to the results of §6 they are finite in number.

Theorem 7.12. *Let K be algebraically closed and suppose $\chi(K) \neq 2$. Then every 4-generator, 2-relation metabelian Lie algebra N is isomorphic to exactly one of $N(2,g)/I_j, j = 1, 2, 3$, where*

$$I_1 = \langle x_1 \wedge x_2, x_3 \wedge x_4 \rangle,$$

$$I_2 = \langle x_1 \wedge x_4 + x_2 \wedge x_3, x_2 \wedge x_4 \rangle,$$

$$I_3 = \langle x_2 \wedge x_4, x_3 \wedge x_4 \rangle.$$

Proof. Let $N = V \oplus \bigwedge^2 V/I$ be a 4-generator, 2-relation algebra. Proceed according to the last remark at the end of §6. Due to the canonical form for skew pencils (Theorem 6.8) and the limitation that a 4×4 skew pencil has at most 2 distinct elementary divisors, the following table lists all possible combinations of "canonical" elementary divisors and minimal indices:

	Elementary divisors	*Minimal indices*
1.	$\{\mu, \mu, \lambda, \lambda\}$	———
2.	$\{\mu^2, \mu^2\}$	———
3.	———	0, 1

The corresponding canonical pencils are

$$\left|\begin{array}{cc|cc} 0 & \mu & & \\ -\mu & 0 & & \\ \hline & & 0 & \lambda \\ & & -\lambda & 0 \end{array}\right|, \quad \left|\begin{array}{cc|cc} & & 0 & \mu \\ & & \mu & \lambda \\ \hline 0 & -\mu & & \\ -\mu & -\lambda & & \end{array}\right|, \quad \left|\begin{array}{c|ccc} 0 & & & \\ \hline & 0 & 0 & \lambda \\ & 0 & 0 & \mu \\ & -\lambda & -\mu & 0 \end{array}\right|.$$

Using equation (4.5), the coresponding canonical relations are $I_1 = \langle x_1 \wedge x_2, x_3 \wedge x_4 \rangle$, $I_2 = \langle x_1 \wedge x_4 + x_2 \wedge x_3, x_2 \wedge x_4 \rangle$ and $I_3 = \langle x_2 \wedge x_4, x_3 \wedge x_4 \rangle$. Apply the Isomorphism Theorem 6.15.

Theorem 7.13. *Let K be algebraically closed and suppose $\chi(K) \neq 2$. Then every 5-generator, 2-relation metabelian Lie algebra N is isomorphic to exactly one of $N(2,g)/I_j, j = 1, \ldots, 5$, where*

$$I_1 = \langle x_1 \wedge x_2, x_3 \wedge x_4 \rangle,$$

$$I_2 = \langle x_1 \wedge x_4 + x_2 \wedge x_3, x_2 \wedge x_4 \rangle,$$

$$I_3 = \langle x_1 \wedge x_2 + x_4 \wedge x_5, x_3 \wedge x_5 \rangle,$$

$$I_4 = \langle x_3 \wedge x_5, x_4 \wedge x_5 \rangle,$$

$$I_5 = \langle x_1 \wedge x_5 + x_2 \wedge x_4, x_2 \wedge x_5 + x_3 \wedge x_4 \rangle.$$

Proof. Proceeding as in the last theorem and realizing that a 5×5 skew pencil has at most 2 distinct elementary divisors, the following table lists all possible combinations of "canonical" elementary divisors and minimal indices:

	Elementary Divisors	*Minimal Indices*
1.	$\{\mu, \mu, \lambda, \lambda\}$	0
2.	$\{\mu^2, \mu^2\}$	0
3.	$\{\mu, \mu\}$	1
4.	———	0, 0, 1
5.	———	2

The corresponding canonical pencils are

$$\left|\begin{array}{cc|cc|c} 0 & \mu & & & \\ -\mu & 0 & & & \\ \hline & & 0 & \lambda & \\ & & -\lambda & 0 & \\ \hline & & & & 0 \end{array}\right|, \quad \left|\begin{array}{cc|cc|c} & & 0 & \mu & \\ & & \mu & \lambda & \\ \hline 0 & -\mu & & & \\ -\mu & -\lambda & & & \\ \hline & & & & 0 \end{array}\right|, \quad \left|\begin{array}{cc|ccc} 0 & \mu & & & \\ -\mu & 0 & & & \\ \hline & & 0 & 0 & \lambda \\ & & 0 & 0 & \mu \\ & & -\lambda & -\mu & 0 \end{array}\right|,$$

$$\left|\begin{array}{cc|ccc} 0 & 0 & & & \\ 0 & 0 & & & \\ \hline & & 0 & 0 & \lambda \\ & & 0 & 0 & \mu \\ & & -\lambda & -\mu & 0 \end{array}\right|, \quad \left|\begin{array}{ccc|cc} & & & 0 & \lambda \\ & & & \lambda & \mu \\ & & & \mu & 0 \\ \hline 0 & -\lambda & -\mu & & \\ -\lambda & -\mu & 0 & & \end{array}\right|,$$

