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GROUPS OF DIFFEOMORPHISMS AND THEIR SUBGROUPS

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ABSTRACT. This paper has two purposes. The first is to prove the existence of a normal coordinate with respect to a connection defined on the group of diffeomorphisms of a closed manifold, relating to an elliptic complex. The second is to prove a Frobenius theorem with respect to a right invariant distribution defined on the group of diffeomorphisms of a closed manifold, relating to an elliptic complex. Consequently, the group of all volume preserving diffeomorphisms and the group of all symplectic diffeomorphisms are Fréchet Lie groups.

Introduction. Although it is well known that the group $\mathfrak{D}$ of the smooth diffeomorphisms of a compact manifold M is a so-called Fréchet Lie group [2], [4], [7], the category of Fréchet Lie groups seems to be still too huge to treat and get some useful results. Compared with Hilbert Lie groups, Fréchet Lie groups have not so nice a property. This is mainly because the implicit function theorem or Frobenius theorem does not necessarily hold in the category of Fréchet manifolds.

On the other hand, if an infinite dimensional Hilbert Lie group acts *effectively and transitively* on a finite dimensional manifold, then the manifold seems to be much restricted and so does the action itself. The reason is the following:

Suppose we have a Hilbert Lie group G acting effectively and transitively on a one dimensional manifold M. Let $\mathfrak{g}$ be the Lie algebra of G. For any $v \in \mathfrak{g}$, this defines a smooth vector field on M. So $\mathfrak{g}$ can be regarded as a subalgebra of the smooth vector fields on M. Fix a point $x_0 \in M$. Let $\mathfrak{g}_n$, $n = 0, 1, 2, \ldots$, denote the subalgebra which consists of all $v \in \mathfrak{g}$ such that the derivatives of v at x_0 vanish up to order n. Then, by the effectivity of G, we have $\bigcap \mathfrak{g}_n = \{0\}$. Therefore, $\mathfrak{g}$ can be regarded as a subalgebra of the formal power series $\mathfrak{F}$ at x_0. Let $i : \mathfrak{g} \to \mathfrak{F}$ be the inclusion. Of course, $\mathfrak{F}$ itself is a Lie algebra and each element f of $\mathfrak{F}$ has the expression $f = \sum a_n x^n\, \partial/\partial x$. Suppose $i(\mathfrak{g})$ contains $x\, \partial/\partial x$, $x^2\, \partial/\partial x$ and $x^3\, \partial/\partial x$. Then, $i(\mathfrak{g})$ must contain all of $x^n\, \partial/\partial x$ because

$$\left[x^2\frac{\partial}{\partial x}, x^3\frac{\partial}{\partial x}\right] = x^4\frac{\partial}{\partial x} \quad \text{and} \quad \left[x^2\frac{\partial}{\partial x}, x^n\frac{\partial}{\partial x}\right] = (n-2)x^{n+1}\frac{\partial}{\partial x}.$$

Let $e_n = i^{-1}(x^n\, \partial/\partial x)$. Then, $[e_1, e_n] = (n-1)e_n$. This is impossible, because the bracket operation $[\ ,\ ]\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is bounded. So if there exists such an example,

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this must be very special from the point of view of G-structures.

In this reason, it seems better to consider the category which lies between Fréchet Lie groups and Hilbert Lie groups. As a matter of fact, these groups $\mathfrak{D}$ have a little nicer property, which is an analogue with the Sobolev chain in function spaces. Namely, let $\Gamma(1_M)$ be the space of the smooth functions of a compact smooth manifold M. Then $\Gamma(1_M)$ is a Fréchet space and leads a Sobolev chain $\{\Gamma^s(1_M)\}$, $s \geq 0$, where $\Gamma^s(1_M)$ is a Hilbert space obtained by the completion of $\Gamma(1_M)$ in the L_2^s-norm, that is, the norm involving the integral of squares of all derivatives up to order s. $\Gamma(1_M)$ is the inverse limit of the system $\{\Gamma^s(1_M)\}$. Similarly, by assuming M is closed (compact without boundary), the connected component $\mathfrak{D}_0$ of $\mathfrak{D}$ in C^∞-topology has the following properties:

(1) $\mathfrak{D}_0$ is a Fréchet Lie group.

(2) There exists a system $\{\mathfrak{D}_0^s\}$, $s \geq \dim M + 5$, of smooth Hilbert manifolds each of which is a topological group and $\mathfrak{D}_0$ is the inverse limit of $\{\mathfrak{D}_0^s\}$ (cf. [2], [7], [9]).

However, the group operations are not so simple. For instance the multiplication $(g, h) \to gh$ in $\mathfrak{D}_0$ can only be extended to the C^ℓ-map of $\mathfrak{D}_0^{s+\ell} \times \mathfrak{D}_0^s$ into $\mathfrak{D}_0^s$ for any $\ell \geq 0$ and the inversion $g \to g^{-1}$ in $\mathfrak{D}_0$ can only be extended to the C^ℓ-map of $\mathfrak{D}_0^{s+\ell}$ into $\mathfrak{D}_0^s$ for any $\ell \geq 0$. These are never $C^{\ell+1}$-maps. On the other hand, the right translation $R_g : \mathfrak{D}_0^s \to \mathfrak{D}_0^s$ is smooth for any $g \in \mathfrak{D}_0^s$.

Now, similar to the Sobolev embedding theorem in function spaces, $\mathfrak{D}_0^s$ is contained in the group of C^{s-k_0} diffeomorphisms with C^{s-k_0} uniform topology and the inclusion is continuous, where $k_0 = [\frac{1}{2} \dim M] + 1$. Therefore the following subgroups are well defined:

(a) Suppose M is oriented. Let μ be a smooth volume element of M.

$$\mathfrak{D}_\mu^s = \{\varphi \in \mathfrak{D}_0^s ; \varphi^*\mu = \mu\}, \qquad s \geq \dim M + 5.$$

(b) Assume M is even dimensional and has a symplectic form w.

$$\mathfrak{D}_w^s = \{\varphi \in \mathfrak{D}_0^s ; \varphi^* w = w\}.$$

According to Ebin and Marsden [2], these two groups satisfy the following:

(A) *$\mathfrak{D}_\mu^s$ and $\mathfrak{D}_w^s$ are topological groups and smooth Hilbert manifolds for any* $s \geq \dim M + 5$, *and the group operations enjoy the same smoothness properties as those of* $\mathfrak{D}_0^s$ (*cf.* [2, *Theorem* 4.2]).

Remark. In their paper, the condition for s is $s > \frac{1}{2} \dim M + 1$. In this paper, *$s$ is always provided* $\geq \dim M + 5$, because the author wants to use the inequalitites established in [9]. Similar results are given in [10] by using a more general method.

Moreover, they defined a C^∞ right-invariant connection on $\mathfrak{D}_\mu^s$ and obtained the following results (B) and (C) by investigating the exponential mapping Exp with respect to this connection:

(B) *Let $x(t)$ be a geodesic in $\mathfrak{D}_\mu^s$ starting at the identity e. Then, putting $u_t = dR_{x(t)}^{-1}\dot{x}(t)$, u_t satisfies the equation*

$$(1) \qquad (\partial/\partial t)u_t + \nabla_{u_t} u_t = \operatorname{grad} p_t, \quad \operatorname{div} u_t = 0,$$

where p_t is an arbitrary function and ∇ is a connection on M defined by a smooth riemannian metric on M. Conversely, if u_t is a solution of (1), *then the equation*

$$(2) \qquad (d/dt)x(t) = dR_{x(t)}u(t)$$

has a unique solution such that $x(0) = e$ (the identity) and then $x(t)$ is a geodesic of $\mathfrak{D}_\mu^s$ starting at e.

(C) *For each s, there exists a neighborhood W^s of* 0 *in the tangent space of $\mathfrak{D}_\mu^s$ at e such that* Exp *is a smooth diffeomorphism of W^s onto an open neighborhood of e.*

Of course this property (C) is easy to prove after establishing the smoothness of the connection. Together with this and (B), we can easily obtain the covering theorem ([2, p. 151]).

However, there is no relation between W^{s+1} and W^s, as far as concerning with W^s in (C). So, $\bigcap W^s$ might be a single point. Therefore, Exp might not give a coordinate for $\mathfrak{D}_\mu = \bigcap \mathfrak{D}_\mu^s$. The same sort of difficulty occurs in using the implicit function theorem. Therefore, in [2] it remains open whether $\mathfrak{D}_\mu$ is a Fréchet Lie group or not. The first purpose of this paper is to strengthen such weak points and to make $\mathfrak{D}_\mu$ a Fréchet Lie group. Namely, the following theorem will be proved in this paper:

Theorem A. *There exists a connection $\hat{\nabla}$ on $\mathfrak{D}_0$ satisfying the following*:

(a) *$\hat{\nabla}$ can be extended to the C^∞ right-invariant connection on $\mathfrak{D}_0^s$ for any* $s \geq \dim M + 7$.

(b) *$\mathfrak{D}_\mu^s$ is a totally geodesic submanifold of $\mathfrak{D}_0^s$.*

(c) *This extended connection coincides to that of Ebin and Marsden* [2], *if we restrict this connection to $\mathfrak{D}_\mu^s$.*

(d) *Let* Exp *be the exponential mapping at e and $\Gamma^s(T_M)$ the tangent space of $\mathfrak{D}_0^s$ at e. Then there exists an open neighborhood V^{k_1} of* 0 *in $\Gamma^{k_1}(T_M)$*, $k_1 = \dim M + 7$, *such that* Exp *is a smooth diffeomorphism of $V^{k_1} \cap \Gamma^s(T_M)$ onto an open neighborhood* $(\operatorname{Exp} V^{k_1}) \cap \mathfrak{D}_0^s$ *of e for any $s \geq k_1$.*

The property (d) implies the *regularity of geodesics*, that is, if a geodesic $x(t)$ in $\mathfrak{D}_0^{k_1}$ satisfies $d/dt|_{t=0}x(t) \in \Gamma^s(T_M)$, then $x(t) \in \mathfrak{D}_0^s$ as far as $x(t)$ is defined as a curve in $\mathfrak{D}_0^{k_1}$. Moreover, this property (d) shows that if a geodesic $x(t)$ in $\mathfrak{D}_0^{k_1}$ satisfies $x(t_0)$, $x(t_0 + \delta) \in \mathfrak{D}_0^s$ for sufficiently small δ and for some t_0, then $x(t)$ is contained in $\mathfrak{D}_0^s$. Since $\mathfrak{D}_\mu^s$ is totally geodesic, (d) also shows that $\mathfrak{D}_\mu$ is a Fréchet manifold and hence a Fréchet Lie group. Therefore, $\mathfrak{D}_\mu$ is a *strong* ILH-Lie group in the sense of [17].

The proof of this theorem will be given in a fairly general manner and as a result, the group $\mathfrak{D}_w$ is completely involved in that argument. So $\mathfrak{D}_w$ is also a strong ILH-Lie group. Specifically, $\mathfrak{D}_w$ is a Fréchet Lie group.

Now, to treat the groups $\mathfrak{D}_\mu$ and $\mathfrak{D}_w$ at the same time, we consider the following things: Suppose M is oriented. Let E and F be smooth finite dimensional vector bundles over M and T_M the tangent bundle of M. Let $\Gamma(E)$, $\Gamma(T_M)$ and $\Gamma(F)$ denote the space of smooth sections of E, T_M and F respectively. Assume M has a smooth riemannian metric and E, F have smooth riemannian inner products. As usual, we define an inner product $\langle\, ,\rangle_s$ on $\Gamma(E)$, $\Gamma(T_M)$ or $\Gamma(F)$ by

$$(3) \qquad \langle u, v\rangle_s = \sum_{t=0}^{s} \int_M \langle \nabla^t u, \nabla^t v\rangle\, \mu,$$

where ∇ is the riemannian connection on E, T_M or F, μ a volume element and $\langle\, ,\rangle$ is the inner product of $T_M^{*t} \otimes E$, $T_M^{*t} \otimes T_M$ or $T_M^{*t} \otimes F$. In fact $\nabla^t u$ can be regarded as a section of one of the above vector bundles. Denote by $\Gamma^s(E)$ (resp. $\Gamma^s(T_M)$, $\Gamma^s(F)$) the completion of $\Gamma(E)$ (resp. $\Gamma(T_M)$, $\Gamma(F)$) under the norm $\|u\|_s$ defined by $\|u\|_s^2 = \langle u, u\rangle_s$.

Suppose $A : \Gamma(E) \to \Gamma(T_M)$ and $B : \Gamma(T_M) \to \Gamma(F)$ are differential operators of order r such that (i) $BA = 0$, (ii) A and B have smooth coefficients and (iii) $AA^* + B^*B : \Gamma(T_M) \to \Gamma(T_M)$ is an elliptic differential operator, where A^* and B^* are formal adjoint operators of A and B respectively. Since M is closed and oriented, we have the following splitting by the same method as the de Rham-Kodaira decomposition theorem:

($\bigstar$) $\Gamma(T_M) = \mathbf{F}_1 \oplus \mathbf{F}_2 \oplus H$, $\Gamma^s(T_M) = F_1^s \oplus F_2^s \oplus H$, $s \geq 0$, *where* $\mathbf{F}_1 = A\Gamma(E)$, $F_1^s = A\Gamma^{s+r}(E)$, $\mathbf{F}_2 = B^*\Gamma(F)$, $F_2^s = B^*\Gamma^{s+r}(F)$ *and H is the kernel of the elliptic operator* $AA^* + B^*B$.

Therefore H is a finite dimensional subspace of $\Gamma(T_M)$ and actually $H = \operatorname{Ker} A^* \cap \operatorname{Ker} B$. Moreover, F_i^s is a closed subspace of $\Gamma^s(T_M)$. (This splitting theorem is not difficult to prove. See Lemma 6 of [10] for a precise proof.)

Define a projection π of $\Gamma(T_M)$ onto $\mathbf{F}_2$ in accordance with this splitting. Then, this π can be extended to the projection of $\Gamma^s(T_M)$ onto F_2^s for any $s \geq 0$. Let ∇ be the riemannian connection defined on M. Consider a bilinear form $\hat{\alpha} : \Gamma(T_M) \times \Gamma(T_M) \to \Gamma(T_M)$ defined by

$$(4) \qquad \hat{\alpha}(u, v) = \pi\nabla_u \pi v + (1 - \pi)\nabla_u(1 - \pi)v.$$

This bilinear form defines a right-invariant connection $\hat{\nabla}$ on $\mathfrak{D}_0$. (See also §3 for a precise definition, because $\hat{\nabla}$ is given, in fact, by a somewhat different manner.) Since $\mathfrak{D}_0$ is a Fréchet Lie group, $\hat{\nabla}$ is a smooth connection on $\mathfrak{D}_0$. Furthermore, $\hat{\nabla}$ has the following property:

Theorem B. (i) $\hat{\nabla}$ *can be extended to the smooth right invariant connection of* $\mathfrak{D}_0^s$ *for any* $s \geq \dim M + 2r + 5$. (ii) *Let* Exp *be the exponential mapping at e defined* *by this connection* $\hat{\nabla}$. *Then, there exists an open neighborhood* V^{k_1} *of* 0 *in* $\Gamma^{k_1}(T_M)$, $k_1 = \dim M + 2r + 5$, *such that* Exp *is a smooth diffeomorphism of* $V^{k_1} \cap \Gamma^s(T_M)$ *onto an open neighborhood* $(\operatorname{Exp} V^{k_1}) \cap \mathfrak{D}_0^s$ *for any* $s \geq k_1$.

Consider subbundles $\tilde{F}_2^s$ and $(F_1^s \oplus H)^{\sim}$ defined by $\tilde{F}_2^s = \{dR_g F_2^s; g \in \mathfrak{D}_0^s\}$, $(F_1^s \oplus H)^{\sim} = \{dR_g(F_1^s \oplus H); g \in \mathfrak{D}_0^s\}$. In general, such bundles are not continuous subbundles of the tangent bundle of $\mathfrak{D}_0^s$. However, in this case, $\tilde{F}_2^s$ and $(F_1^s \oplus H)^{\sim}$ are smooth right-invariant subbundles. (Cf. Lemma 3 and Corollary 3 of [10]. See also §3 in this paper for a precise statement.) Moreover, we have the following

Theorem C. $\tilde{F}_2^s$ *and* $(F_1^s \oplus H)^{\sim}$ *are smooth right-invariant subbundles which are invariant by the parallel displacement defined by* $\hat{\nabla}$.

Assume that $\mathbf{F}_1 \oplus H$ is a subalgebra of $\Gamma(T_M)$, that is, $\operatorname{Ker}\{B : \Gamma(T_M) \to \Gamma(F)\}$ is a subalgebra. Then, it is not hard to verify that $(F_1^s \oplus H)^{\sim}$ is *a smooth involutive distribution on* $\mathfrak{D}_0^s$. (Cf. Proposition A in [10] for a proof. See also §4 in this paper for a simple proof.) Therefore, by the Frobenius theorem in Hilbert manifolds, there exists a unique maximal integral submanifold G^s through the identity. Each G^s is a smooth Hilbert manifold and a topological group and satisfies the same smoothness property as those of $\mathfrak{D}_0^s$. (See §3 in this paper for the precise statements.)

Theorem D. *Suppose* $\mathbf{F}_1 \oplus H$ *is a subalgebra of* $\Gamma(T_M)$. *Then there exists a neighborhood* W^{k_1} *of* 0 *in* $F_1^{k_1} \oplus H$ *such that* $W^{k_1} \subset V^{k_1} \cap (F_1^{k_1} \oplus H)$ *and* Exp *is a* C^∞ *diffeomorphism of* $W^{k_1} \cap (F_1^s \oplus H)$ *onto an open neighborhood* $(\operatorname{Exp} W^{k_1}) \cap G^s$ *of* e *for any* $s \geq k_1$, *where* V^{k_1} *is the same neighborhood as in Theorem* B *and* $k_1 = \dim M + 2r + 5$. *Consequently,* $G = \bigcap G^s$ *is a Fréchet manifold and hence a Fréchet Lie group.*

Theorem B(i), Theorems C and D will be proved in §4, providing Theorem B(ii). An essential part among Theorems A–D is of course Theorem B(ii). This will be proved in §6.

