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SOME STABLE RESULTS ON THE COHOMOLOGY OF THE CLASSICAL INFINITE-DIMENSIONAL LIE ALGEBRAS

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ABSTRACT. In this paper we compute the cohomology of various classical infinite-dimensional Lie algebras generalizing results of Gel'fand-Fuks for the Lie algebra of all formal power series vector fields.

1. Let k be a field of characteristic zero and $x_1, x_2, \ldots, y_1, y_2, \ldots$, etc. indeterminants. For us the classical infinite-dimensional Lie algebras will mean simply the Lie algebras occurring on the following list.

$(\mathrm{I})_n$ *Formal power series vector fields.* This is just all expressions of the form $\sum_{i=1}^n f_i(\partial/\partial x_i)$, where $f_i \in k[[x_1, \ldots, x_n]]$. The bracket is the usual Lie bracket.

$(\mathrm{II})_n$ *Divergence zero formal power series vector fields.* The subalgebra of $(\mathrm{I})_n$ consisting of all $\vec{f} = \sum f_i(\partial/\partial x_i)$ for which $\operatorname{div} \vec{f} = \sum (\partial f_i/\partial x_i) = 0$.

$(\mathrm{III})_n$ *Divergence constant formal power series vector fields.* The subalgebra of $(\mathrm{I})_n$ consisting of all $\vec{f}$ for which $\operatorname{div} \vec{f} \in k$.

$(\mathrm{IV})_{2n}$ *The Poisson algebra.* As a vector space this is $k[[x_1, \ldots, x_n, y_1, \ldots, y_n]]$, and the bracket operation is the Poisson bracket:

$$\{f,g\} = \sum \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i} - \frac{\partial g}{\partial x_i}\frac{\partial f}{\partial y_i}.$$

$(\mathrm{V})_{2n}$ *The Hamiltonian algebra.* The Poisson algebra over its one-dimensional center.

$(\mathrm{VI})_{2n}$ *The derivation algebra of the Hamiltonian algebra.* Since there is just one outer derivation, this contains the Hamiltonian algebra as a codimension one ideal.

$(\mathrm{VII})_{2n+1}$ *The contact algebra.* As a vector space this is $k[[x_1, \ldots, x_n, y_1, \ldots, y_n z]]$. The bracket operation is the Lagrange bracket:

$$[f,g] = \{f,g\}_{x,y} + \frac{\partial f}{\partial z}\left(g - \sum y_i \frac{\partial g}{\partial y_i}\right) - \frac{\partial g}{\partial z}\left(f - \sum y_i \frac{\partial f}{\partial y_i}\right).$$

In this paper we will compute the cohomology of each of these algebras (with coefficients in k) for the range $0 \le i \le n$. It turns out that in this range the answer in each case is independent of n and rather simple. In certain cases our

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results can be considerably improved. For example Gel'fand and Fuks computed the entire cohomology ring of $(\mathrm{I})_n$ in [2]; and, with slight modifications, their computation also works for $(\mathrm{III})_n$. (See [7].) A complete picture of the range $0 \leq i \leq 2n$ has been obtained by the second author for all the above examples except $(\mathrm{II})_n$ and $(\mathrm{VII})_{2n+1}$. (See [7] and the remarks at the end of §4.) However, these improved results seem to require much more delicate arguments than ours. (2)

2. Let L be a Lie algebra over k,(3) A an abelian subalgebra, and $S(A)$ the symmetric algebra over A. We note that L and L^* are both modules for $S(A)$. In fact A acts on both L and L^* by the adjoint representation; therefore, so does the universal enveloping algebra of A, which happens to be $S(A)$ since A is abelian.