Using (4.5), the corresponding canonical relations are

$$I_1 = \langle x_1 \wedge x_2, x_3 \wedge x_4 \rangle, \qquad I_2 = \langle x_1 \wedge x_4 + x_2 \wedge x_3, x_2 \wedge x_4 \rangle,$$

$$I_3 = \langle x_1 \wedge x_2 + x_4 \wedge x_5, x_3 \wedge x_5 \rangle, \qquad I_4 = \langle x_3 \wedge x_5, x_4 \wedge x_5 \rangle,$$

$$I_5 = \langle x_1 \wedge x_5 + x_2 \wedge x_4, x_2 \wedge x_5 + x_3 \wedge x_4 \rangle.$$

Apply Theorem 6.15.

Theorem 7.14. *Let K be algebraically closed and suppose $\chi(K) \neq 2$. Then every 6-generator, 2-relation metabelian Lie algebra N is isomorphic to exactly one of $N(2,g)/I_j$ where $j = 1, \ldots, 11$, and*

$$I_1 = \langle x_1 \wedge x_2 + x_5 \wedge x_6, x_3 \wedge x_4 + x_5 \wedge x_6 \rangle,$$

$$I_2 = \langle x_1 \wedge x_2 + x_3 \wedge x_4, x_5 \wedge x_6 \rangle,$$

$$I_3 = \langle x_1 \wedge x_4 + x_2 \wedge x_3, x_2 \wedge x_4 + x_5 \wedge x_6 \rangle,$$

$$I_4 = \langle x_1 \wedge x_4 + x_2 \wedge x_3 + x_5 \wedge x_6, x_2 \wedge x_4 \rangle,$$

$$I_5 = \langle x_1 \wedge x_6 + x_2 \wedge x_5 + x_3 \wedge x_4, x_2 \wedge x_6 + x_3 \wedge x_5 \rangle,$$

$$I_6 = \langle x_1 \wedge x_2, x_3 \wedge x_4 \rangle,$$

$$I_7 = \langle x_1 \wedge x_4 + x_2 \wedge x_3, x_2 \wedge x_4 \rangle,$$

$$I_8 = \langle x_1 \wedge x_2 + x_5 \wedge x_6, x_4 \wedge x_6 \rangle,$$

$$I_9 = \langle x_5 \wedge x_6, x_4 \wedge x_6 \rangle,$$

$$I_{10} = \langle x_1 \wedge x_3 + x_4 \wedge x_6, x_2 \wedge x_3 + x_5 \wedge x_6 \rangle,$$

$$I_{11} = \langle x_2 \wedge x_6 + x_3 \wedge x_5, x_3 \wedge x_6 + x_4 \wedge x_5 \rangle.$$

Proof. Proceeding as in the last two theorems, and realizing that a 6×6 pencil has at most 3 distinct elementary divisors, the following table lists all possible combinations of "canonical" elementary divisors and minimal indices:

	Elementary Divisors	*Minimal Indices*
1.	$\{\mu, \mu, \lambda, \lambda, \mu + \lambda, \mu + \lambda\}$	———
2.	$\{\mu, \mu, \mu, \mu, \lambda, \lambda\}$	———
3.	$\{\mu^2, \mu^2, \lambda, \lambda\}$	———
4.	$\{\mu^2, \mu^2, \mu, \mu\}$	———
5.	$\{\mu^3, \mu^3\}$	———
6.	$\{\mu, \mu, \lambda, \lambda\}$	0, 0
7.	$\{\mu^2, \mu^2\}$	0, 0
8.	$\{\mu, \mu\}$	0, 1
9.	———	0, 0, 0, 1
10.	———	1, 1
11.	———	0, 2

The corresponding canonical relations are $I_1, \ldots, I_{11}$. Apply Theorem 6.15.

To complete the classification of 7-dimensional metabelian Lie algebras we must classify the 4-generator, 3-relation algebras. In order to establish some conventions for these calculations we make the following definitions. Suppose $V = \langle y_1, \ldots, y_4 \rangle$ and $S = \langle u, v, w \rangle$ is a 3-dimensional subspace of $\wedge^2 V$.

Definition. (1) By the statement "replace y_i by $y_i + ay_j$" is meant to apply θ in $\mathrm{GL}(V)$ to S where $\theta(y_k) = y_k$ $(k \neq i)$ and $\theta(y_i) = y_i + ay_j$.

(2) By the statement "replace $y_i + ay_j$ by y_i" it is meant to apply θ in $\mathrm{GL}(V)$ to S where $\theta(y_i + ay_j) = y_i$, $\theta(y_k) = y_k$ $(k \neq i)$.