To apply these Theorems B–D to our groups $\mathfrak{D}_\mu$ and $\mathfrak{D}_w$ we have to take $\bigwedge^2 T_M^*$ (resp. 1_M) as E and 1_M (resp. $\bigwedge^2 T_M^*$) as F, where $\bigwedge^2 T_M^*$ is the exterior product of the cotangent bundle T_M^* and 1_M is the trivial bundle $M \times R$. Differential operators A, B are the following:

In case of $\mathfrak{D}_\mu$: $Bu = \operatorname{div} u = \delta i u$, where i is the natural identification of T_M with T_M^* through the riemannian metric on M and δ is the formal adjoint operator of the exterior derivative d. $Av = i^{-1}\delta v$.

In case of $\mathfrak{D}_w$: $Bu = d(w \lrcorner u)$ and $Av = j^{-1}dv$, where $\lrcorner$ is the usual inner product and $j : T_M \to T_M^*$ is the mapping defined by $ju = w \lrcorner u$. The nondegeneracy of w shows that j is an isomorphism.

Theorem D implies that the exponential mapping defined by $\hat{\nabla}$ gives a fairly good coordinate at the identity. However, this coordinate is not convenient to consider the factor set $G\backslash\mathfrak{D}_0$. Theorems A–D do not imply $G\backslash\mathfrak{D}_0$ is a Fréchet manifold, even if G is a closed subgroup. To consider factor sets, we have to establish the Frobenius theorem in the group $\mathfrak{D}_0$. The second purpose of this paper is to prove the Frobenius theorem under a suitable condition.

Suppose $A : \Gamma(T_M) \to \Gamma(E)$ and $B : \Gamma(E) \to \Gamma(F)$ are differential operators of order r with smooth coefficients such that (i) $BA = 0$, (ii) $AA^* + B^*B : \Gamma(E) \to \Gamma(E)$ is an elliptic operator, where A^*, B^* are formal adjoint operators of A and B respectively. (Compare these conditions with that of (★).) Let $\mathfrak{g} = \mathrm{Ker}\{A; \Gamma(T_M) \to \Gamma(E)\}$, $\mathfrak{g}^s = \mathrm{Ker}\{A; \Gamma^s(T_M) \to \Gamma^{s-r}(E)\}$, $\mathfrak{m} = A^*\Gamma(E)$ and $\mathfrak{m}^s = A^*\Gamma^{s+r}(E)$. Then, by a similar argument as in (★), we have

(★★) $\mathfrak{m}$ and $\mathfrak{m}^s$ are closed in $\Gamma(T_M)$ and $\Gamma^s(T_M)$ respectively, and $\Gamma(T_M) = \mathfrak{g} \oplus \mathfrak{m}$, $\Gamma^s(T_M) = \mathfrak{g}^s \oplus \mathfrak{m}^s$.

Consider a subbundle $\tilde{\mathfrak{g}}^s$ defined by $\tilde{\mathfrak{g}}^s = \{dR_g \mathfrak{g}^s; g \in \mathfrak{D}_0^s\}$. This becomes a right-invariant *smooth* distribution of $\mathfrak{D}_0^s$. (Cf. Lemma 3 and Corollary 3 of [10]. See also §3 in this paper for precise statements.) Suppose $\mathfrak{g}$ is a subalgebra of $\Gamma(T_M)$. Then $\tilde{\mathfrak{g}}^s$ is a smooth *involutive* distribution on $\mathfrak{D}_0^s$ for $s \geq \dim M + 3r + 5$. (Cf. Proposition A in [10]. See also §4 in this paper.) Therefore, by the Frobenius theorem in Hilbert manifolds, there exists a unique maximal integral submanifold G^s through the identity. Each G^s is a smooth Hilbert manifold and a topological group. Group operations satisfy the same smoothness property as those of $\mathfrak{D}_0^s$. (See §3 in this paper for the precise statements.)

Theorem E (Frobenius theorem). *There exists an open neighborhood U^{k_1} of* 0 *in $\Gamma^{k_1}(T_M)$, $k_1 = \dim M + 3r + 5$, and a mapping $\Psi : U^{k_1} \to \mathfrak{D}_0^{k_1}$ such that* (i) *$\Psi(U^{k_1} \cap (\mathfrak{g}^s + u))$ is an integral submanifold of $\tilde{\mathfrak{g}}^s$ through $\Psi(u)$ for any $u \in U^{k_1} \cap \mathfrak{m}^s$, and* (ii) *$\Psi$ is a smooth diffeomorphism of $U^{k_1} \cap \Gamma^s(T_M)$ onto an open neighborhood $\Psi(U^{k_1}) \cap \mathfrak{D}_0^s$ of the identity for any $s \geq k_1$.*

The mapping Ψ is given in the same manner as in the Frobenius theorem in Hilbert manifolds. Therefore the essential part of this theorem is the second part, that is, the property (ii). The difficulty in proving this is very similar to that of Theorem B(ii). The basic ideas how to manage these difficulties will be given in the next section.

Theorem F. *Suppose G^k $(k \geq k_1)$ is closed in $\mathfrak{D}_0^k$. Then, there exists an open neighborhood V^k of* 0 *in $\Gamma^k(T_M)$ such that the mapping $\tilde{\Psi} : G^s \times V^k \cap \mathfrak{m}^s \to \mathfrak{D}_0^s$ defined by $\tilde{\Psi}(g,u) = g\Psi(u)$ is a homeomorphism of $G^s \times V^k \cap \mathfrak{m}^s$ onto a right-invariant neighborhood $\tilde{\Psi}(G^k \times V^k \cap \mathfrak{m}^k) \cap \mathfrak{D}_0^s$ of G^s for any $s \geq k$.*

This theorem shows that there exists a nice local cross section for the family $\{G_g; g \in \mathfrak{D}_0\}$ of integral submanifolds and consequently the factor set $G\backslash\mathfrak{D}_0$ is a Fréchet manifold. To apply these Theorems E–F to our groups $\mathfrak{D}_\mu$ and $\mathfrak{D}_w$, we have to take 1_M (resp. $\wedge^2 T_M^*$) as E and 0 (resp. $\wedge^3 T_M^*$) as F respectively. Differential operators A, B are given as follows:

In case of $\mathcal{D}_\mu$: $Au = \operatorname{div} u$, $B = 0$.

In case of $\mathcal{D}_w$: $Au = d(w \lrcorner u)$, $Bv = dv$.

Let $\mathfrak{g}_\mu = \{u \in \Gamma(T_M); \operatorname{div} u = 0\}$, $\mathfrak{g}_w = \{u \in \Gamma(T_M); d(w \lrcorner u) = 0\}$. Then, by Theorem D or E, there exist the maximal integral submanifolds $\hat{\mathcal{D}}^s_\mu$ and $\hat{\mathcal{D}}^s_w$ of $\tilde{\mathfrak{g}}^s_\mu$ and $\tilde{\mathfrak{g}}^s_w$ through e respectively for $s \geq \dim M + 8$. However, Theorems B–F do not imply that $\hat{\mathcal{D}}^s_\mu = \mathcal{D}^s_\mu$ or $\hat{\mathcal{D}}^s_w = \mathcal{D}^s_w$. As a matter of course, Theorem D or E implies that $\hat{\mathcal{D}}^s_\mu \subset \mathcal{D}^s_\mu$ and $\hat{\mathcal{D}}^s_w \subset \mathcal{D}^s_w$. Moreover, if $c(t)$ is a smooth curve in $\mathcal{D}^s_0$ such that $c(0) = e$ and $c(t) \in \mathcal{D}^s_\mu$ or $\mathcal{D}^s_w$ for any t, then $c(t) \subset \hat{\mathcal{D}}^s_\mu$ or $\hat{\mathcal{D}}^s_w$ respectively. This is because $\hat{\mathcal{D}}^s_\mu$, $\hat{\mathcal{D}}^s_w$ are integral submanifolds. On the other hand, we have known the following by using the implicit function theorem [2]:

(a) $\mathcal{D}^s_\mu$ *is a smooth Hilbert manifold and connected.* (The connectedness of $\mathcal{D}^s_\mu$ is given by Theorem 2.18 in [7].)

(b) $\mathcal{D}^s_w$ *is a smooth Hilbert manifold.*

Therefore, we have $\hat{\mathcal{D}}^s_\mu = \mathcal{D}^s_\mu$ and $\hat{\mathcal{D}}^s_w =$ the connected component of $\mathcal{D}^s_w$ containing the identity. Thus, $\bigcap \hat{\mathcal{D}}^s_\mu = \mathcal{D}_\mu$ and $\bigcap \hat{\mathcal{D}}^s_w =$ the connected component of $\mathcal{D}_w$ containing the identity. Since $\bigcap \hat{\mathcal{D}}^s_\mu$ and $\bigcap \hat{\mathcal{D}}^s_w$ are Fréchet Lie groups, so also $\mathcal{D}_\mu$ and $\mathcal{D}_w$ are.

A nontrivial problem occurs in case of strictly contact transformation groups. In this case, we can also prove that there exists the maximal integral submanifold of the distribution defined by the infinitesimal strictly contact transformations by assuming some natural conditions. However, in this case the author could not prove the implicit function theorem. So he does not know whether this integral submanifold coincides with the connected component of the contact transformations or not. This sort of problem also occurs in case that w is not a closed form (cf. [10, 2°]). These problems will be treated in the future.

1. **Basic ideas.**

(a) *Theorem* B(ii). Let $\{E^k\}$, $k \in N$ (the nonnegative integers), be a system of separable Hilbert spaces such that (i) $E^{k+1} \subset E^k$ and the inclusion is bounded, (ii) putting $\mathbf{E} = \bigcap E^k$, $\mathbf{E}$ is dense in every E^k. $\mathbf{E}$ is a Fréchet space by defining the inverse limit topology on it. Roughly speaking many local problems of $\mathcal{D}_0$ can be translated to that on an open subset $\mathbf{U}$ of $\mathbf{E}$ such that $\mathbf{U}$ is the intersection of $\mathbf{E}$ and an open subset U of E^0 by considering a coordinate expression. Of course, we have to put $E^k = \Gamma^{k+k_1}(T_M)$, etc. in this case, where $k_1 = \dim M + 2r + 5$ or sometimes $k_1 = \dim M + 3r + 5$.

Assume for simplicity that U is an open neighborhood of $0 \in E^0$. A connection $\hat{\nabla}$ on $U \cap \mathbf{E}$ in the sense of Fréchet manifolds is called an ILH-*connection*, if $\hat{\nabla}$ can be extended to the smooth connection on a Hilbert manifold $U \cap E^k$ for each $k \in N$. Since $U \cap \mathbf{E}$ is an open subset of linear space, $\hat{\nabla}$ defines an $\mathbf{E}$-valued bilinear form Γ by the following:

$$\Gamma_x(u, v) = \hat{\nabla}_u v \quad \text{at } x, \tag{5}$$

where u, v on the right-hand side are identified with vector fields obtained by the parallel translation using the linear structure of E. This bilinear form $\Gamma_x(u, v)$ corresponds to Γ^i_{jk} in the finite dimensional case. Since $\hat{\nabla}$ is an ILH-connection, the bilinear form Γ can be extended to the smooth E^k-valued bilinear form of $U \cap E^k$. As a matter of course, the equation of geodesics is given by

$$x''(t) + \Gamma_{x(t)}(x'(t), x'(t)) = 0.$$

Therefore, if $x(t)$ is geodesic in $U \cap E^k$, then it is so in $U \cap E^{k'}$ for any $k' \leq k$.

Now, assume the following inequalities (I) and (II): Let $\| \, \|_k$ be the norm of E^k, that is, $\|u\|_k^2 = \langle u, u \rangle_k$ using the inner product of E^k.

$$\text{(I)} \qquad \begin{aligned} \|\Gamma_x(u, v)\|_k \leq{} & C\{\|u\|_k \|u\|_0 + \|u\|_0 \|v\|_k\} + C^2 \|x\|_k \|u\|_0 \|v\|_0 \\ & + \gamma_k(\|x\|_{k-1}) \|u\|_{k-1} \|v\|_{k-1}, \qquad k \geq 1, \end{aligned}$$

where C is a constant which does not depend on k and γ_k is a positive coefficients polynomial.

Let $d\Gamma_x(w)(u, v)$ be the derivative of Γ at x evaluated by w, namely,

$$\lim \frac{1}{\|w\|_k} \|\Gamma_{x+w}(u, v) - \Gamma_x(u, v) - d\Gamma_x(w)(u, v)\|_k = 0$$

for each k.

$$\text{(II)} \qquad \begin{aligned} \|d\Gamma_x(w)(u, v)\|_k \leq{} & C^2\{\|u\|_k \|v\|_0 \|w\|_0 + \|u\|_0 \|v\|_k \|w\|_0 + \|u\|_0 \|v\|_0 \|w\|_k\} \\ & + \gamma'_k(\|x\|_k) \|u\|_{k-1} \|v\|_{k-1} \|w\|_{k-1}, \qquad k \geq 1, \end{aligned}$$

where C is the same constant as in (I) and γ'_k is a positive continuous function of $\|x\|_k$. γ_k, γ'_k may depend on k. The constants C^2 used in (I) or (II) are only for convenience of notations. They can be replaced by some other constants as far as they do not depend on k.

Now, let Exp be the exponential mapping at the origin 0. Since $\hat{\nabla}$ is an ILH-connection, for each k, there is an open neighborhood W^k such that Exp is a smooth diffeomorphism of W^k onto an open neighborhood Exp W^k of 0 in E^k. In general, we cannot get any relation between W^k and W^{k+1} from the assumption that $\hat{\nabla}$ is ILH. However, by virtue of inequalities (I) and (II), we have the following theorem. This was the main theorem of [8]:

Theorem (regularity of connections). *Suppose an ILH-connection* $\hat{\nabla}$ *on* $U \cap$ E *satisfies the inequalities* (I) *and* (II). *Let* V *be an open neighborhood of* 0 *in* E^0 *on which the exponential mapping* Exp *can be defined. Then, this mapping can be also defined on* $V \cap E^k$ *as the exponential mapping with respect to the extended connection* $\hat{\nabla}$ *on* $U \cap E^k$. *Moreoever, there exists an open neighborhood* $\acute{V}$ *of* 0 *in* E^0 *such that* (a) $\mathrm{Exp}(\acute{V} \cap E^k) = (\mathrm{Exp}\, \acute{V}) \cap E^k$, (b) *putting* $\tilde{V}^k = (\mathrm{Exp}\, \acute{V}) \cap E^k$, *the exponential mapping is a smooth diffeomorphism of* $\acute{V} \cap E^k$ *onto* $\tilde{V}^k$.

Therefore, to prove Theorem B(ii), it is enough to prove the inequalities (I) and (II) in a coordinate expression of the connection $\hat{\nabla}$ defined by (4). This will be proved in §6. All others in Theorems A–D will be proved in §4 providing Theorem B(ii).

The above theorem was proved in [8] in the following manner: It is easy to see that there exists an open neighborhood V of 0 in E^0 on which Exp is defined (cf. [1]). At first, we have to prove *the regularity of geodesics* (cf. §0). For the proof of this, we need only the inequality (I) and moreover the constant C may depend on k. The regularity of geodesics implies $\operatorname{Exp}(V \cap E^k) \subset (\operatorname{Exp} V) \cap E^k$. Obviously, there exists an open neighborhood $\acute{V}$ of 0 in E^0 such that $\operatorname{Exp} : \acute{V} \to \operatorname{Exp} \acute{V}$ is diffeomorphic, and we may assume that $\acute{V}$ is a star-shaped neighborhood of 0. Secondly, we have to prove *the openness of the exponential mapping* $\operatorname{Exp} : \acute{V} \cap E^k \to (\operatorname{Exp} \acute{V}) \cap E^k$. This was done by proving that the derivative $d\operatorname{Exp}_u$ at u is isomorphic for every $u \in \acute{V} \cap E^k$. This implies that Exp is a diffeomorphism of $\acute{V} \cap E^k$ onto an open neighborhood of 0 in $(\operatorname{Exp} \acute{V}) \cap E^k$. However, this does not imply $\operatorname{Exp}(\acute{V} \cap E^k) = (\operatorname{Exp} \acute{V}) \cap E^k$ in general. So thirdly, we have to prove *the closedness of the image* $\operatorname{Exp}(\acute{V} \cap E^k)$ in $(\operatorname{Exp} \acute{V}) \cap E^k$. Since the connectedness of $\acute{V}$ implies $(\operatorname{Exp} \acute{V}) \cap E^k$ is connected for any k, these three parts complete the proof. To prove these three parts, we had to establish some comparison theorems. For details, see the previous paper [8]. The basic idea of proving the inequalities (I) and (II) will be given in §2.

(b) *Theorem* E (*Frobenius theorem*). As in (a), we start with a system $\{\mathbf{E}, E^k; k \in N\}$. First of all, we assume the following splitting: $\mathbf{E} = \mathbf{S} \oplus \mathbf{T}$, $E^k = S^k \oplus T^k$, where $\mathbf{S}$, $\mathbf{T}$ are closed subspaces of $\mathbf{E}$ and S^k (resp. T^k) is the closure of $\mathbf{S}$ (resp. $\mathbf{T}$) in E^k. Let U be an open neighborhood of 0 in E^0. We may assume without loss of generality that there exist open neighborhoods V, W of 0 in S^0, T^0 respectively such that $U = V \oplus W$ (direct product). Assume furthermore that there exists a mapping Φ of $(U \cap \mathbf{E}) \times \mathbf{S}$ into $\mathbf{T}$ satisfying the following:

(i) $\Phi(0, v) \equiv 0$ and Φ can be extended to the smooth mapping of $(U \cap E^k) \times S^k$ into T^k for each $k \in N$.

(ii) $\Phi(u, v)$ is linear with respect to the second variable v.