We begin with the basic theorem:

Theorem 1. *Let L be a Lie algebra over k, A a finite-dimensional abelian subalgebra, and α an element of L. Assume that L^* is free as a module over $S(A)$. Assume also that α acts semisimply on L and that A is contained in a nonzero eigenspace. Then $H^i(L, k) = 0$ for $0 < i \leq \dim A$.*

Proof. Let $m = \dim A$, and consider the Koszul complex:

$$(2.1)\qquad \begin{aligned} &0 \to S(A) \to S(A) \otimes A^* \to \ldots \to S(A) \otimes_k \Lambda^m(A^*) \\ &\to S(A)/S^+(A) \otimes_k \Lambda^m(A^*) \to 0 \end{aligned}$$

where $S^+(A)$ is the maximal ideal of $S(A)$ of elements of degree > 0. Taking the tensor product over $S(A)$ with $\Lambda^i L^*$, which is free over $S(A)$, we get an exact sequence:

$$(2.2)_i\qquad \begin{aligned} &0 \to \Lambda^i(L^*) \otimes A^* \xrightarrow{d} \ldots \\ &\xrightarrow{d} \Lambda^i L^* \otimes \Lambda^m A^* \to \Lambda^i L^*/S^+\Lambda^i L^* \otimes \Lambda^m A^* \to 0. \end{aligned}$$

Let δ be the Hochschild coboundary operator for ΛL^*. We can amalgamate the complexes $(2.2)_i$ into a double complex, whose jth column is

$$0 \to \Lambda^0 L^* \otimes \Lambda^j A^* \xrightarrow{\delta} \ldots \to \Lambda^k L^* \otimes \Lambda^j A^* \to .$$

We assume α is normalized so that it is $(-1)id$ on A. Then the operator d in $(2.2)_i$ maps $(\Lambda^i L^*)_a \otimes \Lambda^j A^*$ into $(\Lambda^i L^*)_{a+1} \otimes \Lambda^{j+1} A^*$, where the subscript indicates the eigenspace of $\Lambda^i L^*$ with eigenvalue a. Therefore, the double complex we

(2) In [b] Rozenfel'd has announced complete results for $(\mathrm{III})_n$ and $(\mathrm{VI})_{2n}$. However it is not clear as he claims that the methods of proof in [2] extend in $(\mathrm{VI})_{2n}$. In [a] Gel'fand et al. show that $(\mathrm{V})_2$ has nontrivial cohomology in dimensions 7 and 10.

(3) If $\dim L = \infty$ we assume L is topologized and $L^{**} = L$. See, for example, [3].

have just described breaks up into a countable number of subcomplexes. The ath such complex is described in Figure 1.

$$
\begin{array}{ccccccccc}
 & & \uparrow & & \uparrow & & & \uparrow & \\
0 & \rightarrow & (\Lambda^{i+1}L^*)_a & \rightarrow & (\Lambda^{i+1}L^*)_{a+1} \otimes A^* & \rightarrow \dots \rightarrow & \left(\dfrac{\Lambda^{i+1}L^*}{S^+\Lambda^{i+1}L^*}\right)_{a+m} \otimes \Lambda^m A^* & \rightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \rightarrow & (\Lambda^{i}L^*)_a & \rightarrow & (\Lambda^{i}L^*)_{a+1} \otimes A^* & \rightarrow \dots \rightarrow & \left(\dfrac{\Lambda^{i}L^*}{S^+\Lambda^{i}L^*}\right)_{a+m} \otimes \Lambda^m A^* & \rightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \rightarrow & (\Lambda^{i-1}L^*)_a & \rightarrow & (\Lambda^{i-1}L^*)_{a+1} \otimes A^* & \rightarrow \dots \rightarrow & \left(\dfrac{\Lambda^{i-1}L^*}{S^+\Lambda^{i-1}L^*}\right)_{a+m} \otimes \Lambda^m A^* & \rightarrow & 0
\end{array}
$$

Figure 1

As we have already observed, the rows of Figure 1 are exact. We claim that the columns are exact except for the extreme right-hand column and the column indexed by $a + j = 0$. In fact, the standard identity: $(ad\alpha)\omega = \delta(\alpha \lrcorner \omega) + \alpha \lrcorner d\omega$ implies that the only eigenspace of ΛL^* with nontrivial cohomology is the zero eigenspace.