(3) By the statement "scale y_i by a" it is meant to apply θ in $\mathrm{GL}(V)$ to S where $\theta(y_i) = ay_i$, $\theta(y_k) = y_k$ $(k \neq i)$.

(4) By the statement "replace u by au" it is meant to abuse notation and write u for au.

(5) By the phrase "changing bases unimodularly in $\langle y_i, y_j \rangle$" it is meant to apply θ in $\mathrm{GL}(V)$ to S where θ stabilizes $\langle y_i, y_j \rangle$, $\theta(y_k) = y_k$ $(k \neq i,j)$ and $\det \theta = 1$ (for such a θ, $y_i \wedge y_j = \theta(y_i) \wedge \theta(y_j)$).

(6) It is also common that $w = \sum_{i<j} a_{ij} y_i \wedge y_j + z_1 \wedge z_2$, initially. By a "change of bases in $\langle z_1, z_2 \rangle$" we mean to abuse notation and write z_1, z_2 for $az_1 + bz_2$, $cz_1 + dz_2$ respectively where $ad - bc = 1$.

We will also abuse notation by writing u for $\theta(u)$, v for $\theta(v)$, w for $\theta(w)$ and S for $\theta(S)$.

Theorem 7.15. *Let K be algebraically closed and suppose $\chi(K) \neq 2$. Then every 4-generator, 3-relation metabelian Lie algebra N is isomorphic to exactly one of $N(2,g)/I_j$ where $j = 1, \ldots, 6$ and*

$$I_1 = \langle x_1 \wedge x_2, x_1 \wedge x_3, x_2 \wedge x_3 \rangle,$$

$$I_2 = \langle x_1 \wedge x_2, x_1 \wedge x_3, x_1 \wedge x_4 \rangle,$$

$$I_3 = \langle x_1 \wedge x_2, x_1 \wedge x_3, x_2 \wedge x_4 \rangle,$$

$$I_4 = \langle x_1 \wedge x_2, x_3 \wedge x_4, (x_1 + x_3) \wedge (x_2 + x_4) \rangle,$$

$$I_5 = \langle x_1 \wedge x_2 + x_3 \wedge x_4, x_2 \wedge x_4, x_1 \wedge x_4 \rangle,$$

$$I_6 = \langle x_1 \wedge x_2 + x_3 \wedge x_4, x_2 \wedge x_4, x_1 \wedge x_3 \rangle.$$

Proof. Let $N = N(2,g)/S$ where S is a 3-dimensional subspace of $\wedge^2 V$, $\dim V = 4$. The proof is broken up into Lemmas 7.16–7.19.

Lemma 7.16. *If S is spanned by decomposables then S is in the orbit of one of $I_1, \ldots, I_4$.*

Proof. We can write $S = \langle u, v, w \rangle$ where $u = y_1 \wedge y_2$, $v = y_3 \wedge y_4$, $w = z_1 \wedge z_2$. Consider first the subspace $S' = \langle u, v \rangle$ of $\wedge^2 V$. By Theorem 7.12, S' is in the orbit of $\langle x_1 \wedge x_2, x_3 \wedge x_4 \rangle$ or $\langle x_1 \wedge x_2, x_1 \wedge x_3 \rangle$ where $V = \langle x_1, \ldots, x_4 \rangle$. Pick new vectors $y_1, \ldots, y_4$ which are a basis of V and replace x_i by y_i to get $S = \langle y_1 \wedge y_2, y_1 \wedge y_3, z_1 \wedge z_2 \rangle$ or $S = \langle y_1 \wedge y_2, y_3 \wedge y_4, z_1 \wedge z_2 \rangle$.

1. Suppose $S = \langle y_1 \wedge y_2, y_1 \wedge y_3, z_1 \wedge z_2 \rangle$. Let $V' = \langle y_1, y_2, y_3 \rangle \cap \langle z_1, z_2 \rangle$. Since $\dim V = 4$, $\dim V' = 1$ or 2.

1.1. Suppose $\dim V' = 2$. If y_1 is in V', by changing bases in $\langle z_1, z_2 \rangle$ we can assume $z_1 = y_1$, and thus $S = \langle y_1 \wedge y_2, y_1 \wedge y_3, y_1 \wedge (ay_1 + by_2) \rangle$ which is only 2-dimensional. Thus $y_1 \notin V'$. Write $z_1 = ay_1 + by_2 + cy_3$, $z_2 = dy_1 + ey_2 + fy_3$. Since $y_1 \notin V'$, we must have $bf - ce \neq 0$. Thus $\langle y_1 \wedge y_2, y_1 \wedge y_3 \rangle = \langle y_1 \wedge (by_2 + cy_3), \quad y_1 \wedge (ey_2 + fy_3) \rangle = \langle y_1 \wedge (ay_1 + by_2 + cy_3), y_1 \wedge (dy_1 + ey_2 + fy_3) \rangle$. Replacing $by_2 + ay_1 + cy_2$ by y_2 and then $dy_1 + ey_2 + fy_3$ by y_3 we achieve $S = I_1$.