(iii) Putting $D_u^k = \{(v, \Phi(u, v)); v \in S^k\}$, D_u^k is a closed subspace of E^k for each $u \in U \cap E^k$ and $D^k = \{D_u^k; u \in U \cap E^k\}$ gives a smooth involutive distribution on $U \cap E^k$.

Roughly speaking, the local problem about the Frobenius theorem begins with setting this situation. Of course, there are several problems in changing "distributions" to the above local shape. However, different from the case of Banach manifolds, such setting of problems does not necessarily imply existence of integral manifolds. (See [5] to know what are the difficulties in proving the Frobenius theorem.) The difficulties in this case are similar to those in (a).

However, this case is much simpler, because the equation is order 1, that is, differential operator of order 1.

Anyway, we assume the following inequality (I′):

$$\text{(I}'\text{)}\qquad \|\Phi(u,v)\|_k \leq C_k\{\|u\|_0\|v\|_k + \|u\|_k\|v\|_0\} + \gamma_k(\|u\|_{k-1})\|v\|_{k-1}, \qquad k \geq 1,$$

where C_k is a constant which may or may not depend on k and γ_k is a positive continuous function.

Now as in [1, p. 305] consider the following equation:

$$(d/dt)y(t) = \Phi(tx + y(t), x), \tag{6}$$

where $x \in V \cap \mathbf{S}, y(t) \in W \cap \mathbf{T}$. By the above property (i), this equation can be regarded as that on $U \cap E^k$ for any $k \in N$. Assume for a moment that $x \in V \cap S^k$ and $y(0) \in W \cap T^k$ (resp. $x \in V \cap \mathbf{S}$, $y(0) \in W \cap \mathbf{T}$). Then for any s satisfying $s \leq k$ (resp. $s < \infty$), there exists $t_s > 0$ such that $y(t)$, $0 \leq t < t_s$, is the solution of (6) with the initial condition $y(0)$ and contained in $W \cap T^s$. Assume t_s is the maximal number in such t_s. By the property (i) again, it is easy to see that $t_s \leq t_{s-1}$. However, we cannot conclude $t_s = t_{s-1}$ in general without the inequality (I′).

Lemma 1 (regularity of solutions). *Let $x \in V \cap S^k$, $y(0) \in W \cap T^k$ (resp. $x \in V \cap \mathbf{S}, y(0) \in W \cap \mathbf{T}$). Assume $y(t)$ is a solution of (6) in W with the initial condition $y(0)$. Then $y(t)$ is contained in $W \cap T^k$ (resp. $W \cap \mathbf{T}$).*

Proof. Assume $t_0 = \ldots = t_{s-1} > t_s$. Then, obviously $y(t_s) \notin W \cap T^s$, while $y(t_s) \in W \cap T^{s-1}$. If $\|y(t)\|_s$ is bounded in $t \in [0, t_s)$, then so is $\|(d/dt)y(t)\|_s$ also because of the equation (6) and (I′). Hence $\lim_{t \to t_s} y(t)$ exists in $W \cap T^s$. Therefore, we can extend the solution beyond t_s. This is contradiction. Thus, $\|y(t)\|_s$ is unbounded. Let $z(t) = tx + y(t)$. Obviously, $\|z(t)\|_s^2$, $t \in [0, t_s)$, is unbounded.

On the other hand,

$$(d/dt)\|z(t)\|_s^2 \leq 2\|z(t)\|_s\{\|\Phi(z(t), x)\|_s + \|x\|_s\}.$$

Since $\|z(t)\|_{s-1}$ is bounded in $[0, t_s]$ and x is fixed, the inequality (I′) implies that there exist constants C' and K such that

$$(d/dt)\|z(t)\|_s^2 \leq C'\|z(t)\|_s^2 + K\|z(t)\|_s.$$

This shows that $\|z(t)\|_s^2$ is bounded in $[0, t_s)$, contradicting the above argument. Therefore $t_s = t_{s-1}$. Q.E.D.

By considering the equation

$$(d/dt)y(1 - t) = \Phi((1 - t)x + y(1 - t), -x),$$

we have the following

Corollary 1. *Let $y(t)$ be a solution of* (6) *in W. Assume $x \in V \cap S^k$ and $y(1) \in W \cap T^k$ (resp. $x \in V \cap \mathbf{S}$, $y(1) \in W \cap \mathbf{T}$). Then, $y(t)$ is contained in $W \cap T^k$ (resp. $W \cap \mathbf{T}$).*

Now, by the Frobenius theorem Hilbert manifolds [1], there exist open star-shaped neighborhoods V_1, W_1 of 0 in V, W respectively and a smooth diffeomorphism Ψ of $V_1 \times W_1$ onto an open neighborhood of 0 in $V \times W$ such that $\Psi(V_1, w)$ is an integral manifold of the involutive distribution D^0 through w. This mapping Ψ is given in the following manner: Let $\bar{y}(x,y,t)$ be the solution of (6) with the initial condition y. Then Ψ is given by $\Psi(x,y) = (x, \bar{y}(x,y,1))$. Therefore, by Lemma 1, Ψ can be defined as a mapping of $V_1 \cap S^k \times W_1 \cap T^k$ into $U \cap E^k$ and this is smooth for every k. Obviously, $\Psi : V_1 \cap S^k \times W_1 \cap T^k \to U \cap E^k$ is injective, and Corollary 1 shows that $\Psi(V_1 \cap S^k, W_1 \cap T^k) = \Psi(V_1, W_1) \cap E^k$.

Proposition 1. *Ψ is a smooth diffeomorphism of $V_1 \cap S^k \times W_1 \cap T^k$ onto $\Psi(V_1, W_1) \cap E^k$.*

Proof. It is enough to prove that the derivative $d\Psi_{(x,y)}$ at (x,y) is an isomorphism of E^k onto itself. Since $d\Psi_{(x,y)}$ is injective, we have only to prove surjectivity. The derivative is given by

$$d\Psi_{(x,y)} = \begin{pmatrix} \mathrm{id} & 0 \\ d_1\bar{y}_{(x,y,1)} & d_2\bar{y}_{(x,y,1)} \end{pmatrix},$$

where $d_i\bar{y}_{(x,y,1)}$ is the derivative of $\bar{y}$ at $(x,y,1)$ with respect to the ith variable, $i = 1, 2$. Therefore, we have only to show that $d_2\bar{y}_{(x,y,1)}$ is surjective.

Put $z(t) = d_2\bar{y}_{(x,y,t)}z$. Then $z(t)$ satisfies the equation

$$(d/dt)z(t) = d_1\Phi_{\Psi(tx,y)}(z(t), x), \tag{7}$$

where $d_1\Phi_u(u',v)$ is the derivative of Φ with respect to the first variable at u, that is,

$$\lim \frac{1}{\|u'\|_s}\|\Phi(u+u',v) - \Phi(u,v) - d_1\Phi_u(u',v)\|_s = 0.$$

Since (7) is a linear equation and x, y are fixed, for any given $z(1)$ we can find a solution $z(t)$. Put $z = z(0)$. Then, $z(1) = d_2\bar{y}_{(x,y,1)}z$. Thus $d\Psi_{(x,y)}$ is an isomorphism. Q.E.D.

(c) *Distributions defined by kernel of operations.* The assumptions which are imposed on Φ in the previous section (b) are fairly strong. So it is hard to apply Proposition 1 directly. Here a sufficient condition will be given under which the conditions (i)–(iii) in (b) are satisfied. Let $\mathbf{F}$ be another Fréchet space satisfying the same properties (i), (ii) as $\mathbf{E}$ in the first part of (a). Let U' be an open neighborhood of 0 in E^0. Assume there is a mapping $L : U' \cap \mathbf{E} \times \mathbf{E} \to \mathbf{F}$ satisfying the following:

(1) L can be extended to the smooth mapping of $U' \cap E^k \times E^k$ into F^k, $k \geq 0$.

(2) $L(u,v)$ is linear with respect to the second variable and putting $L_u v = L(u,v)$, $L_0 : E^k \to F^k$ is surjective for any k.

(3) $\mathrm{Ker}\{L_u : E^k \to F^k\}$ gives a smooth involutive distribution of $U' \cap E^k$ for each k and $S^k = \mathrm{Ker}\{L_0 : E^k \to F^k\}$, where S^k is the same space as in (b)(that is, we assume the splitting property $\mathbf{E} = \mathbf{S} \oplus \mathbf{T}$, $E^k = S^k \oplus T^k$).

Notice that $u \to L_u$ is also a smooth mapping of $U' \cap E^k$ into $L(E^k, F^k)$. More generally, if a mapping $\varphi : U^k \times E^k \to F^k$ is a C^l-mapping and linear with respect to the second variable, then φ_u defined by $\varphi_u v = \varphi(u,v)$, is a C^{l-1}-mapping with respect to u. Namely $u \to \varphi_u$ is a C^{l-1}-mapping of U^k into $L(E^k, F^k)$.

If φ is C^0-mapping, then $u \to \varphi_u$ is not necessarily continuous. However, since φ is continuous at $(u, 0)$, this mapping is locally bounded, namely, there exist a neighborhood W of u in U^k and a constant K such that $\|\varphi_w v\|_k \leq K\|v\|_k$ for any $w \in W$.

Anyway, many distributions are given locally by such way. However, these properties (1)–(3) do not necessarily imply the existence of Φ satisfying (i)–(iii) in (b). So, we assume the following conditions:

Denote by $\tilde{L}_u$ the restriction of L_u on T^k. By property (3), $\tilde{L}_0 : T^k \to F^k$ is an isomorphism for every k.

$$\text{(I}''\text{)} \qquad \|\tilde{L}_0 v\|_k \geq C\|v\|_k - \gamma_k \|v\|_{k-1}, \qquad k \geq 1,$$

$$\text{(II}''\text{)} \qquad \begin{aligned} \|L_{u+w} v - L_u v\|_k &\leq \bar{C}\{\|w\|_0 \|v\|_k + \|w\|_k \|v\|_0\} \\ &\quad + \bar{\gamma}_k(\|u\|_k, \|w\|_{k-1})\|v\|_{k-1}, \qquad k \geq 1, \end{aligned}$$

where C, $\bar{C}$ are constants which are independent from k, γ_k is a constant depending on k and $\bar{\gamma}_k$ is a nonnegative continuous function such that $\bar{\gamma}_k(*\ ,0) \equiv 0$.

Let U be an open connected neighborhood of 0 in U' such that $U \subset \{x \in U'; \|x\|_0 < C/2\bar{C}\}$. Then, for any $u \in U \cap E^k$,

$$\begin{aligned} \|\tilde{L}_u v\|_k &\geq \|\tilde{L}_0 v\|_k - \|\tilde{L}_u v - \tilde{L}_0 v\|_k \\ &\geq (C/2)\|v\|_k - \bar{C}\|u\|_k \|v\|_0 - \gamma'_k(\|u\|_{k-1})\|v\|_{k-1}, \end{aligned}$$

where γ'_k is a positive continuous function. Assume furthermore that $\tilde{L}_u : T^0 \to F^0$ is an isomorphism for any $u \in U$.

Lemma 2. *For any $u \in U \cap E^k$, the mapping $\tilde{L}_u : T^k \to F^k$ is an isomorphism. Especially $L_u : E^k \to F^k$ is surjective for any $u \in U \cap E^k$.*

Proof. Assume that $\tilde{L}_u : T^s \to F^s$ is isomorphic for any $u \in U \cap E^s$ and $s \leq k-1$. Let W^k be the subset of $U \cap E^k$ such that $\tilde{L}_u : T^k \to F^k$ is isomorphic for $u \in W^k$. W^k is an open subset. By the above inequality $\tilde{L}_u T^k$ is a closed subspace of F^k for any $u \in U \cap E^k$. $\tilde{L}_u : T^k \to F^k$ is of course injective, because $\tilde{L}_u : T^0 \to F^0$ is injective.

Let u be a boundary point of W^k and u_n a sequence in W^k which converges to u in E^k. Take an element $w \notin \tilde{L}_u T^k$. Then for any n, there is $v_n \in T^k$ such that $\tilde{L}_{u_n} v_n = w$. Thus

$$\|w\|_k \geq (C/2)\|v_n\|_k - \bar{C}\|u_n\|_k\|v_n\|_0 - \gamma'_k(\|u_n\|_{k-1})\|v_n\|_{k-1}.$$

By the assumption, $\|v_n\|_{k-1}$ is bounded, because $v_n = \tilde{L}_{u_n}^{-1} w$ in F^{k-1}. Therefore $\|v_n\|_k$ is bounded. So by the inequality (II″), we have

$$\|\tilde{L}_u v_n - \tilde{L}_{u_n} v_n\|_k \leq \bar{C}\{\|u - u_n\|_0\|v_n\|_k + \|u - u_n\|_k\|v_n\|_0\} + \bar{\gamma}_k(\|u\|_k, \|u - u_n\|_{k-1})\|v_n\|_{k-1}.$$

Hence, $\lim_{n\to\infty} \tilde{L}_u v_n = w$ in E^k. Since $\tilde{L}_u T^k$ is closed, we have $w \in \tilde{L}_u T^k$. This is a contradiction. Therefore $\tilde{L}_u$ is isomorphic, so that $W^k = U \cap E^k$, because $U \cap E^k$ is connected. The desired result is obtained by induction.

Let $G_u : F^k \to T^k$ be the inverse of $\tilde{L}_u : T^k \to F^k$ for $u \in U \cap E^k$. Then, for any $w \in E^k$, $w - G_u L_u w$ is contained in $\mathrm{Ker}\{L_u : E^k \to F^k\}$. It is easy to see that $\{w - G_u L_u w; w \in E^k\} = \{v - G_u L_u v; v \in S^k\}$. On the other hand, $w = G_u L_u w + (w - G_u L_u w)$ and $T^k \cap \{w - G_u L_u w; w \in E^k\} = \{0\}$. Thus, we have

$$\mathrm{Ker}\{L_u : E^k \to F^k\} = \{v - G_u L_u v; v \in S^k\}.$$

Since $L_u : T^0 \to F^0$ is isomorphic by assumption, so is G_u. Assume furthermore that $\|G_u L_u v\|_0 \leq C_0\|v\|_0$ for any $u \in U$ and for some constant C_0. This assumption is satisfied if we take U very small in U'.

Lemma 3. *Put $\Phi(u, v) = -G_u L_u v$ for any $u \in U \cap E^k$, $v \in S^k$. Then, there is a constant C' independent from k such that*

$$\|\Phi(u, v)\|_k \leq C'\{\|u\|_k\|v\|_0 + \|u\|_0\|v\|_k\} + \gamma_k(\|u\|_{k-1})\|v\|_{k-1},$$

where γ_k is a positive continuous function.

Proof. By the inequality just above Lemma 2,

$$(C/2)\|G_u L_u v\|_k \leq \|L_u v\|_k + \bar{C}\|u\|_k\|G_u L_u v\|_0 + \gamma'_k(\|u\|_{k-1})\|G_u L_u v\|_{k-1}.$$

Since $L_0 v \equiv 0$ for any $v \in S^k$, $\|L_u v - L_0 v\|_k = \|L_u v\|_k$. On the other hand, $\|G_u L_u v\|_0 \leq C_0 \|v\|_0$ by assumption. Therefore, there exists C'' independent from k such that

$$(C/2)\|G_u L_u v\|_k \leq C''\{\|u\|_k \|v\|_0 + \|u\|_0 \|v\|_k\} + \gamma'_k(\|u\|_{k-1})\|G_u L_u v\|_{k-1}.$$

The desired inequality is obtained by induction. Q.E.D.

Since $G_u : T^k \to F^k$ can be extended to an isomorphism of T^s onto F^s for each $s \leq k$, G_u can be defined as an isomorphism of $\mathbf{T}$ onto $\mathbf{F}$, if $u \in U \cap \mathbf{E}$. Therefore, Φ can be regarded as a map of $U \cap \mathbf{E} \times \mathbf{S}$ into $\mathbf{T}$ which can be extended to the smooth mapping of $U \cap E^k \times S^k$ into T^k for each $k \in N$.

2. **Some remarks on elliptic operators.** As we see in (a) and (c) in the previous section, the independency of constants C, $\bar{C}$, etc. from k is very important in this paper. The Frobenius theorem or the regularity of connections which will be proved by using this fact is much clearer and stronger than the theorems corresponding to those in case of Fréchet manifolds, because in this case the neighborhoods are not only those in Fréchet manifolds but also those given by the intersection of neighborhoods of smooth Hilbert manifolds in the Sobolev chain (cf. §0) and the Fréchet manifolds which are obtained by the inverse limit.

Such independency of C, $\bar{C}$ from k is fairly difficult to prove in general, but several inequalities which involve constants not depending on k have already been proved in [9], [10]. These will be illustrated in §5. Here, first of all, another kind of inequality relating (I″) in (c) will be proved.

Let E be a finite dimensional smooth vector bundle over a closed, oriented and smooth manifold M. Let $\Gamma(E)$, $\Gamma^s(E)$ be the same space as defined in §0.

Lemma 4. *Suppose $D : \Gamma(E) \to \Gamma(E)$ is a differential operator of order m with smooth coefficients. If D is elliptic, then the following inequality holds*:

$$\|Du\|_s \geq C\|u\|_{s+m} - \gamma_s \|u\|_{s+m-1},$$

where C, γ_s are constants such that C is independent from s (C depends only on the highest term of D) and γ_s depends on s.

Of course, this is the usual Gårding inequality. However, independency of C from s is essential in this paper. Since in many books the authors do not care about this point, a brief proof of this lemma will be given below.