Let us consider Figure 1 with $a = 0$. All the rows are exact, and all the columns are exact except for the extreme right-hand column. Therefore, we get an isomorphism between the cohomology of the first column in position i (which is just $H^i(L, k)$) and the cohomology of the last column in position $i - m$. If $i \leq m$ this cohomology is zero. (It is zero in dimension zero since the complex in question is a relative complex.) Thus $H^i(L, k) = 0$ for $i \leq m$, proving our theorem.

As corollaries we get:

Corollary 1. *The cohomology of the algebra* $(\mathrm{I})_n$ *is zero in dimensions* $0 < i \leq n$.

Proof. Just take A to be the subalgebra consisting of the constant vector fields, $\sum c_i(\partial/\partial x_i)$, $c_i \in k$, and take $\alpha = \sum_{i=1}^{n} x_i(\partial/\partial x_i)$.

Corollary 2. *The cohomology of the algebra* $(\mathrm{IV})_{2n}$ *is zero in dimensions* $0 < i \leq n$.

Proof. Take A to be the subalgebra consisting of all linear forms $\sum_{i=1}^{n} c_i x_i$, $c_i \in k$, and take $\alpha = \sum x_i y_i$.

Corollary 3. *The cohomology of the algebra* $(\mathrm{VII})_{2n+1}$ *is zero in dimensions* $0 < i \leq n$.

Proof. Take A and α to be the same as in the preceding example.

3. To apply Theorem 1 to example $(\mathrm{II})_n$ we will show that the dual algebra is free over $S(A)$ where A is the subalgebra spanned by $\partial/\partial x_i$, $i = 1, \ldots, n-1$. It is not hard to prove this directly, but we prefer to deduce it from a result which is applicable to other examples besides those discussed here.

We begin with a standard theorem in commutative algebra.

Theorem. *Let M be a graded module over the polynomial ring $k[x_1, \ldots, x_n]$. Then the following are equivalent.*

(a) *The Koszul cohomology of M with respect to $x_1, \ldots, x_n$ is zero except in dimension zero.*

(b) *M is free as a module over $k[x_1, \ldots, x_n]$.*

(c) *For each j, $0 \leq j \leq n$, x_j is a nonzero divisor of $M/(x_1, \ldots, x_{j-1})M$.*

See [1, Chapter VIII, Theorem 6.1] and [6, Chapter IV, Proposition 3].

Using the equivalence of (a) and (b) one can prove some general results relating the vanishing of the cohomology of a Lie algebra, L, to the vanishing of its Koszul cohomology with respect to an abelian subalgebra A. The Koszul cohomology is in turn closely related to the Spencer cohomology of L. (See, for example, [4].)

To show that the cohomology of example $(\mathrm{II})_n$ vanishes in dimensions $0 < i < n$ we will use the equivalence of (b) and (c).

Condition (c) dualizes to the following condition:

For each j, let $L_j = \{\vec{f}, \operatorname{div} \vec{f} = \partial\vec{f}/\partial x_1 = \cdot = \partial\vec{f}/\partial x_{j-1} = 0\}$; then the map $L_j \to L_j$, $\vec{f} \to \partial\vec{f}/\partial x_j$, is surjective. To check this condition we note first of all that L_j is just all divergence free vector fields in the variables $x_j, \ldots, x_n$; so it is enough to check the condition for $j = 1$. Given $\vec{f} \in L_1$ we can find a $\vec{g}$ such that $(\partial/\partial x_1)\vec{g} = \vec{f}$. Since $\operatorname{div} \vec{f} = 0$, $(\partial/\partial x_1)\operatorname{div} \vec{g} = 0$; so $\operatorname{div} \vec{g}$ is a power series in $x_2, \ldots, x_n$. Let $\vec{g}_1$ be a power series vector field in $x_2, \ldots, x_n$ such that $\operatorname{div} \vec{g}_1 = \operatorname{div} \vec{g}$, then $(\partial/\partial x_1)(\vec{g} - \vec{g}_1) = \vec{f}$ and $\operatorname{div}(\vec{g} - \vec{g}_1) = 0$. We have, therefore, proved