1.2. Suppose $\dim V' = 1$. Changing bases in $\langle z_1, z_2 \rangle$ we can assume $z_1 \in \langle y_1, y_2, y_3 \rangle$ while z_2 is independent of y_1, y_2, y_3. Set $y_4 = z_2$ and note that $y_1, \ldots, y_4$ is a basis of V. Write $z_1 = ay_1 + by_2 + cy_3$.

1.2.1. Suppose $a \neq 0$. Scale y_1 by a^{-1} and scale u, v by a to get $z_1 = y_1 + by_2 + cy_3$.

1.2.1.1. Suppose $b = c = 0$. Then $S = I_2$.

1.2.1.2. Suppose $b \neq 0 = c$. By symmetry this is identical to $b = 0 \neq c$. Scale y_2 by b^{-1} and u by b to get $z_1 = y_1 + y_2$. Replacing $y_2 + y_1$ by y_2 we accomplish $z_1 = y_2$, $w = y_2 \wedge y_4$, $S = I_3$.

1.2.1.3. Suppose $b \neq 0 \neq c$. Scale y_2 by b^{-1}, y_3 by c^{-1}, u by b, v by c to get $z_1 = y_1 + y_2 + y_3$. Then $S = \langle y_1 \wedge y_2, y_1 \wedge y_3, (y_1 + y_2 + y_3) \wedge y_4 \rangle = \langle y_1 \wedge (y_1 + y_2 + y_3), y_1 \wedge y_3, (y_1 + y_2 + y_3) \wedge y_4 \rangle$, so replacing $y_2 + y_1 + y_3$ by y_2 we get $S = I_3$.

1.2.2. Suppose $a = 0$. We can assume $b \neq 0$, $z_1 = by_2 + cy_3$. Scaling y_2 by b^{-1} and u by b we achieve $z_1 = y_2 + cy_3$, $S = \langle y_1 \wedge y_2, y_1 \wedge y_3, (y_2 + cy_3) \wedge y_4 \rangle = \langle y_1 \wedge (y_2 + cy_3), y_1 \wedge y_3, (y_2 + cy_3) \wedge y_4 \rangle$. Replacing $y_2 + cy_3$ by y_2 we see that $S = I_3$.

2. Suppose $S = \langle y_1 \wedge y_2, y_3 \wedge y_4, z_1 \wedge z_2 \rangle$. Write $V = \langle y_1, y_2 \rangle \oplus \langle y_3, y_4 \rangle$ and let z_i^j denote the jth projection of z_i in the splitting of V.

2.1. Suppose $\{z_1^1, z_2^1\}$, $\{z_1^2, z_2^2\}$ are sets of independent vectors. Changing bases in $\langle y_1, y_2 \rangle$ and $\langle y_3, y_4 \rangle$ and scaling u, v we can assume $z_1^1 = y_1$, $z_2^1 = y_2$, $z_1^2 = y_3$, $z_2^2 = y_4$; $S = I_4$.

2.2. Suppose $\{z_1^1, z_2^1\}$, $\{z_1^2, z_2^2\}$ are dependent sets. Since $z_1 \wedge z_2 \neq 0$ we cannot have $\langle z_1^1, z_2^1 \rangle = 0$ or $\langle z_1^2, z_2^2 \rangle = 0$. Changing bases in $\langle z_1, z_2 \rangle$ we can assume $z_1^1 \neq 0 = z_2^1$. Changing bases unimodularly in $\langle y_1, y_2 \rangle$ we can insure $z_1^1 = y_1$. Now $z_2 \neq 0$ so $z_2^2 \neq 0$. Changing bases unimodularly in $\langle y_3, y_4 \rangle$ we can assume $z_2^2 = y_3$, $z_1^2 = ay_3$. Thus $z_1 \wedge z_2 = (y_1 + ay_3) \wedge y_3 = y_1 \wedge y_3$ and $S = I_3$.

2.3. Suppose $\{z_1^1, z_2^1\}$ is independent while $\{z_1^2, z_2^2\}$ is dependent. Changing bases in $\langle z_1, z_2 \rangle$ we can get $z_1^2 \neq 0 = z_2^2$, and changing unimodularly in $\langle y_3, y_4 \rangle$ we can insure $z_1^2 = y_3$. Changing bases in $\langle y_1, y_2 \rangle$ and scaling u we can achieve $z_1^1 = y_1$, $z_2^1 = y_2$. Replace w by $w - u$ to achieve $w = y_3 \wedge y_2$. Apply θ in $\mathrm{GL}(V)$ to S where $\theta(y_1) = x_3$, $\theta(y_2) = x_1$, $\theta(y_3) = x_2$, $\theta(y_4) = x_4$ to get $S = I_3$.