Proof of Lemma 4. Let $\sigma(D)$ be the symbol of D. The ellipticity implies that $|\sigma(D)\xi| \geq C_0 |\xi|^m$ for a positive constant C_0, where ξ is any element in the cotangent bundle T^*_M of M. Now, the following is known (cf. [11, Lemma 3.11 and its proof, pp. 196-197]): For every $s \geq 0$, there exists $\delta(s) > 0$, depending on s, such that if the diameter of the support of φ, $\varphi \in \Gamma(E)$, is less than $\delta(s)$, then

$$\|D\varphi\|_s \geq (C_0/4)\|\varphi\|_{s+m}.$$

Let $\{f_\alpha^2\}$ be a partition of unity on M such that the diameter of the support of f_α is less than $\delta(s)$, that is, $\sum_\alpha f_\alpha^2 \equiv 1$ and $\{f_\alpha\}$ depends on s. Let $\mathbf{C} = C_0/4$. By the above inequality, we have

$$\sum_\alpha \|Df_\alpha u\|_s^2 \geq C^2 \sum_\alpha \|f_\alpha u\|_{s+m}^2$$

for any $u \in \Gamma(E)$. Since $\nabla^t f_\alpha u = f_\alpha \nabla^t u +$ lower terms, there is a constant K_s, depending on s, such that

$$\sum_\alpha \|f_\alpha u\|_{s+m}^2 = \sum_\alpha \langle f_\alpha u, f_\alpha u\rangle_{s+m} = \sum_\alpha \sum_t^{s+m} \int_M \langle \nabla^t f_\alpha u, \nabla^t f_\alpha u\rangle \mu$$
$$\geq \|u\|_{s+m}^2 - K_s \|u\|_{s+m} \|u\|_{s+m-1}.$$

By similar argument, we have

$$\sum_\alpha \|Df_\alpha u\|_s^2 = \sum_\alpha \langle Df_\alpha u, Df_\alpha u\rangle_s \leq \|Du\|_s^2 + K_s' \|u\|_{s+m} \|u\|_{s+m-1},$$

for some constant K_s' depending on s. Combining these two inequalities, we have the desired result.

As a matter of fact, the constant C in this lemma can be so chosen that it may satisfy $C_0 - \varepsilon \leq C$. However, the constant γ_s in this case also depends on ε and if $\varepsilon \to 0$, then $\gamma_s \to \infty$.

Now, let $J^m E$ be the mth jet bundle of E. Then, there exists a smooth section α of $(J^m E)^* \otimes E$ such that $D = \alpha j^m$. Namely, $\alpha \in \Gamma((J^m E)^* \otimes E)$. Let $k_0 = [\frac{1}{2} \dim M] + 1$. Then, by the Sobolev embedding theorem, we have that for any $\beta \in \Gamma^{k_0}((J^m E)^* \otimes E)$ and $\tilde{v} \in \Gamma^{k_0}(J^m E)$, there exist e, e' (> 0) such that

$$\max_x |\beta(x)| \leq e\|\beta\|_{k_0}, \qquad \max_x |\tilde{v}(x)| \leq e'\|\tilde{v}\|_{k_0}.$$

Therefore, for any $\beta \in \Gamma^s((J^m E)^* \otimes E)$, we have

$$\|\beta j^m v\|_s \leq C\{\|\beta\|_s \|v\|_{k_0+m} + \|\beta\|_{k_0} \|v\|_{s+m}\} + K_s \|\beta\|_{s-1} \|v\|_{s+m-1},$$

where $s \geq \dim M + 3$, C is a constant depending only on m and K_s is a constant depending on s. (See Lemma 13(ii) in [9] for a precise proof.)

Therefore,

$$\|(\beta - \alpha) j^m v\|_s \leq C\{\|\beta - \alpha\|_s \|v\|_{k_0+m} + \|\beta - \alpha\|_{k_0} \|v\|_{s+m}\}$$
$$+ K_s \|\beta - \alpha\|_{s-1} \|v\|_{s+m-1}.$$

This inequality is related to (II$''$) in §1(c). So by the same method as in the inequality stated just above Lemma 2, if $\|\beta - \alpha\|_{k_0}$ is small enough, then

$$\|\beta j^m v\|_s \geq \bar{C}\|v\|_{s+m} - C\|\beta - \alpha\|_s \|v\|_{k_0+m} - K_s \|\beta - \alpha\|_{s-1} \|v\|_{s+m-1},$$

where $s \geq \dim M + 3$ and $\bar{C}$, C are constants which are independent from s.

So taking the Poincaré inequality into account, we have the following lemma:

Lemma 5. *If $\|\beta - \alpha\|_{k_0}$ is small enough, then there exists $\delta(s, \beta) > 0$ such that*

$$\|\beta j^m v\|_s \geq (\bar{C}/2)\|v\|_{s+m}$$

for any $v \in \Gamma^{s+m}(E)$, $s \geq \dim M + 3$, satisfying that the diameter of the support of v is less than $\delta(s, \beta)$.

Obviously, if $\|\beta - \alpha\|_{k_0}$ is small enough, then βj^m is an elliptic differential operator. By a completely parallel argument as in [15], the inequality in the above lemma yields the following:

Corollary 2. *Let $\beta \in \Gamma^s((J^m E)^* \otimes E)$. Suppose βj^m is an elliptic differential operator. If $\beta j^m v \in \Gamma^s(E)$ for an element $v \in \Gamma^{k_1}(E)$, $k_1 = \dim M + m + 3$, then $v \in \Gamma^{s+m}(E)$.*

Of course, if $\beta \in \Gamma((J^m E)^* \otimes E)$, then this is the well-known regularity of solutions of elliptic differential operators.

3. **Review of the smooth extension theorem and the definition of $\hat{\nabla}$.** Before we start to define the connection $\hat{\nabla}$ on $\mathfrak{D}_0$ mentioned in §0, we have to establish our notation and recall some results which will be used later. In fact the definition of $\hat{\nabla}$ is given by a somewhat different manner from that using $\hat{\alpha}$ in §0. In this section, we discuss local properties of $\mathfrak{D}_0$ and what we have to prove.

Let $\Gamma(T_M)$, $\hat{\Gamma}^1(T_M)$, $\hat{\mathfrak{D}}^1$ denote the space of smooth sections of T_M, the space of C^1 sections of T_M with C^1 uniform topology, the group of C^1 diffeomorphisms of M with C^1 uniform topology respectively. Then, there is a homeomorphism ξ of a bounded open neighborhood $\hat{U}$ of 0 in $\hat{\Gamma}^1(T_M)$ onto an open neighborhood $\tilde{U}$ of the identity e in $\hat{\mathfrak{D}}^1$. Actually this homeomorphism ξ is given by $\xi(u)(x) = \mathrm{Exp}_x u(x)$, where Exp is the exponential mapping defined by a fixed smooth riemannian metric on M. Since $\hat{\mathfrak{D}}^1$ is a topological group, there exists a bounded open neighborhood $\hat{V}$ of 0 in $\hat{\Gamma}^1(T_M)$ such that $\xi(\hat{V})^2 \subset \tilde{U}$. Put $\mathbf{U} = \hat{U} \cap \Gamma(T_M)$, $\mathbf{V} = \hat{V} \cap \Gamma(T_M)$, $U^s = \hat{U} \cap \Gamma^s(T_M)$ and $V^s = \hat{V} \cap \Gamma^s(T_M)$. Since $\Gamma^s(T_M)$ is contained in $\hat{\Gamma}^1(T_M)$ for any $s \geq [\frac{1}{2}\dim M] + 2$ by the Sobolev embedding theorem, the definition of U^s, V^s makes sense. Define a mapping $\eta : \mathbf{V} \times \mathbf{V} \to \mathbf{U}$ by $\eta(u, v) = \xi^{-1}(\xi(u)\xi(v))$. Then, η has the following properties (cf. [9, Corollary 3]):

(η, i) *η can be extended to the C^l-mapping of $V^{s+l} \times V^s$ into U^s.*

(η, ii) *Put $\eta_u(v) = \eta(v, u)$ and η_u is a smooth map of V^s into U^s for any $u \in V^s$.*

(η, iii) *Put $\zeta(v, u) = (d\eta_u)_0 v$ (the derivative of η_u at 0) and ζ can be extended to the C^l-mapping of $\Gamma^{s+l}(T_M) \times V^s$ into $\Gamma^s(T_M)$.*

(η, iv) *There is an open neighborhood* $\hat{W}$ *of* 0 *in* $\hat{V}$ *and a continuous mapping* $\psi : \hat{W} \cap \Gamma^s(T_M) \to V^s$ *such that* $\eta(\psi(x), x) \equiv 0$.

(η, v) *There is a neighborhood* W^s *of* 0 *in* V^s *and a constant* K_s *such that* $\|\eta(v', u) - \eta(v, u)\|_s \leq K_s \|v' - v\|_s$, $u, v, v' \in W^s$.

So these properties are regarded as local properties of $\mathfrak{D}_0^s$. Actually, the group $\mathfrak{D}_0^s$ is defined in the following manner: Let $\mathfrak{N}^s$ be a basis of neighborhoods of 0 in $\Gamma^s(T_M)$. Then, $\{\xi(W \cap \mathbf{U}); W \in \mathfrak{N}^s\}$ is a family of open neighborhoods of the identity e in $\mathfrak{D}_0$ and defines a new, weaker topology for $\mathfrak{D}_0$ under which $\mathfrak{D}_0$ becomes a topological group. $\mathfrak{D}_0^s$ is the completion of $\mathfrak{D}_0$ by the uniform topology defined by this new topology.

The properties of the system $\{\mathfrak{D}_0, \mathfrak{D}_0^s, s \geq \dim M + 5\}$ are the following (cf. [9]):

($\mathfrak{D}_0$, 1) $\mathfrak{D}_0^s$ is a smooth Hilbert manifold.

($\mathfrak{D}_0$, 2) $\mathfrak{D}_0^{s+1} \subset \mathfrak{D}_0^s$ and the inclusion is smooth.

($\mathfrak{D}_0$, 3) $\mathfrak{D}_0 = \bigcap \mathfrak{D}_0^s$ and the original topology (C^∞-topology) of $\mathfrak{D}_0$ is the same as the inverse limit topology.

($\mathfrak{D}_0$, 4) The multiplication $\mathfrak{D}_0 \times \mathfrak{D}_0 \to \mathfrak{D}_0$, $(g, h) \to gh$, can be extended to the C^l-mapping of $\mathfrak{D}_0^{s+l} \times \mathfrak{D}_0^s$ into $\mathfrak{D}_0^s$.

($\mathfrak{D}_0$, 5) The mapping $\mathfrak{D}_0 \to \mathfrak{D}_0$, $g \to g^{-1}$, can be extended to the C^l-mapping of $\mathfrak{D}_0^{s+l}$ into $\mathfrak{D}_0^s$.

($\mathfrak{D}_0$, 6) For any $g \in \mathfrak{D}_0^s$, the right translation $R_g : \mathfrak{D}_0^s \to \mathfrak{D}_0^s$ is smooth.

($\mathfrak{D}_0$, 7) The mapping $dR : \Gamma^{s+1}(T_M) \times \mathfrak{D}_0^s \to T\mathfrak{D}_0^s$ defined by $dR(v, g) = dR_g v$ (the derivative of R_g at e) is a C^l-mapping.

These are the properties which the author called ILH-Lie groups in [7] (see also [9]). The remarkable thing here is that all of these properties are immediate conclusions obtained by fundamental techniques in Hilbert manifolds (cf. [9], [10]) from (η, i – v) and the fact that $\mathfrak{D}_0 = \bigcup_n \tilde{W}^n$ for any open neighborhood $\tilde{W}$ of e in $\mathfrak{D}_0$.

Let $\mathbf{V}$, V^s be the same neighborhoods as above. Put $\tilde{\mathbf{V}} = \{dR_g \mathbf{V}; g \in \mathfrak{D}_0\}$, $\tilde{V}^s = \{dR_g V^s; g \in \mathfrak{D}_0^s\}$. Then $\tilde{\mathbf{V}}$ and $\tilde{V}^s$ are open neighborhoods of zero sections of $T\mathfrak{D}_0$ and $T\mathfrak{D}_0^s$ respectively, where $T\mathfrak{D}_0$, $T\mathfrak{D}_0^s$ are tangent bundles of $\mathfrak{D}_0$, $\mathfrak{D}_0^s$ respectively. Of course $\tilde{\mathbf{V}}$ and $\tilde{V}^s$ are right invariant. Let p be the projection of $T\mathfrak{D}_0$ onto $\mathfrak{D}_0$. Define a mapping $\Xi : \tilde{\mathbf{V}} \to \mathfrak{D}_0$ by

$$\Xi(v)(x) = \mathrm{Exp}_{(pv)(x)} v(x). \tag{8}$$

Then, obviously, $\Xi(dR_g v) = R_g \Xi(v)$. The properties of Ξ are the following (cf. Proposition 1 in [9]):

(Ξ, 1) $\Xi dR_g = R_g \Xi$.

(Ξ, 2) *Ξ can be extended to the smooth mapping of $\tilde{V}^s$ into $\mathfrak{D}_0^s$.*

(Ξ, 3) *Let $V_g^s = dR_g V^s$ for $g \in \mathfrak{D}_0^s$ and Ξ_g be the restriction of Ξ to V_g^s. Then Ξ_g is a smooth diffeomorphism of V_g^s onto an open neighborhood of g in $\mathfrak{D}_0^s$, where throughout* (Ξ, 1 – 3) *s is provided* $\geq \dim M + 5$.

By this property of Ξ, the pair (Ξ_g, V_g) can be regarded as a coordinate (or smooth chart) around $g \in \mathfrak{D}_0$ for every $g \in \mathfrak{D}_0$. So consider this coordinate as a normal coordinate at $g \in \mathfrak{D}_0$ and we get the concept of right-invariant connection on $\mathfrak{D}_0$, namely, the covariant derivative at g is the usual derivative with respect to this coordinate (Ξ_g, V_g). Let $\overset{\circ}{\nabla}$ denote the obtained right invariant connection of $\mathfrak{D}_0$. Since Ξ can be extended to the smooth mapping of $\tilde{V}^s$ into $\mathfrak{D}_0^s$, this connection $\overset{\circ}{\nabla}$ can be extended to the smooth connection of $\mathfrak{D}_0^s$ for $s \geq \dim M + 5$, namely, $\overset{\circ}{\nabla}$ is an ILH-connection in the sense of [8].

The connection $\hat{\nabla}$ mentioned in §0 is given by using $\overset{\circ}{\nabla}$. Namely, let $\pi : \Gamma(T_M) \to \mathbf{F}_2$ be the projection defined in §0. Define a bundle morphism $\tilde{\pi} : T\mathfrak{D}_0 \to T\mathfrak{D}_0$ by $\tilde{\pi} = dR_g \pi dR_g^{-1}$. Then $\tilde{\pi}$ is a smooth projection, because $\mathfrak{D}_0$ is a Fréchet Lie group. So, define the connection $\hat{\nabla}$ by

$$\hat{\nabla}_u v = \tilde{\pi}\overset{\circ}{\nabla}_u \tilde{\pi} v + (1 - \tilde{\pi})\overset{\circ}{\nabla}_u (1 - \tilde{\pi}) v. \tag{9}$$

Obviously, $\hat{\nabla}$ satisfies the axioms of connections and is right-invariant. By this definition, to prove that $\hat{\nabla}$ can be extended to the smooth connection on $\mathfrak{D}_0^s$ for $s \geq \dim M + 2r + 5$, it is enough to prove that $\tilde{\pi}$ can be extended to the smooth projection of $T\mathfrak{D}_0^s$ into itself for $s \geq \dim M + 2r + 5$. Such a property of $\tilde{\pi}$ has been already proved in [10] (cf. Corollary 3 in [10]).

To explain how to prove this property, we have to begin with vector bundles over $\mathfrak{D}_0$ (cf. §2 in [9]). Let E be a finite dimensional smooth riemannian vector bundle over M, $\Gamma(E)$ the space of smooth sections of E and $\Gamma_g(E)$ the space of smooth sections of the pull back $g^{-1}E$ of E by $g \in \mathfrak{D}_0$. Put $\gamma(E) = \bigcup\{\Gamma_g(E); g \in \mathfrak{D}_0\}$. This is a sort of vector bundle over $\mathfrak{D}_0$, if we forget about the topology. Obviously $\gamma(T_M) = T\mathfrak{D}_0$ (the tangent bundle of $\mathfrak{D}_0$). Denote by p the projection of $\gamma(E)$ onto $\mathfrak{D}_0$. Since g is a diffeomorphism, for every $v \in \Gamma_g(E)$ there is a unique $v' \in \Gamma(E)$ such that $v(x) = v'(g(x))$ and conversely $v'(g(x))$ is an element of $\Gamma_g(E)$ for any $v' \in \Gamma(E)$. Moreover, $v(g(x))$ is an element of $\Gamma_{hg}(E)$ for every $v \in \Gamma_h(E)$. Therefore $\gamma(E)$ is a vector bundle on which $\mathfrak{D}_0$ acts as bundle automorphism. Denote by R_g^* the action of g, that is, $(R_g^* v)(x) = v(g(x))$. Obviously, $R_{hg}^* = R_g^* R_h^*$. Recollect the definition of $\xi : \mathbf{U} \to \mathfrak{D}_0$. Using this notation, $\tau'(\xi(u)(x))v(x)$ denotes the parallel translation of $v(x) \in E$ along the curve $\xi(tu)(x)$, $t \in [0, 1]$, in M. Of course $\tau'(\xi(u)(x))v(x)$ is an element of the tangent space of M at $\xi(u)(x)$. So, if we regard x as a variable, $\tau'(\xi(u)(x))v(x)$ is an element of $\Gamma_{\xi(u)}(E)$ for every $v \in \Gamma(E)$ and $u \in \mathbf{U}$. Put $\tau(u, v)(x) = \tau'(\xi(u)(x))v(x)$, $\tau_{\xi(u)} v = \tau(u, v)$. Then, $\tau : \mathbf{U} \times \Gamma(E) \to \gamma(E)$ is a bijection of $\mathbf{U} \times \Gamma(E)$ onto $p^{-1}\xi(\mathbf{U})$ and $\tau_{\xi(u)}$ is a linear bijection of $\Gamma(E)$ onto $\Gamma_{\xi(u)}(E)$. So τ can be regarded as a local trivialization of $\gamma(E)$.