Proposition 1. *If L is the Lie algebra $(\mathrm{II})_n$ then L^* is free over $S(A)$ where A is the subalgebra spanned by $\partial/\partial x_i$, $i = 1, \ldots, n-1$.*

If $\alpha = (\sum_{i=1}^{n-1} x_i(\partial/\partial x_i)) - (n-1)x_n(\partial/\partial x_n)$, the hypotheses of Theorem 1 are satisfied so we have:

Corollary. *The cohomology of the algebra $(\mathrm{II})_n$ is zero in dimensions $0 < i < n$.*

Remark. One can show that the nth cohomology group of $(\mathrm{II})_n$ is one-dimensional with the volume form, $dx_1 \wedge \ldots \wedge dx_n$, as its generator. See [7].

4. We will finally compute the cohomology of the algebras $(\mathrm{III})_n$, $(\mathrm{V})_{2n}$ and $(\mathrm{VI})_{2n}$. We begin by describing a spectral sequence due to Hochschild and Serre: Let L be a Lie algebra over k and M an ideal in L. The Hochschild-Serre spectral sequence has as its E_∞ term $H(L, k)$ and has $E_2^{j,i}$ equal to $H^j(L/M, H^i(M))$. (See for example [1, Chapter XVI, §6]).

If L is the Poisson algebra and M is its one-dimensional center, then $H^i(M) = k$ when $i = 0, 1$ and zero otherwise; so $E_2^{j,0} = E_2^{j,1} = H^j(L/M, k)$, and the other terms are zero. Since $H^j(L, k) = 0$ in dimensions $0 < j \leq n$ the only way these E_2 terms can cancel out is for $d_2 : E_2^{j,1} \to E_2^{j+2,0}$ to be bijective for $j + 2 \leq n$. So we have proved

Proposition 2. *The cohomology of the Hamiltonian algebra* $(\mathrm{V})_{2n}$ *is equal to* k *in all even dimensions in the range* $0 \leq i \leq n$ *and equal to zero in all odd dimensions in this range.*

Remark. The Poisson algebra is a nontrivial central extension of the Hamiltonian algebra, so it defines an element in H^2 of the Hamiltonian algebra. (See, for example, [5].) It is not hard to see that the ith power of this 2-dimensional element is a basis for the cohomology in dimension $2i$ when $2i \leq n$.

Next we apply the Hochschild-Serre spectral sequence to the Hamiltonian algebra and to its derivation algebra. Let M be the Hamiltonian algebra and L its derivation algebra. It is easy to see that the two-dimensional generator of $H(M)$ is not invariant with respect to L/M, so $H^i(L/M, H^j(M))$ is zero except when $j = 0$; and $H^i(L/M, H^0)$ is equal to k for $i = 0, 1$ and zero otherwise, since L/M is one-dimensional. We conclude with

Proposition 3. *The cohomology of the algebra* $(\mathrm{VI})_{2n}$ *is equal to* k *in dimensions zero and one and is zero in the range* $1 < i \leq n$.

A simpler computation of the same kind shows

Proposition 4. *The cohomology of the algebra* $(\mathrm{III})_n$ *is equal to* k *in dimensions zero and one, and is zero in the range* $1 < i \leq n$.

Remark. The generator of the first cohomology group of $(\mathrm{III})_n$ is the one cocycle $\vec{f} \to \operatorname{div} \vec{f}$. The restriction of this is the generator of the first cohomology group of $(\mathrm{VI})_{2n}$.

To conclude, we note that Gel'fand and Fuks have proved the cohomology of $(\mathrm{I})_n$ vanishes in the range $0 < i \leq 2n$. See [2]. The second author, in his thesis, has proved that the cohomology of $(\mathrm{IV})_{2n}$ also vanishes in this range. (The proofs require rather complicated techniques from classical invariant theory.)

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