Lemma 7.17. *S contains at least 2 independent decomposable vectors.*

Proof. This follows from several applications of the easily proven result: any 2-dimensional subspace T of $\wedge^2 V$ ($\dim V = 4$) contains a nonzero decomposable vector.

Lemma 7.18. *If S is not spanned by decomposables then it is in the orbit of I_5 or I_6.*

Proof. By Lemma 7.17 we can write $S = \langle u, v, w \rangle$ where $u = y_1 \wedge y_2 + y_3 \wedge y_4$, $v = z_1 \wedge z_2$, $w = z_3 \wedge z_4$ and $V = \langle y_1, \ldots, y_4 \rangle$. Set $S' = \langle u, v \rangle$. By Theorem 7.12, S' is in the orbit of $\langle x_1 \wedge x_2 + x_3 \wedge x_4, x_2 \wedge x_4 \rangle$. Replacing x_i by y_i we see we can initially assume $S = \langle u, v, w \rangle$ where $u = y_1 \wedge y_2 + y_3 \wedge y_4$, $v = y_2 \wedge y_4$, $w = z_1 \wedge z_2$ and $y_1, \ldots, y_4$ is a basis of V. Again, we write $V = \langle y_1, y_2 \rangle \oplus \langle y_3, y_4 \rangle$ and let z_i^j denote the jth projection of z_i in this splitting.

1. Suppose $\{z_1^1, z_2^1\}$, $\{z_1^2, z_2^2\}$ are independent sets. Changing bases in $\langle z_1, z_2 \rangle$ and scaling w, we can assume $z_1^1 = y_1$, $z_2^1 = y_2$, that is, $z_1 = y_1 + ay_3 + by_4$, $z_2 = y_2 + cy_3 + dy_4$ where $ad - bc \neq 0$.

1.1. Suppose $b \neq 0$. Scaling y_4 by b^{-1}, y_3 by b, and v by b, we can assume $z_1 = y_1 + ay_3 + y_4$, $z_2 = y_2 + cy_3 + dy_4$ (we abuse notation here writing a for ab, c for cb, etc. and will follow this convention throughout the remaining calculations).

1.1.1. Suppose $d \neq 0$. Scale y_1, y_4 by d^{-1}, and u, v, z_1 by d to get $z_1 = y_1 + ay_3 + y_4$, $z_2 = y_2 + cy_3 + y_4$. Consider the equation $(Xu + w)^2 = 0$. This is equivalent to the equation $2X^2 + 2X(a - c + 1) = 0$ over $K[X]$. Since S is not spanned by decomposables $a - c + 1 = 0$ or $a = c - 1$. (Note: if $\chi(K) \neq 2$, a vector $w \in \bigwedge^2 V$ is decomposable if and only if $w^2 = w \wedge w = 0$.) Replacing z_1 by $z_1 - z_2$, we get $z_1 = y_1 - y_2 - y_3$, $z_2 = y_2 + cy_3 + y_4$. Consider the equation $(Xu + v + w)^2 = 0$. This is equivalent to $2X^2 + 2X(-1 + 1) + 2(-c) = 0$ over $K[X]$. Since S is not spanned by decomposables we have $c = 0$. Replacing y_4 by $y_4 - y_2$ and then replacing $y_1 - y_3$ by y_1, we get $z_1 = y_1 - y_2$, $z_2 = y_4$. Replacing $y_1 - y_2$ by y_1 we get $S = I_5$.

1.1.2. Suppose $d = 0$. Then $z_1 = y_1 + ay_3 + y_4$, $z_2 = y_2 + cy_3$ where $c \neq 0$. Replace z_1 by $z_1 - ac^{-1}z_2$ and then replace $y_1 - ac^{-1}y_2$ by y_1 to get $z_1 = y_1 + y_4$, $z_2 = y_2 + cy_3$. Consider $(Xu + w)^2 = 0$. This is equivalent to $2X^2 + 2X(-c + 1) = 0$ over $K[X]$. Thus, since S is not spanned by decomposables, $c = 1$. Replace y_3 by $y_3 - y_2$ and then $y_1 + y_4$ by y_1 to get $z_1 = y_1$, $z_2 = y_3$, $S = I_6$.

1.2. Suppose $b = 0$. Then $z_1 = y_1 + ay_3$, $z_2 = y_2 + cy_3 + dy_4$ and $ad \neq 0$. Scaling y_1, y_4 by d^{-1} and scaling z_1, u, v by d, we achieve $z_1 = y_1 + ay_3$, $z_2 = y_2 + cy_3 + y_4$. Consider the equation $(Xu + w)^2 = 0$, which is equivalent to $2X^2 + 2X(a + 1) = 0$. Since S is not spanned by decomposables $a = -1$, $z_1 = y_1 - y_3$, $z_2 = y_2 + cy_3 + y_4$.