Let

$$R'(w, u) = \tau_{\xi(u)}^{-1} R_{\xi(u)}^* w, \qquad T'(u, w, v) = \tau_{\xi(u)\xi(v)}^{-1} R_{\xi(v)}^* \tau_{\xi(u)} w,$$

where $\tau_{\xi(u)\xi(v)}$ implies $\tau_{\xi(\eta(u,v))}$. Then we have

(τ, 1) *R' : $\mathbf{\Gamma}(E) \times \mathbf{U} \to \mathbf{\Gamma}(E)$ can be extended to the C^l-mapping of $\Gamma^{s+l}(E) \times U^t$ into $\Gamma^s(E)$ for any $t \geq s \geq$* dim M + 5 [9, *Lemma* 5].

(τ, 2) *T' : $\mathbf{V} \times \mathbf{\Gamma}(E) \times \mathbf{V} \to \mathbf{\Gamma}(E)$ can be extended to the C^l-mapping of $V^{t+l} \times \Gamma^{s+l}(E) \times V^t$ into $\Gamma^s(E)$ for any $t \geq s \geq$* dim M + 5, *where* $\mathbf{V}$ and *V^s, V^t are the same neighborhood as in* (η, i–ii) (*cf.* [9, *Proposition* 2]).

(τ, 3) *There exists a mapping $\tilde{\tau}_\Delta$: $\mathbf{V} \times \mathbf{V} \to \mathbf{\Gamma}(E^* \otimes E)$ which can be extended to the smooth mapping of $V^s \times V^s$ into $\Gamma^s(E^* \otimes E)$. $\tilde{\tau}_\Delta$ satisfies*

$$T'(u, w, v) = \tilde{\tau}_\Delta(u, \zeta(u, v))R'(w, v),$$

where ζ is the same mapping as in (η, iii). (*Cf. Lemma* 6 *of* [9]. *$T' = \tau_\Delta \circ R'$ in the notation of that paper.*)

Especially if v is fixed in V^s, then $R''_{\xi(v)}(w, u)$ defined by $R''_{\xi(v)}(w, u) = T'(u, w, v)$ is a smooth map of $V^s \times \Gamma^s(E)$ onto $\Gamma^s(E)$ and continuous with respect to $v \in V^s$.

Since $\mathfrak{D}_0^{n+5}$, $n = \dim M$, is a topological group by the properties ($\mathfrak{D}_0$, 4 – 5), there exists an open neighborhood W of 0 in $\Gamma^{n+5}(T_M)$ such that $\xi(W)\xi(W)^{-1} \subset \xi(V^{n+5})$. Put $\mathbf{W} = W \cap \mathbf{\Gamma}(T_M)$, $W^s = W \cap \Gamma^s(T_M)$. Consider the disjoint union $\bigcup\{\xi(W^t)g \times \Gamma^s(E); g \in \mathfrak{D}_0^t\}$ for any fixed t, s, where $t \geq s \geq n + 5$. We define an equivalence relation $\sim$ as follows: $(\xi(u)g, w) \sim (\xi(u')g', w')$ if and only if $\xi(u)g = \xi(u')g'$ and $w = R'_{g'g^{-1}}(w', u')$. Since $g'g^{-1} = \xi(u)\xi(u')^{-1} \in \xi(V^t)$, this equivalence relation makes sense. So, put

$$\gamma^{t,s}(E) = \bigcup\{\xi(W^t)g \times \Gamma^s(E); g \in \mathfrak{D}_0^t\}/\sim .$$

Then, the properties of $\gamma^{t,s}(E)$ are the following (cf. (γ, 1–7) in [9, §2]):

(γ, 1) $\gamma^{t,s}(E)$ is a smooth vector bundle over $\mathfrak{D}_0^t$ with the fibre $\Gamma^s(E)$.

(γ, 2) $\gamma^{t+1,s}(E) \subset \gamma^{t,s}(E)$, $\gamma^{t,s+1}(E) \subset \gamma^{t,s}(E)$ (where $t \geq s + 1$) and the inclusion is smooth.

(γ, 3) $\gamma(E) = \bigcap \gamma^s(E)$, where $\gamma^{s,s}(E) = \gamma^s(E)$.

(γ, 4) $\gamma^{t,s}(E)$ is the pull back of $\gamma^s(E)$ by the inclusion $\mathfrak{D}_0^t \subset \mathfrak{D}_0^s$.

(γ, 5) The right translation R_g^* can be defined for $g \in \mathfrak{D}_0^t$ and is a smooth map of $\gamma^{t,s}(E)$ onto itself. Moreover, this is continuous with respect to $g \in \mathfrak{D}_0^t$.

(γ, 6) Put $R^*(w, g) = R_g^* w$ for $w \in \mathbf{\Gamma}(E)$, $g \in \mathfrak{D}_0$. Then, the mapping R^* : $\mathbf{\Gamma}(E) \times \mathfrak{D}_0 \to \gamma(E)$ can be extended to the C^l-mapping of $\Gamma^{s+l}(E) \times \mathfrak{D}_0^s$ into $\gamma^s(E)$.

All of these properties (γ, 1–6) are immediate conclusions obtained from (τ, 1–3) by using fundamental techniques in Hilbert manifolds.

Now, let E, F be smooth finite dimensional riemannian vector bundles over M. Consider a linear mapping A : $\mathbf{\Gamma}(E) \to \mathbf{\Gamma}(F)$ which can be extended to the bounded linear map of $\Gamma^s(E)$ into $\Gamma^{s-r}(F)$ for every $s \geq n + 5 + r$. Then, by using (γ, 5–6) above, the mapping $\tilde{A}$ defined by $\tilde{A} = R_g^* A R_g^{*-1}$ is a continuous mapping of $\gamma^s(E)$ into $\gamma^{s,s-r}(F)$. Obviously, $\tilde{A}$ is right-invariant. The following was the main theorem of [10]:

Theorem (smooth extension theorem). *Let A be a differential operator of order r with smooth coefficients. Then $\tilde{A}$ can be extended to the smooth bundle morphism of $\gamma^s(E)$ into $\gamma^{s,s-r}(F)$ for any $s \geq n + 5 + r$.*

4. **Reduction of Theorems A–D to Theorem B(ii).**

(a) *Proof of Theorem* B(i). Recall definitions of $\gamma(T_M)$, $\gamma^s(T_M)$, $T\mathfrak{D}_0$ and $T\mathfrak{D}_0^s$. Obviously, $\gamma(T_M) = T\mathfrak{D}_0$ and $\gamma^s(T_M) = T\mathfrak{D}_0^s$. The right translation R_g^* in $(\gamma, 5)$ is equal to the derivative of R_g. Consider the differential operators A, B and the splitting $\mathbf{\Gamma}(T_M) = \mathbf{F}_1 \oplus \mathbf{F}_2 \oplus H$ as in ($\bigstar$) in §0. Since $H \subset \mathbf{\Gamma}(T_M)$ and is finite dimensional, the subbundle $\tilde{H}$ defined by $\{dR_g H; g \in \mathfrak{D}_0^s\}$ is a smooth distribution of $\mathfrak{D}_0^s$ by virtue of the property $(\gamma, 6)$ in §3.

Put $\gamma_\square^{t,s}(T_M) = \{dR_g(F_1^s \oplus F_2^s); g \in \mathfrak{D}_0^t\}$. Since $F_1^s \oplus F_2^s$ is perpendicular to H in the inner product $\langle\,,\rangle_0$ (cf. the defining equality (3)), $\gamma_\square^{t,s}(T_M)$ is also a smooth subbundle of $\gamma^{t,s}(E)$ (cf. Theorem 1 and Lemma 3 in [10]). Let $\square = AA^* + B^*B$, where A^*, B^* are formal adjoint operators of A, B respectively. Then, $\tilde{\square}$ is a smooth bundle morphism of $\gamma_\square^s(T_M) (= \gamma_\square^{s,s}(T_M))$ onto $\gamma_\square^{s,s-2r}(T_M)$. Since $\square : F_1^s \oplus F_2^s \to F_1^{s-2r} \oplus F_2^{s-2r}$ is isomorphic, $\tilde{\square}$ is in fact a smooth bundle isomorphism. Therefore $\tilde{\square}^{-1}$ is a smooth bundle isomorphism of $\gamma^{s,s-2r}(T_M)$ onto $\gamma^s(T_M)$.

The projection $\pi : \mathbf{\Gamma}(T_M) \to \mathbf{F}_2$ is given by $\pi = \square^{-1} B^*B$. Since $\tilde{\pi} = \tilde{\square}^{-1} \tilde{B}^*\tilde{B}$, the smooth extension theorem shows that $\tilde{\pi}$ can be extended to the smooth projection of $\gamma^s(T_M)$ into itself. Recall the definition of $\hat{\nabla}$ (9). This defining equality (9) shows that $\hat{\nabla}$ can be extended to the smooth right-invariant connection of $\mathfrak{D}_0^s$ for $s \geq \dim M + 2r + 5$.

(b) *Relation between* (4) *and* (9). The definitions of $\hat{\nabla}$ in §0 and §3 are somewhat different. In this section, we discuss the relation between the defining equalities (4) and (9).

For any $v \in \Gamma^{s+1}(T_M)$, let $\tilde{v}$ denote the C^1 vector field defined by $\{dR_g v; g \in \mathfrak{D}_0^s\}$. Recall the definition of $\hat{\nabla}$ in §3. It is easy to see that $\hat{\nabla}_u \tilde{v} = \nabla_u v$, where $u \in \Gamma^s(T_M)$ and ∇ is the original connection on M by which the normal coordinate $(\Xi g, V_g)$ and hence $\hat{\nabla}$ is defined.

Obviously, $\widetilde{\pi v} = \tilde{\pi}\tilde{v}$ and $((1-\pi)v)^\sim = (1-\tilde{\pi})\tilde{v}$. Therefore

$$\begin{aligned}\hat{\nabla}_u \tilde{v} &= \tilde{\pi}\hat{\nabla}_u(\pi v)^\sim + (1-\tilde{\pi})\hat{\nabla}_u((1-\pi)v)^\sim \\ &= \pi\hat{\nabla}_u(\pi v)^\sim + (1-\pi)\hat{\nabla}_u((1-\pi)v)^\sim \\ &= \pi\nabla_u \pi v + (1-\pi)\nabla_u(1-\pi)v = \hat{\alpha}(u,v).\end{aligned}$$

Assume there is another right-invariant connection $\hat{\nabla}'$ on $\mathfrak{D}_0$ such that $\hat{\nabla}_u \tilde{v} \equiv \hat{\nabla}_u{}'\tilde{v}$. In the following part of this section, $\hat{\nabla} = \hat{\nabla}'$ will be proved. Recall the definition of $\xi : \mathbf{U} \to \mathfrak{D}_0$. Define $\mathbf{\Gamma}(T_M)$-valued bilinear forms $\hat{\Gamma}$, $\hat{\Gamma}'$ defined on U by

$$\hat{\Gamma}_x(u,v) = d\xi_x^{-1}\, \hat{\nabla}_{d\xi_x u} d\xi v, \qquad \hat{\Gamma}'_x(u,v) = d\xi_x^{-1}\, \hat{\nabla}_{d\xi_x u} d\xi v,$$

where $d\xi_x$ is the derivative of ξ at x and $d\xi v$ in the right-hand side of each defining equality is the image by $d\xi$ of the vector field v which is defined from $v \in \Gamma(T_M)$ by the parallel displacement using the linear structure of $\Gamma(T_M)$. v is regarded as a vector field on $\mathbf{U}$. To prove $\hat{\nabla} = \hat{\nabla}'$, it is enough to prove $\hat{\Gamma} = \hat{\Gamma}'$. Let $\{e_n\}$, $n = 1, 2, \dots$, be a basis of $\Gamma(T_M)$. Since $\hat{\nabla}$, $\hat{\nabla}'$ are right-invariant and $\hat{\nabla}_u \tilde{v} = \hat{\nabla}'_u \tilde{v}$, we have $\hat{\nabla}_{\tilde{e}_n} \tilde{e}_m = \hat{\nabla}'_{\tilde{e}_n} \tilde{e}_m$. Let $\hat{e}_n(x) = d\xi_x^{-1} \tilde{e}_n$. Then $\{\hat{e}_n(x)\}$ is a basis of $\Gamma(T_M)$ for any $x \in \mathbf{U}$. By the above equality, we have $\hat{\Gamma}_x(\hat{e}_n(x), \hat{e}_m(x)) = \hat{\Gamma}'_x(\hat{e}_n(x), \hat{e}_m(x))$. Therefore, $\hat{\Gamma}_x = \hat{\Gamma}'_x$ for any x.

(c) *Proof of Theorem* C. Since $\tilde{\pi} : \gamma^s(T_M) \to \gamma^s(T_M)$ is smooth and the image is closed (in fact, $\tilde{\pi}\gamma^s(T_M) = \tilde{F}_2^s$), Ker $\tilde{\pi}$ is a smooth subbundle of $\gamma^s(T_M)$. (This is not difficult to prove. For a precise proof, see Lemma 3 in [10].) Obviously, Ker $\tilde{\pi}$ is given by $(F_1^s \oplus H)^\sim$. By the same reason,

$$\tilde{F}_2^s = \operatorname{Ker}\{1 - \tilde{\pi} : \gamma^s(T_M) \to \gamma^s(T_M)\}$$

is a smooth subbundle of $\gamma^s(T_M)$.

Let $c(t)$ be a C^1 curve in $\mathfrak{D}_0^s$ such that $c(0) = e$. Let $v_0 \in \Gamma^s(T_M)$ and $v(t)$ a parallel displacement of v_0 along $c(t)$, namely $\hat{\nabla}_{\dot{c}(t)} v(t) = 0$ and $v(0) = v_0$. Since $\tilde{\pi}\hat{\nabla}_u v = \hat{\nabla}\tilde{\pi} v$ by definition, $\tilde{\pi}v(t)$ and $(1 - \tilde{\pi})v(t)$ are also parallel displacements of πv_0, $(1 - \pi)v_0$ respectively. Thus, if $v_0 = \pi v_0$ (resp. $v_0 = (1 - \pi)v_0$), then $v(t) = \tilde{\pi}v(t)$ (resp. $v(t) = (1 - \tilde{\pi})v(t)$). This is because of the uniqueness of parallel displacements in the Hilbert manifold $\mathfrak{D}_0^s$. Therefore $\tilde{F}_2$ and $(F_1 \oplus H)^\sim$ are parallel.

(d) *Proof of Theorem* D *providing Theorem* B(ii). For any C^1 vector fields $\tilde{u}$, $\tilde{v}$ on $\mathfrak{D}_0^s$, the torsion tensor $\hat{T}$ is given by $\hat{T}(\tilde{u}, \tilde{v}) = \hat{\nabla}_{\tilde{u}} v - \hat{\nabla}_{\tilde{v}} \tilde{u} - [\tilde{u}, \tilde{v}]$. Since $(F_1^s \oplus H)^\sim$ is parallel, if $\tilde{u}, \tilde{v} \in (F_1^s \oplus H)^\sim$ then $\hat{\nabla}_{\tilde{u}} \tilde{v}$ and $\hat{\nabla}_{\tilde{v}} \tilde{u}$ are contained in $(F_1^s \oplus H)^\sim$.

For any $w, w' \in F_1^{s+1} \oplus H$, define vector fields $\tilde{w}, \tilde{w}'$ by $\tilde{w} = \{dR_g w; g \in \mathfrak{D}_0^s\}$, $\tilde{w}' = \{dR_g w'; g \in \mathfrak{D}_0^s\}$. These are C^1 vector fields by $(\gamma, 6)$. It is easy to see that $[\tilde{w}, \tilde{w}'] = ([w, w'])^\sim \in (F_1^s \oplus H)^\sim$. Therefore, we have $\hat{T}(\tilde{w}, \tilde{w}') \in (F_1^s \oplus H)^\sim$, for any $w, w' \in F_1^s \oplus H$. Since $\hat{T}$ is a tensor field in $\mathfrak{D}_0^s$ and $F_1^{s+1} \oplus H$ is dense in $F_1^s \oplus H$, we have $\hat{T}(\tilde{u}, \tilde{v}) \in (F_1^s \oplus H)^\sim$ for any C^1 vector field $\tilde{u}, \tilde{v} \in (F_1^s \oplus H)^\sim$. So, the equality $[\tilde{u}, \tilde{v}] = \hat{\nabla}_{\tilde{u}} \tilde{v} - \hat{\nabla}_{\tilde{u}} \tilde{u} - \hat{T}(\tilde{u}, \tilde{v})$ shows that $(F_1^s \oplus H)^\sim$ is involutive.

Therefore, for each s there exists the maximal integral submanifold G^s through the identity. Since $(F_1^s \oplus H)^\sim$ is invariant, we have $G^s \cdot G^s = G^s$. Therefore, G^s is a subgroup. So this is a topological group by the relative topology. However, the manifold structure might be given by a stronger topology. Recall that the Frobenius theorem shows the existence of slices of integral manifolds. So the usual technique as in finite dimensional manifolds yields that $\{G^s\}$ satisfies all of the properties of $(\mathfrak{D}_0, 1 - 7)$. (Cf. Lemma 4 and (G, 1–7) in [10] for the precise proof.) Of course G is defined by $\bigcap G^s$ with the inverse limit topology. Therefore $(\mathfrak{D}_0, 3)$ should be changed slightly.

Let Exp be the exponential mapping at e defined by $\hat{\nabla}$. Since $(F_1^s \oplus H)^\sim$ is parallel, $\operatorname{Exp} u \in G^s$ for any $u \in F_1^s \oplus H$, whenever this is defined. Thus $\operatorname{Exp}(V^{k_1} \cap F_1^s \oplus H) \subset G^s$ (cf. the statements of Theorem B in §0). Since exponential mapping is right-invariant, this implies that G^s is totally geodesic in $\mathfrak{D}_0^s$.