1.2.1. Suppose $c = 0$. Replace y_4 by $y_4 - y_2$ and then $y_1 - y_3$ by y_1 to get $z_1 = y_1$, $z_2 = y_4$, $S = I_5$.

1.2.2. Suppose $c \neq 0$. Scaling y_1, y_3 by c^{-1} and scaling u, z_1 by c, we get $z_1 = y_1 - y_3$, $z_2 = y_2 + y_3 + y_4$. Replace y_4 by $y_4 - y_2$ and then $y_1 - y_3$ by y_1 to get $z_1 = y_1$, $z_2 = y_3 + y_4$. Replace $y_3 + y_4$ by y_3 to get $z_1 = y_1$, $z_2 = y_3$, $S = I_6$.

2. Suppose $\{z_1^1, z_2^1\}$ is independent and $\{z_1^2, z_2^2\}$ is dependent. Changing bases in $\langle z_1, z_2 \rangle$ and scaling w we can assume $z_1^1 = y_1$, $z_2^1 = y_2$. That is, we have $z_1 = y_1 + ay_3 + by_4$, $z_2 = y_2 + cy_3 + dy_4$, rank $\left|\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right| = 0$ or 1. If the rank is zero, S is spanned by decomposables contrary to assumption. Thus the rank is one.

2.1. Suppose $ay_3 + by_4 \neq 0$. Since $z_2^2 = \lambda z_1^2$ we can replace z_2 by $z_2 - \lambda z_1$ to achieve $z_1 = y_1 + ay_3 + by_4$, $z_2 = -\lambda y_1 + y_2$. Consider the equation $(Xu + w)^2 = 0$ which is equivalent to $2X^2 + 2X = 0$. This equation has a nonzero solution $X = -1$ so S is spanned by decomposables.

2.2. Suppose $ay_3 + by_4 = 0$. Then $cy_3 + dy_4 \neq 0$. Consider $(Xu + w)^2 = 0$, or equivalently, $2X^2 + 2X = 0$ which has the solution $X = -1$. This is impossible.

3. Suppose both $\{z_1^1, z_2^1\}$ and $\{z_1^2, z_2^2\}$ are dependent. If either $\langle z_1^1, z_2^1 \rangle$ or $\langle z_1^2, z_2^2 \rangle$ is zero, so is $z_1 \wedge z_2$, hence each is 1-dimensional. Changing bases in $\langle z_1, z_2 \rangle$ we can assume $z_1^1 \neq 0 = z_2^1$. But then $z_2 \neq 0$ forces $z_2^2 \neq 0$, and replacing z_1 by $z_1 - \lambda z_2$ for suitable λ we can insure $z_1 = z_1^1 = ay_1 + by_2$, $z_2 = z_2^2 = cy_3 + dy_4$.

3.1. Suppose $a \neq 0$. Scaling z_1 by a^{-1} and replacing $y_1 + ba^{-1}y_2$ by y_1 we get $z_1 = y_1$.

3.1.1. Suppose $c \neq 0$. Scaling z_2 by c^{-1} and replacing $y_3 + dc^{-1}y_4$ by y_3 we insure $z_2 = y_3$, $S = I_6$.

3.1.2. Suppose $c = 0$. Then $d \neq 0$, so scaling w we can get $w = z_1 \wedge z_2 = y_1 \wedge y_4$, $S = I_5$.

3.2. Suppose $a = 0$. Scaling w by b^{-1} we get $z_1 = y_2$.

3.2.1. Suppose $c \neq 0$. Scaling z_2 by c^{-1} and replacing $y_3 + dc^{-1}y_4$ by y_3 we get $z_2 = y_3$. Apply θ in $\mathrm{GL}(V)$ to S where $\theta(y_1) = x_3$, $\theta(y_2) = x_4$, $\theta(y_3) = x_1$, $\theta(y_4) = x_2$ to get $S = I_5$.

3.2.2. Suppose $c = 0$. Then $w = y_2 \wedge dy_4 = dv$ and S is not 3-dimensional.

Lemma 7.19. *The subspaces $I_1, \ldots, I_6$ lie in distinct orbits.*

Proof. A Zariski-closed subset of an affine variety can be written uniquely as an irredundant finite union of irreducible closed subsets–so-called components. Let $\mathfrak{D}(S)$ be the closed subset of $\bigwedge^2 V$ consisting of all decomposable vectors lying in the subspace S. Then if T lies in the orbit of S, $\mathfrak{D}(T)$ must have the same number of components as $\mathfrak{D}(S)$ and they must match up in isomorphic pairs.

Now $\mathfrak{D}(I_1) = I_1$, $\mathfrak{D}(I_2) = I_2$, $\mathfrak{D}(I_3)$ consists of two 2-dimensional subspaces, $\mathfrak{D}(I_4) = \{xu + yv + zw \mid xy + xz + yz = 0\}$, $\mathfrak{D}(I_5)$ is a 2-dimensional subspace and $\mathfrak{D}(I_6) = \{xu + yv + zw \mid -x^2 + yz = 0\}$. Furthermore, if $V' = \langle x_1, x_2, x_3 \rangle$ and we view $\bigwedge^2 V'$ as a subspace of $\bigwedge^2 V$ then $I_1 \subset \bigwedge^2 V'$, while there is no 3-dimensional subspace V'' of V with $I_2 \subset \bigwedge^2 V''$. Thus no two of $I_1, \ldots, I_6$ lie in the same orbit.

We can now prove

Theorem 7.20. *There are exactly* 14 *7-dimensional metabelian Lie algebras over any algebraically closed field of characteristic not* 2. *All of these algebras have forms over their respective prime fields.*

Proof. The duals of the 7-dimensional algebras are the 4-generator, 3-relation, 5-generator, 2-relation and 6-generator, 1-relation metabelian algebras. The exact number of the algebras in each case together with generators and relations are given by Theorem 7.15, Theorem 7.13 and Theorem 5.2 respectively. The second statement can be deduced from the theorems mentioned and the following proposition.

Proposition 7.21. *Let F be the prime field of an arbitrary field K. Let I be an ideal of $N(2,g)$ in $\mathcal{I}(2,g)$ (i.e. I is a subspace of $\bigwedge^2 V$) which has a basis consisting of F-linear combinations of the basis vectors $x_i \wedge x_j$ $(i < j)$. Then $N = N(2,g)/I$ has an F-form. Also N has an F-form if and only if its dual has an F-form.*

Proof. Let $N(2,g)_F$ denote the F-subalgebra of $N(2,g)$ generated by $x_1, \ldots, x_g$ and let I_F be the F-subspace of I spanned by the mentioned basis of I. Then $N(2,g)_F/I_F$ is an F-form of N.

If L/K is an extension of fields and N is a K-algebra, let N_L denote the usual operation of extending the scalars of N to L. By the results of §3 it can be shown

easily that $(N_L)^0 \cong (N^0)_L$, hence the final statement.

To complete the classification of 8-dimensional metabelian Lie algebras it remains to calculate the 5-generator, 3-relation algebras. The computations are similar to the 4-generator, 3-relation calculations. Unfortunately, it is not presently possible to determine the exact number of algebras due to the large number of possible orbit representatives yielded by my ad hoc techniques. In addition there possibly remain some uncalculated orbits in the case $\dim V = 5$ and S a 3-dimensional subspace of $\bigwedge^2 V$. In [12] I established the following result which I include without proof.

Theorem 7.22. *Suppose $N = N(2,5)/I$ is a 5-generator, 3-relation algebra over an algebraically closed field K of characteristic not 2 (i.e. $\dim I = 3$). There are at most 42 orbits under $\mathrm{GL}(V)$ of subspaces I of $\bigwedge^2 V$ which contain nonzero decomposable vectors. If $V = \langle x_1, \ldots, x_5 \rangle$, the orbit of the subspace $\langle x_1 \wedge x_2 + x_3 \wedge x_4, x_2 \wedge x_3 + x_4 \wedge x_5, x_1 \wedge x_4 + x_3 \wedge x_5 \rangle$ (which has no nonzero decomposables) is Zariski-open (dense) in the Grassmann variety of 3-dimensional subspaces of $\bigwedge^2 V$.*

Remarks. (1) It was shown that if the number of orbits were finite one of them was dense. Were the converse true, we would have a finite upper bound for the number of 8-dimensional metabelian algebras, and so 9 would be the first dimension of an infinite family of algebras.

(2) Due to Proposition 7.21, all the metabelian algebras of dimensions 7 and 8 calculated have forms over their respective prime fields. Recall that at dimension 6 over $\mathbf{Q}$ Scheuneman [24] constructed an infinite family of metabelian Lie algebras. Thus the following interesting conjecture appears possible.

Conjecture. *For any dimension n there are at most finitely many $\mathbf{C}$-metabelian Lie algebras having $\mathbf{Q}$-forms.*

However, the algebras $N_a(g)$ of Theorem 7.10 have $\mathbf{Q}$-forms when $a \in \mathbf{Q}$ by Proposition 7.21. Thus $\{N_a(8) \mid a \in \mathbf{Q} - \{0, 1\}\}$ is an infinite family of 34-dimensional, $\mathbf{C}$-metabelian Lie algebras having $\mathbf{Q}$-forms and the conjecture is false. The duals of these algebras are 10-dimensional.

8. **Derivations.** For the following all modules are finite-dimensional and the algebra acts on the right. Unless it is mentioned otherwise the field K is arbitrary.

Definition. Let L be a Lie algebra and let M be a right L-module. A linear mapping $\theta : L \to M$ satisfying $\theta([k, l]) = (\theta(k))l - (\theta(l))k$ for all k, l in L is called a 1-cocycle.