Let $W^{k_1} \subset V^{k_1} \cap F_1^{k_1} \oplus H$ be an open neighborhood of 0 such that Exp is a smooth diffeomorphism of W^{k_1} into an open neighborhood of e in G^{k_1}. It is easy to see that

$$(\operatorname{Exp} W^{k_1}) \cap G^s \supset \operatorname{Exp}(W^{k_1} \cap F_1^s \oplus H).$$

Assume $g \in (\operatorname{Exp} W^{k_1}) \cap G^s$. Then there exists uniquely an element $u \in W^{k_1}$ such that $g = \operatorname{Exp} u$. Since $g \in G^s \subset \mathfrak{D}_0^s$, u must be contained in $\Gamma^s(T_M)$ by Theorem B(ii). Thus, $u \in W^{k_1} \cap F_1^s \oplus H$, so that

$$(\operatorname{Exp} W^{k_1}) \cap G^s = \operatorname{Exp}(W^{k_1} \cap F_1^s \oplus H).$$

Obviously, $\operatorname{Exp} : W^{k_1} \cap F_1^s \oplus H \to (\operatorname{Exp} W^{k_1}) \cap G$ is smooth, and the derivative $(d \operatorname{Exp})_u$ at the point u is injective, because of the assumption $\operatorname{Exp} : W^{k_1} \to G^{k_1}$ is diffeomorphic. Suppose w is a tangent vector of G^s at $\operatorname{Exp} u$. There is an element $v \in F_1^{k_1} \oplus H$ such that $(d \operatorname{Exp}_u)v = w$. By Theorem B(ii) again, v must be contained in $\Gamma^s(T_M)$ and hence $v \in F_1^s \oplus H$. Thus, Exp is a diffeomorphism of $W^{k_1} \cap F_1^2 \oplus H$ onto $(\operatorname{Exp} W^{k_1}) \cap G^s$. Consequently, $\operatorname{Exp} : W^{k_1} \cap \mathbf{F}_1 \oplus H \to (\operatorname{Exp} W^{k_1}) \cap G$ gives a local coordinate of G and hence G is a Fréchet manifold by the inverse limit topology. Therefore, G is a Fréchet Lie group.

5. **Some inequalities.** As one can guess in the arguments in §3–§4, to get inequalities of types (I), (II) or (I″) in §1 which are essential in proofs of Theorem B(ii) or Theorem E, one has to know explicit local expressions of bundle morphisms $\tilde{A}$ defined by differential operators $A : \Gamma(E) \to \Gamma(F)$ with smooth coefficients. Recollect the definitions of $\xi : \mathbf{U} \to \mathfrak{D}_0$ and $\tau : \mathbf{U} \times \Gamma(E) \to \gamma(E)$ in §3. Let $\tau'' : \mathbf{U} \times \Gamma(F) \to \gamma(F)$ be the same mapping "τ", replacing E by F. By definition, the local expressions of $\tilde{A} : \gamma(T_M) \to \gamma(F)$ and $\tilde{A} : \gamma(E) \to \gamma(F)$ are $\tau''^{-1}_{\xi(u)} \tilde{A} d\xi_u v$ and $\tau''^{-1}_{\xi(u)} \tilde{A}\tau(u,v)$ respectively, where $\tau''_{\xi(u)} v = \tau''(u,v)$ and $d\xi_u$ is the derivative of ξ at u. Of course, these are only definitions and have nothing to do with the inequalities (I), (II) or (I″), (II″) directly. In case of $E = T_M$, we may use the local trivialization τ instead of $d\xi$. However, to consider the Frobenius theorem, we have to translate the distributions defined in $\mathfrak{D}_0$ into those on $\mathbf{U}$ through the coordinate mapping ξ. At that time, we have to use $d\xi$ as a local trivilization of the tangent bundle. This is the reason why we use $d\xi$ in the case of $E = T_M$.

Let m be the order of the differential operator A, and $J^m T_M$, $J^m E$ the mth jet bundle of T_M, E respectively. Let W^m be a relatively compact and open tubular neighborhood of zero section of $J^m T_M$. Denote

$$\Gamma(W^m) = \{u \in \Gamma(T_M); (j^m u)(x) \in W^m \text{ for every } x \in M\}.$$

This is well defined for $s \geq [\frac{1}{2}\dim M] + m + 1$ by virtue of the Sobolev embedding theorem.

By the arguments in §4(b) in [10], we see that there exists a fibre preserving smooth mapping ψ (resp. ψ') of W^m into $(J^m T_M)^* \otimes F$ (resp. $(J^m E)^* \otimes F$) such that

$$(10) \qquad (\tau''^{-1}_{\xi(u)} \tilde{A} d\xi_u v)(x) = \psi((j^m u)(x))(j^m v)(x),$$

$$(11) \qquad (\tau''^{-1}_{\xi(u)} \tilde{A}\tau(u, v))(x) = \psi'((j^m u)(x))(j^m v)(x),$$

for any $u \in U^s \cap \Gamma^s(W^m)$ and $v \in \Gamma^s(T_M)$, where $s \geq [\frac{1}{2}\dim M] + m + 1$ and U^s is the same neighborhood as in the first part of §3. In this range of s, $U^s \cap \Gamma^s(W^m)$ is open in $\Gamma^s(T_M)$ because of the Sobolev embedding theorem. Explicit expressions of ψ and ψ' are much complicated, using higher order connections defined on jet spaces of maps. (As for the explicit expressions, see §4(b) in [10].) However, these explicit expressions have never been used but only the facts (10) and (11) stated above. In fact, even the main theorem of [10](smooth extension theorem) is an immediate conclusion from these facts together with Theorem B in [9], which was one of the main theorems of [9]. Similarly, inequalities which will be used in this paper are given by using only these facts together with Theorem A in [9]. Moreover in this case, we have only to use a special case of Theorems A and B [9] as follows:

Theorem (Theorems A, B [9]**).** *Let f be a smooth fibre preserving mapping of a relatively compact tubular neighborhood W of zero section of a smooth finite dimensional vector bundle E over M into a tensor bundle $F^* \otimes G$ of a smooth finite dimensional vector bundle over M. Assume f can be extended to the smooth map of $\bar{W}$ (closure of W) into $F^* \otimes G$. Then the mapping $\Phi : \Gamma(W) \times \Gamma(F) \to \Gamma(G)$ defined by $(\Phi(u)v)(x) = f(u(x))v(x)$ can be extended to the smooth map of $\Gamma^s(W) \times \Gamma^s(E)$ into $\Gamma^s(G)$ for $s \geq n + 5$ and satisfies the inequality*

$$\|\Phi(u)v\|_s \leq C\|u\|_s\|v\|_{k_0} + D\|v\|_s + \gamma_s(\|u\|_{s-1})\|v\|_{s-1},$$

where $n = \dim M$, $k_0 = n + 5$ and u is restricted in a bounded set in $\Gamma^{2k_0}(E)$. $\Gamma(W)$ is of course $\{u \in \Gamma(E); u(x) \in W\}$ and the constants C, D are independent from s and γ_s is a polynomial with positive coefficients (cf. Theorem A, *Theorem* B *in* [9]).

Now back to our case; put $(\Psi(u)v)(x) = \psi((j^m u)(x))(j^m v)(x)$ and $(\Psi'(u)v)(x) = \psi'((j^m u)(x))(j^m v)(x)$. Notice the following simple fact: There exists a constant e_m such that $\|j^m v\|_s \leq e_m\|v\|_{s+m}$ for every s (cf. Lemma 16 of [10] for the proof).

Combining this fact with Theorems A, B [9], we have the following for $s \geq n + m + 5$.

$$(12) \qquad \|\Psi(u)v\|_{s-m} \leq C\|u\|_s\|v\|_{k_0+m} + D\|v\|_s + \gamma_s(\|u\|_{s-1})\|v\|_{s-1},$$

where u is restricted in a bounded set in $\Gamma^{2k_0+m}(T_M)$, C, D are constants independent from s and γ_s is a polynomial with positive coefficients. The same inequality holds for Ψ'.

Let $\partial\psi_p$ be the partial derivative of ψ at p along the fibre. Then, the derivative $d\Psi_u$ of Ψ at u is given by $(d\Psi_u u')(x) = \partial\psi_{(j^m u)(x)}(j^m u')(x)$, where Ψ is regarded as a mapping of $\Gamma(W^m)$ into $\Gamma((J^m T_M)^* \otimes F)$. Notice that $p \to \partial\psi_p$ is a smooth fibre preserving mapping of W^m into

$$(J^m T_M)^* \otimes (J^m T_M)^* \otimes F = (J^m T_M \otimes J^m T_M)^* \otimes F.$$

Let $d\Psi_u(u', v)$ denote the (partial) derivative of $\Psi(u)v$ at u with respect to the first variable u. Then, we have

$$(d\Psi_u(u', v))(x) = \partial\psi_{(j^m u)(x)}((j^m u')(x), (j^m v)(x)), \tag{13}$$

where $\partial\psi_p(X, Y)$ is the (partial) derivative of $\psi(p)Y$ at p along the fibre.

Notice the following simple fact: *If $s \geq \dim M + 3$, then*

$$\|u \otimes v\|_s \leq C\{\|u\|_s \|v\|_{k_0} + \|u\|_{k_0} \|v\|_s\} + \gamma_s \|u\|_{s-1} \|v\|_{s-1}$$

for any $u \in \Gamma(E)$, $v \in \Gamma(F)$, where $u \otimes v$ is defined by $(u \otimes v)(x) = u(x) \otimes v(x)$, C is a constant independent from s and γ_s a constant depending on s (cf. Lemma 13, [9]).

Combining this with the equality (13) and Theorems A, B [9], we have the following inequality for $s \geq \dim M + m + 5$ and u restricted in a bounded set in $\Gamma^{2k_0+m}(T_M)$:

$$\begin{aligned} \|d\Psi_u(u', v)\|_{s-m} \leq\; & C\{\|u'\|_s \|v\|_{k_0+m} + \|u'\|_{k_0+m} \|v\|_s\} \\ & + D\|u\|_s \|u'\|_{k_0+m} \|v\|_{k_0+m} \\ & + \gamma_s(\|u\|_{s-1}) \|u'\|_{s-1} \|v\|_{s-1}, \end{aligned} \tag{14}$$

where C, D are constants independent from s and γ_s is a polynomial with positive coefficients. The same inequality holds for ψ'.

Let $d^2\Psi_u$ be the second derivative of Ψ at u. $d^2\Psi_u(w, u', v)$ is defined by natural manner. Then, we have

$$(d^2\Psi_u(w, u', v))(x) = \partial^2\psi_{(j^m u)(x)}((j^m w)(x), (j^m u')(x), (j^m v)(x)),$$

where $\partial^2\psi_p(X, Y, Z)$ is the (partial) second derivative of $\psi(p)Z$ at p along the fibre. Since $p \to \partial^2\psi_p$ is a smooth fibre preserving mapping of W^m into

$$(J^m T_M \otimes J^m T_M \otimes J^m T_M)^* \otimes F,$$

we have the following inequality (15) by Theorem A [9] combined with the fact that

$$\begin{aligned} \|u \otimes v \otimes w\|_s \leq\; & C\{\|u\|_s \|v\|_{k_0} \|w\|_{k_0} + \|u\|_{k_0} \|v\|_s \|w\|_{k_0} + \|u\|_{k_0} \|v\|_{k_0} \|w\|_s\} \\ & + D_s \|u\|_{s-1} \|v\|_{s-1} \|w\|_{s-1} \end{aligned}$$

for $s \geq \dim M + 3$ (cf. [9, Lemma 13]):

$$
\begin{aligned}
\|d^2\Psi_u(w,u',v)\|_{s-m} \leq\ & C\{\|w\|_s\|u'\|_{k_0+m}\|v\|_{k_0+m} + \|w\|_{k_0+m}\|u'\|_s\|v\|_{k_0+m} \\
& \qquad + \|w\|_{k_0+m}\|u'\|_{k_0+m}\|v\|_s\} \\
& + D\|u\|_s\|w\|_{k_0+m}\|u'\|_{k_0+m}\|v\|_{k_0+m} \\
& + \gamma_s(\|u\|_{s-1})\|w\|_{s-1}\|u'\|_{s-1}\|v\|_{s-1},
\end{aligned} \tag{15}
$$

where $s \geq \dim M + m + 5$ and u is restricted in a bounded set in $\Gamma^{2k_0+m}(T_M)$. The same inequality holds for Ψ'.

Lemma 6. *Suppose $s \geq \dim M + m + 5$ and u, w are restricted in a bounded set in $\Gamma^{2k_0+m}(T_M)$. Then*

$$
\begin{aligned}
\|\Psi(u+w)v - \Psi(u)v\|_{s-m} \leq\ & C\{\|w\|_s\|v\|_{k_0+m} + \|w\|_{k_0+m}\|v\|_s\} \\
& + P_s(\|u\|_s, \|w\|_{s-1})\|w\|_{s-1}\|v\|_{s-1},
\end{aligned}
$$

where C is independent from s and P_s is a polynomial with positive coefficients. The same inequality holds for Ψ'.

Proof.

$$\Psi(u+w)v - \Psi(u)v = \int_0^1 d\Psi_{u+\theta w}(w,v)\,d\theta.$$

By (14), we have

$$
\begin{aligned}
\|\Psi(u+w)v - \Psi(u)v\|_{s-m} \leq\ & C'\{\|w\|_s\|v\|_{k_0+m} + \|w\|_{k_0+m}\|v\|_s\} \\
& + D'(\|u\|_s + \|w\|_s)\|w\|_{k_0+m}\|v\|_{k_0+m} \\
& + \gamma'_s(\|u\|_{s-1}, \|w\|_{s-1})\|w\|_{s-1}\|v\|_{s-1},
\end{aligned}
$$

where γ'_s is a polynomial with positive coefficients. Since $\|w\|_{k_0+m} \leq \|w\|_{2k_0+m} \leq \|w\|_{s-1}$, putting

$$P_s(\|u\|_s, \|w\|_{s-1}) = C''\|u\|_s + \gamma'_s(\|u\|_{s-1}, \|w\|_{s-1}),$$

we have the desired inequality, where C'' is some constant.

6. **Local expression of the connection and proof of Theorem B(ii).**

(a) *Local expression of* $\overset{0}{\nabla}$ *and* $\overset{1}{\nabla}$. In this section, s is always provided $\geq \dim M + 2r + 5$, where $2r$ is the order of the differential operator $\square$ in §0. Recall the definition of $\xi : \mathbf{U} \to \mathfrak{D}_0$. This gives a smooth chart at e and so also $\xi : U^s \to \mathfrak{D}_0^s$ does. For any $v \in \mathbf{\Gamma}(T_M)$, let $\tilde{v}$ denote the vector field on $\mathbf{U}$ defined by parallel displacement of v using the linear structure of $\mathbf{\Gamma}(T_M)$. So $d\xi\tilde{v}$ is a smooth vector field on $\xi(U)$, where $d\xi$ is the derivative of ξ. Let $d\xi_u$ be the derivative at u. For the connections $\tilde{\nabla}$, $\hat{\nabla}$ (cf. §3), define $\tilde{\Gamma}_u(w,v)$, $\hat{\Gamma}_u(w,v)$ as follows:

$$\tilde{\Gamma}_u(w,v) = d\xi_u^{-1}\tilde{\nabla}_{d\xi_u w}d\xi\tilde{u}, \tag{16}$$

$$\hat{\Gamma}_u(w,v) = d\xi_u^{-1}\hat{\nabla}_{d\xi_u w}d\xi\tilde{v}. \tag{17}$$

Recollect the defining equality (9) of $\hat{\nabla}$. Let $\pi' = d\xi^{-1}\tilde{\pi}d\xi$ (the local expression of $\tilde{\pi}$). Then, by a simple computation together with the fact

$$d\xi_u^{-1}\tilde{\nabla}_{d\xi_u w}d\xi\pi'\tilde{v} = d'_u(w)v + \tilde{\Gamma}_u(w,\pi'v),$$

we have the following

Lemma 7.

$$\hat{\Gamma}_u(w,v) = \pi'\tilde{\Gamma}_u(w,\pi'v) + (1-\pi')\tilde{\Gamma}_u(w,(1-\pi')v) + (2\pi'-1)d\pi'_u(w)v,$$

where $d\pi'_u$ is the derivative of π' at u. Taking the derivative,

$$\begin{aligned} d\hat{\Gamma}_u(u')(w,v) &= \pi' d\tilde{\Gamma}_u(u')(w,\pi'v) + (1-\pi')d\tilde{\Gamma}_u(u')(w,(1-\pi')v) \\ &\quad + d\pi'_u(u')\tilde{\Gamma}_u(w,(2\pi'-1)v) + (2\pi'-1)\tilde{\Gamma}_u(w,d\pi'_u(u')v) \\ &\quad + 2d\pi'_u(u')d\pi'_u(w)v + (2\pi'-1)d^2\pi'_u(u',w)v, \end{aligned}$$

where $d^2\pi'_u$ is the second derivative of π' at u.

Let Exp_x be the exponential mapping defined by the original connection on M from which the mapping Ξ and hence $\tilde{\nabla}$ is defined (cf. §3). Let $(d\,\mathrm{Exp}_x)_{u(x)}$, $(d^2\mathrm{Exp}_x)_{u(x)}$ be the first and second derivatives of Exp_x at $u(x)$. Then, by Lemma 4 in [9], we have the following fact:

$$\tilde{\Gamma}_u(w,v)(x) = d\,\mathrm{Exp}_x^{-1}(d^2\mathrm{Exp}_x)_{u(x)}((d\,\mathrm{Exp}_x)_{u(x)}w(x), v(x)).$$

As a matter of course u is restricted in the following set: Let W be a relatively compact tubular neighborhood of zero section of T_M such that $\mathrm{Exp}_x : \bar{W}_x \to M$ is a diffeomorphism of $\bar{W}_x$ onto an open neighborhood of $x \in M$ for each x, where W_x is the fibre of W at x and $\bar{W}_x$ is its closure. Then, u is restricted in the set $\Gamma^{k_0}(W) = \{u \in \Gamma^{k_0}(T_M); u(x) \in W \text{ for every } x\}$, where $k_0 = [\frac{1}{2}\dim M] + 1$.