Proposition 8.1. *Let M be a module for the free Lie algebra $\mathcal{F}$ on g-generators $y_1, \ldots, y_g$. Any mapping $\theta : \{y_i\}_i \to M$ extends uniquely to 1-cocycle.*

Proof. Form the split extension $\mathcal{F} \oplus M$ [15, p. 17]. Consider $\alpha : \mathcal{F} \to \mathcal{F} \oplus M$ where α is the unique homomorphism extending the correspondence $\alpha(y_i)$

$= y_i + \theta(y_i)$. For each y in $\mathcal{G}$ write $\alpha(y) = \phi(y) + \theta(y)$ where $\phi(y) \in \mathcal{G}$ and $\theta(y) \in M$. Equating projections on $\mathcal{G}$ and M of the identity $\alpha([x,y]) = [\alpha(x), \alpha(y)]$, we find that ϕ is the identity homomorphism and θ is a 1-cocycle extending the desired map. The uniqueness follows since the y_i generate $\mathcal{G}$.

Proposition 8.2. *Any mapping $d : \{x_i\}_{i=1}^{g} \to N(l,g)$ extends uniquely to a derivation.*

Proof. Recall that $N(l,g) = \mathcal{G}/\mathcal{G}^{l+1}$ and x_i is the image of y_i under the natural map $\pi : \mathcal{G} \to N(l,g)$. Consider $N(l,g)$ as a right $\mathcal{G}$-module in the natural way and consider the map $\alpha : \{y_i\}_i \to N(l,g)$ given by $\alpha(y_i) = d(x_i)$. Extend α to a 1-cocycle. By induction it follows that $\alpha(\mathcal{G}^n) \subseteq N(l,g)^n$, so $\alpha(\mathcal{G}^{l+1}) = 0$. Hence α factors through $N(l,g)$ by a map $d : N(l,g) \to N(l,g)$ which extends the original map d. If $x, y \in \mathcal{G}$, then $d([\pi(x), \pi(y)]) = d(\pi[x,y]) = \alpha([x,y]) = (\alpha(x))y - (\alpha(y))x = [d(\pi(x)), \pi(y)] - [d(\pi(y)), \pi(x)]$. The surjectivity of π implies d is a derivation and its uniqueness is for the obvious reason.

Proposition 8.3. *Suppose $I \in \mathcal{I}(l,g)$ and $N = N(l,g)/I$. Any derivation d of N lifts to a derivation D of $N(l,g)$ (i.e. if $\rho : N(l,g) \to N$ is the natural map, then $\rho \circ D = d \circ \rho$).*

Proof. Let v_i be in $\rho^{-1}(d(\rho(x_i)))$ and let D be the unique derivation of $N(l,g)$ satisfying $D(x_i) = v_i$.

Jacobson [17] has shown that when $\chi(K) = 0$, any Lie algebra having an injective derivation is nilpotent. He noted the converse was open. Shortly thereafter, Dixmier and Lister [10] constructed a 3-step nilpotent Lie algebra all of whose derivations were nilpotent linear transformations. We will obtain a criterion which insures injective derivations for a large class of nilpotent Lie algebras including all metabelian Lie algebras in characteristic different from 2.

Recall from §1 that $N(l,g)$ inherits a grading from $\mathcal{G}$ and hence we have a notion of homogeneous ideals in $N(l,g)$.

Theorem 8.4. *Suppose $\chi(K) > l$ or $\chi(K) = 0$ and suppose the ideal I in $\mathcal{I}(l,g)$ is homogeneous. Then $N = N(l,g)/I$ has an injective derivation.*

Proof. Extend the correspondence $x_i \to x_i$ to a derivation D of $N(l,g)$. Then D stabilizes $N(l,g)_p$ (see §1) for all p. In fact, it can be shown inductively that the restriction of D to $N(l,g)_p$ is p times the identity. Since I is homogeneous $D(I) \subseteq I$, and since the characteristic is bigger than l, or zero, we have D injective. Let d be the injective linear map induced on N by D. It is easily checked that d is also a derivation.

Corollary 8.5. *If $\chi(K) \neq 2$ any metabelian Lie algebra has an injective derivation.*

Proof. An ideal I is in $\mathcal{I}(2,g)$ only if $I \subset N(2,g)_2$, so I is homogeneous.

Corollary 8.6. *Not every nilpotent Lie algebra is isomorphic to an algebra of the type $N(l,g)/I$ for some homogeneous ideal I in $\mathcal{J}(l,g)$.*

Proof. The example of Dixmier and Lister [10] is conclusive.

Corollary 8.7. *Let N be as in the theorem. Then N is algebraic and has a faithful representation on a* $(\dim N + 1)$*-dimensional space.*

Proof. Let D be an injective derivation of N and form the split extension $S = \langle D \rangle \oplus N$. Then $S^2 = N$ and the adjoint representation of S restricted to N is faithful. It is well known [25, p. 142] that any derived algebra is algebraic.

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