Let p be the projection of T_M. Define a smooth mapping γ of $\bar{W}$ into $T_M^* \otimes T_M^* \otimes T_M$ by

$$\gamma(X)(Y,Z) = d\,\mathrm{Exp}_{pX}^{-1}(d^2\mathrm{Exp}_{pX})_X((d\,\mathrm{Exp}_{pX})_X Y, Z).$$

Then, γ is a fibre preserving mapping and obviously

$$\tilde{\Gamma}_u(w, v)(x) = \gamma(u(x))(w(x), v(x)).$$

The derivative $d\tilde{\Gamma}_u$ of $\tilde{\Gamma}$ at u is given by

$$d\tilde{\Gamma}_u(u')(w, v)(x) = \partial\gamma_{u(x)}(u'(x))(w(x), v(x)),$$

where $\partial\gamma_{u(x)}$ is the derivative of γ at $u(x)$ along the fibre. Obviously, the important thing is not the explicit expression of $\tilde{\Gamma}$ but the fact that $\tilde{\Gamma}$ is defined from a smooth fibre preserving mapping γ.

By Theorems A, B [9], we have the following (cf. Corollary 4 in [9]):

Lemma 8. *Let* $\Gamma^s(W) = \Gamma^{k_0}(W) \cap \Gamma^s(T_M)$. *Let* $u \in \Gamma^s(W)$, u', v, $w \in \Gamma^s(T_M)$ *for* $s \geq \dim M + 5$. *Then*

$$(\Gamma, 1) \qquad \begin{aligned} \|\tilde{\Gamma}_u(w, v)\|_s \leq\ & C\{\|w\|_s\|v\|_{k_0} + \|w\|_{k_0}\|v\|_s\} \\ & + C^2\|u\|_s\|w\|_{k_0}\|v\|_{k_0} + P_s(\|u\|_{s-1})\|w\|_{s-1}\|v\|_{s-1}, \end{aligned}$$

$$(\Gamma, 2) \qquad \begin{aligned} \|d\tilde{\Gamma}_u(u')(w, v)\|_s \leq\ & C\{\|u'\|_s\|w\|_{k_0}\|v\|_{k_0} + \|u'\|_{k_0}\|w\|_s\|v\|_{k_0} + \|u'\|_{k_0}\|w\|_{k_0}\|v\|_s\} \\ & + D\|u\|_s\|u'\|_{k_0}\|w\|_{k_0}\|v\|_{k_0} \\ & + Q_s(\|u\|_{s-1})\|u'\|_{s-1}\|w\|_{s-1}\|v\|_{s-1}, \end{aligned}$$

where C, D *are constants independent from* s *and* P_s, Q_s *are polynomials with positive coefficients.*

$(\Gamma, 1)$, $(\Gamma, 2)$ imply that the connection $\tilde{\nabla}$ satisfies (I), (II) in §1(a), putting $k = s - k_0$.

(b) *Proof of Theorem* B(ii). For the proof of Theorem B(ii) we have only to establish some inequalities of types (I), (II) with respect to $\tilde{\Gamma}$. Therefore taking Lemmas 7–8 into account, we have only to estimate $\|\pi'v\|_s$, $\|d\pi'_u(u')v\|_s$ and $\|d^2\pi'_u(u', w)v\|_s$. Let $\pi'_u = d\xi_u^{-1}\tilde{\pi}d\xi_u$, that is, the mapping π' at u. Since $\tilde{\pi} = \tilde{\square}^{-1}(B^*B)^{\sim}$ (cf. §4(a)), we see

$$\pi'_u = d\xi_u^{-1}\tilde{\square}^{-1}\tau_{\xi(u)}\tau_{\xi(u)}^{-1}(B^*B)^{\sim}d\xi_u,$$

where τ is the mapping defined in the first part of §5 (originally in §3) and $\tilde{\square}^{-1}$ is defined as a mapping of $\gamma_{\square}^{s,s-2r}(T_M)$ onto $\gamma_{\square}^{s}(T_M)$. So, to estimate $\|\pi'_u v\|_s$, etc., we have to estimate $\|d\xi_u^{-1}\tilde{\square}^{-1}\tau_{\xi(u)}v\|_s$, etc. This is not so simple, because $d\xi_u^{-1}\tilde{\square}^{-1}\tau_{\xi(u)}$ is defined not on $\Gamma(T_M)$ but on a finite codimensional subspace of $\Gamma(T_M)$, which depends on u.

Now, consider $\tau_{\xi(u)}^{-1}d\xi_u$ and $d\xi_u^{-1}\tau_{\xi(u)}$. The first one is the case $A = \mathrm{id}$ in the argument of §5. Let $\tau'_{\mathrm{Exp}\,X}Y$ be the parallel displacement of the tangent vector Y

of M along the curve $\operatorname{Exp} tX$, $0 \leq t \leq 1$, in M. Let p be the projection of T_M. Define a fibre preserving smooth mapping ψ of W into $T_M^* \otimes T_M$ by

$$\psi(X)Y = d\,\operatorname{Exp}_{pX}^{-1} \tau_{\operatorname{Exp} X} Y,$$

where W is the same tubular neighborhood as in Lemma 8 and Exp_{pX} implies the exponential mapping at pX. Then, obviously, $(d\xi_u^{-1}\tau_{\xi(u)}v)(x) = \psi(u(x))v(x)$.

Assume u is restricted in a bounded set in $\Gamma^{2k_0}(T_M)$. Let $\Gamma^s(W)$ be the same set as in Lemma 8. Then, by the inequalities (12), (14) and (15), we have

Lemma 9. *Let $\Psi(u)v$ be $d\xi_u^{-1}\tau_{\xi(u)}v$ or $\tau_{\xi(u)}^{-1}d\xi_u v$. Suppose furthermore that $u \in \Gamma^s(W)$, u', v, $w \in \Gamma^s(T_M)$. Then,*

(a) $$\|\Psi(u,v)v\|_s \leq C\|u\|_s\|v\|_{k_0} + D\|v\|_s + \gamma_s(\|u\|_{s-1})\|v\|_{s-1},$$

(b) $$\begin{aligned}\|d\Psi_u(u',v)\|_s \leq{}& C\{\|u'\|_s\|v\|_{k_0} + \|u'\|_{k_0}\|v\|_s\} + D\|u\|_s\|u'\|_{k_0}\|v\|_{k_0}\\ &+ \gamma_s(\|u\|_{s-1})\|u'\|_{s-1}\|v\|_{s-1},\end{aligned}$$

(c) $$\begin{aligned}\|d^2\Psi_u(w,u',v)\|_s \leq{}& C\{\|w\|_s\|u'\|_{k_0}\|v\|_{k_0} + \|w\|_{k_0}\|u'\|_s\|v\|_{k_0} + \|w\|_{k_0}\|u'\|_{k_0}\|v\|_s\}\\ &+ D\|u\|_s\|w\|_{k_0}\|u'\|_{k_0}\|v\|_{k_0}\\ &+ \gamma_s(\|u\|_{s-1})\|w\|_{s-1}\|u'\|_{s-1}\|v\|_{s-1},\end{aligned}$$

where C, D are independent from s and γ_s is a polynomial with positive coefficients. Notations $d\Psi_u d^2\Psi_u$ have the same meaning as in (12), (14), (15).

Now, let t_j be a fixed element of $\Gamma(T_M)$. Define a mapping $T^j : \mathbf{U} \to \Gamma(T_M)$ by $T^j(u) = \tau_{\xi(u)}^{-1} dR_{\xi(u)} t_j$. By the property $(\mathfrak{D}_0, 7)$ in §3, T^j can be extended to the smooth mapping of U^s into $\Gamma^s(T_M)$ for any $s \geq \dim M + 5$. Let $\psi : W \to T_M$ be the smooth fibre preserving mapping defined by

$$\psi(X) = \tau'^{-1}_{\operatorname{Exp} X} t^j(\operatorname{Exp} X).$$

Then, obviously $T^j(u)(x) = \psi(u(x))$. Therefore, applying Theorem A [9] (cf. §5) in the case $F = M \times R$ and $v \equiv 1$, we have

Lemma 10. *Assumptions being as in Lemma* 9, *we have*

(a) $$\|T^j(u)\|_s \leq C\|u\|_{s-1} + \gamma_s(\|u\|_{s-1}),$$

(b) $$\|dT_u^j(u')\|_s \leq C\|u\|_s\|u'\|_{k_0} + D\|u'\|_{k_0} + \gamma_s(\|u\|_{s-1})\|u'\|_{s-1},$$

(c) $$\begin{aligned}\|d^2T_u^j(u',v)\|_s \leq{}& C\{\|u'\|_s\|v\|_{k_0} + \|u'\|_{k_0}\|v\|_s\} + D\|u\|_s\|u'\|_{k_0}\|v\|_{k_0}\\ &+ \gamma_s(\|u\|_{s-1})\|u'\|_{s-1}\|v\|_{s-1}.\end{aligned}$$

Now, recollect the definition of $\mathbf{F}_1$, $\mathbf{F}_2$ and H in ($\bigstar$), §0. Let $\tilde{H}^s = \{dR_g H; g \in \mathfrak{D}_0^s\}$. Then $\tilde{H}^s$ is a smooth right-invariant subbundle of $\gamma^{s,s'}(T_M)$ for any $s' \leq s$ (cf. (γ, 6) in §3). Obviously, $\gamma^{s,s'}(T_M) = \gamma_{\square}^{s,s'}(T_M) \oplus \tilde{H}^s$ (cf. §4(a)). Let P be the projection of $\Gamma(T_M)$ onto H. Define a bundle morphism $\tilde{P}$ by $dR_g P dR_g^{-1}$. Then, this mapping $\tilde{P} : \gamma(T_M) \to \tilde{H}$ ($\tilde{H} = \{dR_g H; g \in \mathfrak{D}_0\}$) can be extended to the smooth mapping of $\gamma^{s,s'}(T_M)$ onto $\tilde{H}^s$. This is because $\gamma_{\square}^{s,s'}(T_M)$ and $\tilde{H}^s$ are smooth subbundles of $\gamma^{s,s'}(T_M)$. Let $P''_u = \tau_{\xi(u)}^{-1} \tilde{P} \tau_{\xi(u)}$. Then P'' is a mapping of $U^s \times \Gamma^{s'}(T_M)$ into $\Gamma^{s'}(T_M)$.

Now let $t_j, j = 1, 2, \ldots, m$, be a basis of H. Then, P is given by

$$Pv = \sum_j \int_M \langle v(x), t_j(x) \rangle_x \mu(x)\, t_j,$$

where $\langle\, , \rangle_x$ is the riemannian inner product of T_M. Therefore, for any $v' \in T_g \mathfrak{D}_0$, we have

$$\begin{aligned} \tilde{P}v' &= \sum_j \int_M \langle (dR_g^{-1} v')(x), t_j(x) \rangle_x \mu(x)\, dR_g t_j \\ &= \sum_j \int_M \langle v'(x), (dR_g t_j)(x) \rangle_{gx} \mu(gx)\, dR_g t_j. \end{aligned}$$

Notice that $(dR_g w)(x) = w(gx)$. Therefore, we have

$$P''_u v = \sum \int \langle \tau_{\xi(u)} v(x), \tau_{\xi(u)} T^j(u)(x) \rangle_{\xi(u)(x)} \mu(\xi(u)(x)) T^j(u).$$

Since parallel displacements preserve the inner product, we have

$$P''_u v = \sum_j \int_M \langle v(x), T^j(u)(x) \rangle_x J(u) \mu(x)\, T^j(u) \tag{18}$$

where $J(u)(x)$ is the Jacobian of $\xi(u)$ at x.

Now, let W^1 be a sufficiently small tubular neighborhood of zero section of $J^1 T_M$. Let $\Gamma(W^1) = \{v \in \Gamma(T_M); (j^1 v)(x) \in W^1\}$ and $\Gamma^s(W^1) = \{v \in \Gamma^s(T_M); (j^1 v)(x) \in W^1\}$. This is well defined for $s \geq k_0 + 1$. If we take W^1 very small, then there exists a constant C such that

(a)
$$\max_x |T^j(u)(x)| \leq C, \qquad \max_x |J(u)(x)| \leq C,$$

(b)
$$\begin{aligned} \max_x |dT_u^j(u')(x)| &\leq C \max_x |u'(x)| \\ \max_x |dJ_u(u')(x)| &\leq C \max_x |(j^1 u')(x)|, \end{aligned}$$

(c)
$$\begin{aligned} \max_x |d^2 T_u^j(u', v)(x)| &\leq C \max_x |u'(x)| \max_x |v(x)|, \\ \max_x |d^2 J_u(u', v)(x)| &\leq C \max_x |(j^1 u')(x)| \max_x |(j^1 v)(x)|, \end{aligned}$$

where $|\ |$ implies the length of the vector at that point with respect to some inner products defined on T_M, $J^1 T_M$, etc.

Notice that $\max_x |w(x)| \leq e_{k_0} \|w\|_{k_0}$ by the Sobolev embedding theorem and $\|j^1 w\|_{k_0} \leq e' \|w\|_{k_0+1}$ for some constants e_{k_0}, e'. Thus, combining these facts with Lemma 10, we have

Lemma 11. *Suppose u is restricted in a bounded set in $\Gamma^{2k_0}(T_M)$. Then, for any u', $v \in \Gamma^s(T_M)$ and for $u \in \Gamma^s(W) \cap \Gamma^s(W^1)$,*

$$\text{(a)} \qquad \|P''_u v\|_s \leq C\|u\|_s \|v\|_{k_0} + \gamma_s(\|u\|_{s-1})\|v\|_{k_0},$$

$$\text{(b)} \qquad \begin{aligned} \|dP''_u(u')v\|_s \leq{} & C\|u'\|_s \|v\|_{k_0} + D\|u\|_s \|u'\|_{k_0} \|v\|_{k_0} \\ & + \gamma_s(\|u\|_{s-1})\|u'\|_{s-1} \|v\|_{k_0}, \end{aligned}$$

$$\text{(c)} \qquad \begin{aligned} \|d^2 P''_u(w, u')v\|_s \leq{} & C\{\|w\|_s \|u'\|_{k_0} + \|w\|_{k_0} \|u'\|_s\}\|v\|_{k_0} \\ & + D\|u\|_s \|w\|_{k_0} \|u'\|_{k_0} \|v\|_{k_0} \\ & + \gamma_s(\|u\|_{s-1})\|w\|_{s-1} \|u'\|_{s-1} \|v\|_{k_0}, \end{aligned}$$

where C, D are independent from s and γ_s is a polynomial with positive coefficients.

Now, by Theorems A, B [9] in §5, for the differential operators $\Box$, B^*B, there exists a tubular neighborhood W^{2r} of zero section of $J^{2r}T_M$ such that the inequalities (12), (14), (15) hold for any $u \in \Gamma^s(W^{2r}) \cap U^s$ as far as u is restricted in a bounded set of $\Gamma^{2k_0+2r}(T_M)$ (cf. §5).

To simplify the argument below, we put $k_1 = \dim M + 2r + 5$, $V_\delta = \{v \in \Gamma^{k_1}(T_M); \|v\|_{k_1} < \delta\}$. Then, there is $\delta > 0$ such that $V_\delta \subset U^{k_1} \cap \Gamma^{k_1}(W) \cap \Gamma^{k_1}(W^1) \cap \Gamma^{k_1}(W^{2r})$. Put $V_\delta^s = V_\delta \cap \Gamma^s(T_M)$ for any $s \geq k_1$. Since $k_0 \leq k_0 + 2r \leq 2k_0 + 2r \leq k_1$ and hence $\|v\|_{k_0} \leq \|v\|_{k_0+2r} \leq \|v\|_{2k_0+2r} \leq \|v\|_{k_1}$, all the inequalities obtained in the above arguments hold by changing $\|v\|_{k_0}$, $\|v\|_{k_0+2r}$ to $\|v\|_{k_1}$.

Lemma 12. *There exists $\delta > 0$ such that*

$$\|\tau_{\xi(u)}^{-1} \tilde{\Box}\, d\xi_u v\|_{s-2r} \geq C\|v\|_s - D\|u\|_s \|v\|_{k_1} - \gamma_s(\|u\|_{s-1})\|v\|_{s-1}$$

holds for any $u \in V_\delta^s$, $v \in \Gamma^s(T_M)$.

Proof. By Lemma 4, we have

$$\|\Box\, v\|_{s-2r} \geq C'\|v\|_s - \gamma'_s \|v\|_{s-1}.$$

On the other hand, Lemma 6 shows

$$\|\tau_{\xi(u)}^{-1} \tilde{\Box}\, d\xi_u v - \Box\, v\|_{s-2r} \leq C''\{\|u\|_s \|v\|_{k_1} + \|u\|_{k_1} \|v\|_s\} + \gamma''_s(\|u\|_{s-1})\|u\|_{s-1} \|v\|_{s-1}.$$

So, put $\delta = C'/2C''$ and we get the desired inequality using the relation

$$\|\tau_{\xi(u)}^{-1} \tilde{\Box}\, d\xi_u v\|_{s-2r} \geq \|\Box\, v\|_{s-2r} - \|\tau_{\xi(u)}^{-1} \tilde{\Box}\, d\xi_u v - \Box\, v\|_{s-2r}.$$

Now, let P, $\tilde{P}$ be the same projections defined between Lemma 10 and Lemma 11. Let $L : \Gamma(T_M) \to \Gamma(T_M)$ be the mapping defined by $Lv = \Box\,(1 - P)v + Pv$. Then, L can be extended to the isomorphism of $\Gamma^s(T_M)$ onto $\Gamma^{s-2r}(T_M)$. For simplicity, put $\Box'_u = \tau_{\xi(u)}^{-1} \tilde{\Box}\, d\xi_u$, $I_u = \tau_{\xi(u)}^{-1} d\xi_u$ and $I_u^{-1} = d\xi_u^{-1} \tau_\xi(u)$. Let $\tilde{L}$ be the bundle isomorphism defined by $dR_g L dR_{g^{-1}}$. Let L'_u denote the mapping defined by $\tau_{\xi(u)}^{-1} \tilde{L} d\xi_u$. L'_u is smooth with respect to u, because $\tilde{L} : \gamma^s(T_M) \to \gamma^{s,s-2r}(T_M)$ is. By a simple computation, we have

$$L'_u v = \Box'_u\,(1 - I_u^{-1} P''_u I_u)v + P''_u I_u v.$$

Lemma 13. *There exist $\delta > 0$, C, D and γ_s such that*

$$\|L'_u v\|_{s-2r} \geq C\|v\|_s - D\|u\|_s \|v\|_{k_1} - \gamma_s(\|u\|_{s-1})\|u\|_{s-1}$$

for any $u \in V_\delta^s$, $v \in \Gamma^s(T_M)$.

Proof. For sufficiently small δ, we may assume that there is a constant $K > 0$ such that

$$\|I_u v\|_{k_1} \leq K\|v\|_{k_1}, \quad \|I_u^{-1} v\|_{k_1} \leq K\|v\|_{k_1}, \quad \|P''_u v\|_{k_1} \leq K\|v\|_{k_1}$$

for any $u \in V_\delta$. This is because all maps are smooth with respect to $u \in \Gamma^{k_1}(T_M)$. On the other hand,

$$\|L'_u v\|_{s-2r} \geq \|\Box'_u v\|_{s-2r} - \|\Box'_u I_u^{-1} P''_u I_u v\|_{s-2r} - \|P''_u I_u v\|_{s-2r}.$$

By using Lemma 12 and inequality (12), we have

$$\begin{aligned}\|L'_u v\|_{s-2r} \geq\ & C\|v\|_s - D\|u\|_s \|v\|_{k_1} - \gamma_s(\|u\|_{s-1})\|v\|_{s-1} \\ & - C'\|I_u^{-1} P''_u I_u v\|_s - D'\|u\|_s \|v\|_{k_1} \\ & - \gamma'_s(\|u\|_{s-1})\|I_u^{-1} P''_u I_u v\|_{s-1} - \|P''_u I_u v\|_{s-2r}.\end{aligned}$$

By Lemma 9, we have

$$\|I_u^{-1} P''_u I_u v\|_s \leq C''\|P''_u I_u v\|_s + D''\|u\|_s \|v\|_{k_1} + \gamma''_s(\|u\|_{s-1})\|P''_u I_u v\|_{s-1}.$$

On the other hand, Lemma 11 shows

$$\|P''_u I_u v\|_s \le C\|u\|_s \|I_u v\|_{k_1} + \gamma_s(\|u\|_{s-1})\|I_u v\|_{k_1}.$$

Therefore, we have the desired inequality.

Let $G_u = L_u'^{-1}$. Then, G_u is smooth with respect to u. For sufficiently small δ, there is $K > 0$ such that $\|G_u v\|_{k_1} \le K\|v\|_{k_1-2r}$ for any $u \in V_\delta$. By Lemma 13, we have

$$\|G_u v\|_s \le C'\|v\|_s + D''\|u\|_s \|v\|_{k_1-2r} + \gamma_s(\|u\|_{s-1})\|G_u v\|_{s-1}.$$

Therefore, by induction, we have

$$\|G_u v\|_s \le C'\|v\|_{s-2r} + D'\|u\|_s \|v\|_{k_1-2r} + \gamma'_s(\|u\|_{s-1})\|v\|_{s-2r-1}. \tag{19}$$

Let $\beta_u v = \tau_{\xi(u)}^{-1}(B^*B)^{\sim} d\xi_u v$. Since $\pi'_u = G_u \beta_u$ and if δ is sufficiently small, then $\|\beta_u v\|_{k_1-2r} \le K\|v\|_{k_1}$ for any $u \in V_\delta$, combining (12) and (19), we have

$$\|\pi'_u v\|_s \le C\|v\|_s + D\|u\|_s \|v\|_{k_1} + \gamma_s(\|u\|_{s-1})\|\beta_u v\|_{s-2r-1}.$$

By using (12) successively, we have

$$\|\pi'_u v\|_s \le C\|v\|_s + D\|u\|_s \|v\|_{k_1} + \gamma_s(\|u\|_{s-1})\|v\|_{s-1}. \tag{20}$$

Lemma 14. *For a sufficiently small $\delta > 0$,*

$$\begin{aligned}\|dL'_u(u')v\|_{s-2r} \le{}& C\{\|u'\|_s \|v\|_{k_1} + \|u'\|_{k_1} \|v\|_s\} + D\|u\|_s \|u'\|_{k_1} \|v\|_{k_1} \\ &+ \gamma_s(\|u\|_{s-1})\|u'\|_{s-1} \|v\|_{s-1},\end{aligned}$$

$$\begin{aligned}\|d^2 L'_u(u', w)v\|_{s-2r} \le{}& C\{\|u'\|_s \|w\|_{k_1} \|v\|_{k_1} + \|u'\|_{k_1} \|w\|_s \|v\|_{k_1} + \|u'\|_{k_1} \|w\|_{k_1} \|v\|_s\} \\ &+ D\|u\|_s \|u'\|_{k_1} \|w\|_{k_1} \|v\|_{k_1} \\ &+ \gamma_s(\|u\|_{s-1})\|u'\|_{s-1} \|w\|_{s-1} \|v\|_{s-1}\end{aligned}$$

hold for any $u \in V_\delta^s$, u', w, $v \in \Gamma^s(T_M)$, where C, D are independent from s.

Proof. Since $L'_u v = \Box'_u (1 - I_u^{-1} P''_u I_u v) + P''_u I_u v$, inequalities (12), (14), (15) and inequalities in Lemma 9 and Lemma 11 yield a desired result by simple computations.

Since $dG_u(u') = -G_u dL'_u(u')G_u$ and

$$\begin{aligned}d^2 G_u(u', w) ={}& G_u dL'_u(u')G_u dL'_u(w)G_u + G_u dL'_u(w)G_u dL'_u(u')G_u \\ &- G_u d^2 L'_u(u', w)G_u,\end{aligned}$$

inequality (19) and Lemma 14 give the following:

Lemma 15. *For sufficiently small* $\delta > 0$,

$$\|G_u v\|_s \leq C\|v\|_{s-2r} + D\|u\|_s \|v\|_{k_1-2r} + \gamma_s(\|u\|_{s-1})\|v\|_{s-2r-1},$$

$$\begin{aligned}\|dG_u(u')v\|_s \leq\ & C\{\|u'\|_s \|v\|_{k_1-2r} + \|u'\|_{k_1} \|v\|_{s-2r}\} \\ & + D\|u\|_s \|u'\|_{k_1} \|v\|_{k_1-2r} + \gamma_s(\|u\|_{s-1})\|u'\|_{s-1} \|v\|_{s-2r-1},\end{aligned}$$

$$\begin{aligned}\|d^2 G_u(u', w)v\|_s \leq\ & C\{\|u'\|_s \|w\|_{k_1} \|v\|_{k_1-2r} + \|u\|_{k_1} \|w\|_s \|v\|_{k_1-2r} \\ & + \|u'\|_{k_1} \|w\|_{k_1} \|v\|_{s-2r}\} \\ & + D\|u\|_s \|u'\|_{k_1} \|w\|_{k_1} \|v\|_{k_1-2r} \\ & + \gamma_s(\|u\|_{s-1})\|u'\|_{s-1} \|w\|_{s-1} \|v\|_{s-2r-1},\end{aligned}$$

hold for any $u \in V_\delta^s$, u', $w \in \Gamma^s(T_M)$, $v \in \Gamma^{s-2r}(T_M)$.

Now, since $\pi'_u = G_u \beta_u$, one can estimate $\|\pi'_u v\|_s$, $\|d\pi_u(u')v\|_s$, $\|d^2\pi'_u(u', w)v\|_s$ by the similar method as above. Combining those resulting inequalities with Lemmas 7–8, we obtain finally the following

Proposition 2. *There exists* $\delta > 0$ *such that the following inequalities hold for* $u \in V_\delta^s$, u', v, $w \in \Gamma^s(T_M)$:

$$(\hat{\Gamma}, 1) \qquad \begin{aligned}\|\hat{\Gamma}_u(w, v)\|_s \leq\ & C\{\|w\|_s \|v\|_{k_1} + \|w\|_{k_1} \|v\|_s\} + C^2 \|u\|_s \|w\|_{k_1} \|v\|_{k_1} \\ & + P_s(\|u\|_{s-1})\|w\|_{s-1} \|v\|_{s-1},\end{aligned}$$

$$(\hat{\Gamma}, 2) \qquad \begin{aligned}\|d\hat{\Gamma}_u(u')(w, v)\|_s \leq\ & C\{\|u'\|_s \|w\|_{k_1} \|v\|_{k_1} + \|u'\|_{k_1} \|w\|_s \|v\|_{k_1} + \|u'\|_{k_1} \|w\|_{k_1} \|v\|_s\} \\ & + D\|u\|_s \|u'\|_{k_1} \|w\|_{k_1} \|v\|_{k_1} \\ & + Q_s(\|u\|_{s-1})\|u'\|_{s-1} \|w\|_{s-1} \|v\|_{s-1},\end{aligned}$$

where C, D *are constants independent from* s *and* P_s, Q_s *are polynomials with positive coefficients depending on* s.

These $(\hat{\Gamma}, 1)$, $(\hat{\Gamma}, 2)$ imply that the connection $\hat{\nabla}$ satisfies (I) and (II) by putting $k = s - k_1$. Therefore, the regularity of connection is ensured (cf. Theorem, §1(a)). This completes the proof of Theorem B(ii).

7. **Local expression of** $\mathfrak{g}^s$ **and proof of Theorem E.**

(a) *Local expression of* $\mathfrak{g}^s$. Recall the statement of Theorem E, and a basic idea (b), (c) in §1. To prove Theorem E, we have only to check (1), (2), (3) in (c) and establish inequalities (I″), (II″) for the mapping L which will be defined below.

Now, recollect the splitting (★) in §0. In a similar manner, we have the following splitting (cf. Lemma 6 of [10]):

$$\Gamma(E) = \mathbf{F}_1 \oplus \mathbf{F}_2 \oplus H, \quad \Gamma^s(E) = F_1^s \oplus F_2^s \oplus H, \qquad s \geq 0,$$

$$\mathbf{F}_1 = A\Gamma(T_M), \quad F_1^s = A\Gamma^{s+r}(T_M), \quad \mathbf{F}_2 = B^*\Gamma(F), \quad F_2^s = B^*\Gamma^{s+r}(F),$$

$$H = \text{Ker of } AA^* + B^*B.$$

Let $\square = AA^* + B^*B$. By assumption, this is an elliptic operator. Notice that $\square : F_i^{s+2r} \to F_i^s$ is an isomorphism for $i = 1, 2$ and the splitting ($\bigstar\bigstar$) in §0 is given by

$$w = (w - A^* \square^{-1} Aw) + A^* \square^{-1} Aw.$$

Closedness of $\mathfrak{m}^s = A^*\Gamma^{s+r}(E)$ follows from $\square : F_1^{s+2r} \to F_1^s$ is an isomorphism.

Similarly, let $\mathfrak{h} = \operatorname{Ker}\{B^* : \Gamma(F) \to \Gamma(E)\}$, $\mathfrak{h}^s = \operatorname{Ker}\{B^*\Gamma^s(F) \to \Gamma^{s-r}(E)\}$, $\mathfrak{k} = B\Gamma(E)$ and $\mathfrak{k}^s = B\Gamma^{s+r}(E)$. Then, $\mathfrak{k}^s$ is also closed and $\Gamma(F)$ has the splitting $\Gamma(F) = \mathfrak{h} \oplus \mathfrak{k}$, $\Gamma^s(F) = \mathfrak{h}^s \oplus \mathfrak{k}^s$. This is given by

$$u = (u - B\square^{-1} B^*u) + B\square^{-1} B^*u.$$

Let q and P be projections of $\Gamma(E)$ onto F_1 and H respectively. Then, these can be extended to the projections of $\Gamma^s(E)$ onto F_1^s and H respectively. Since $q = AA^* \square^{-1} (1 - P)$, inequality (12) (or directly, the inequality just before Lemma 5) and Lemma 4 show that there is a constant C' independent from s such that

$$\|qu\|_s \leq C'\|u - Pu\|_s + D'_s\|u - Pu\|_{s-1}.$$

However, it is easy to see that Lemma 11 holds also in this case. Therefore, there is a constant C such that

$$\|qu\|_s \leq C\|u\|_s + D_s\|u\|_{s-1}. \tag{21}$$

Now, let $k_1 = \dim M + 2r + 5$. Recall the definition of local trivialization $d\xi$, τ, τ'' in the first part of §5. For simplicity we denote $\tau_{\xi(u)}^{-1}\tilde{A}d\xi_u$, $\tau_{\xi(u)}^{-1}\tilde{B}\tau_{\xi(u)}$ by A'_u, B'_u respectively.

Let W be an open neighborhood of 0 in $\Gamma^{k_1+r}(T_M)$ satisfying the following:

(a) $$P\tau_{\xi(u)}^{-1}R_{\xi(u)}^* : H \to H \text{ is isomorphic for any } u \in W,$$

(b) $$\tilde{q}B'_u : F_2^{k_1+r} \to \mathfrak{k}^{k_1} \text{ is isomorphic for any } u \in W,$$

where $\tilde{q}$ is the projection of $\Gamma^{k_1}(F)$ onto $\mathfrak{k}^{k_1}$. Such W exists because these mappings are smooth with respect to (u, v), $v \in H$ or $F_2^{k_1}$, and $H \subset \Gamma(E)$. (Cf. notes, between (1)–(3) and (I″), (II″) in §1(c).)

Since $\mathfrak{D}_0^{k_1}$ is a topological group, we may assume that W satisfies $\xi^{-1}(\xi(W)^{-1}) = W$. Let W^s (resp. $\mathbf{W}$) denote the intersection $W \cap \Gamma^s(T_M)$ (resp. $W \cap \Gamma(T_M)$).

Define a mapping $L : W \times \Gamma(T_M) \to \Gamma(E)$ by $L(u, v) = qA'_u v$, Then, we have

Lemma 16. *L satisfies the conditions* (1), (2), *and* (3) *in* §1(c) *by putting* $E^k = \Gamma^{k+k_1+r}(T_M)$, $F^k = F_1^{k+k_1}$.

Proof. The properties (1) and (2) are trivial, because of the smooth extension theorem. We have only to show (3). Since $\tilde{A}$ is a smooth bundle morphism of $\gamma^s(T_M)$ into $\gamma^{s,s-r}(E)$ and the image is closed, Theorem 2 in [10] shows that the kernel of $\tilde{A}$ forms a smooth subbundle of $\gamma^s(T_M)$. So it is enough to prove that $\mathrm{Ker}\{A'_u : \Gamma^s(T_M) \to \Gamma^{s-r}(E)\} = \mathrm{Ker}\{L_u : \Gamma^s(T_M) \to F_2^{s-r}\}$ for any $s \geq k_1 + r$. The direction $\subset$ is trivial, so we have to prove $\supset$. Suppose there is v such that $L_u v = 0$ but $A'_u v \neq 0$ for some $u \in W^s$. Then, $A'_u v \in F_2^{s-r} \oplus H$. Since $\tilde{q}B'_u : F_2^{s-r} \to \mathfrak{k}^{s-2r}$ is injective by assumption, $B'_u A'_u v = 0$ implies that $A'_u v \in H$. Thus, $p\tau_{\xi(w)}^{-1} R^*_{\xi(w)} A'_u v \neq 0$ for any $w \in W^s$, because of the assumption. Put $w = \xi^{-1}(\xi(u)^{-1})$. Then,

$$P\tau_{\xi(w)}^{-1} R^*_{\xi(w)} A'_u v = P\tau_{\xi(w)}^{-1} R^*_{\xi(w)} \tau_{\xi(u)}^{-1} R^*_{\xi(u)} A R^*_{\xi(u)} d\xi_u v.$$

Since $\tau_{\xi(w)}^{-1} R^*_{\xi(w)} \tau_{\xi(u)}^{-1} R^*_{\xi(u)} = \mathrm{id}$ (this is ensured by using the explicit defining equalities given in §3), the fact $PA \equiv 0$ gives a contradiction, Thus $\supset$ is proved.

(b) *Proof of Theorem* E. By the above Lemma 16 and (b), (c) in §1, it is enough to prove the inequalities (I″), (II″) in §1(c). Let L_u denote the mapping defined by $L_u v = L(u, v)$. Since, **S**, **T** corresponds to $\mathfrak{g}$, $\mathfrak{m}$ respectively, we have to prove the inequality (I″) for the map $\tilde{L}_0 : \mathfrak{m} \to \mathbf{F}_1$. Recall that $\mathfrak{m} = A^*\Gamma(E)$ and $L_0 = A'_0 = A$. Notice that $A^*\Gamma(E) = A^*\mathbf{F}_1$. Therefore, using Lemma 4, we have the following for $v \in \mathfrak{m}$.

$$\|Av\|_{s-r} = \|AA^*u\|_{s-r} = \|\Box\, u\|_{s-r} \geq C\|u\|_{s+r} - \gamma_s \|u\|_{s+r-1},$$

where $u \in \mathbf{F}_1$. Since $v = A^*u$, inequality (12) gives

$$\|v\|_s \leq C'\|u\|_{s+r} + D'_s \|u\|_{s+r-1}.$$

Therefore

$$\|Av\|_{s-r} \geq C''\|v\|_s - D''_s \|u\|_{s+r-1}.$$

However, since $A^* : F_1^{s+r-1} \to \mathfrak{m}^{s-1}$ is isomorphic, there is $K_s > 0$ such that $\|v\|_{s-1} \geq K_s \|u\|_{s+r-1}$. Therefore, we have

$$\text{(I}''\text{)} \qquad \|L_0 v\|_{s-r} \geq C''\|v\|_s - K'_s \|v\|_{s-1}, \qquad s \geq k_1 + r.$$

Inequality (II″) is an immediate conclusion of Lemma 6 and (21). Thus, Theorem E is proved (cf. §1(b) and (c)).

8. **Proof of Theorem F.** Recollect, at first, the statements of Theorems E and F in §0. We use the same notation. Assume there exists $k \geq k_1$ such that G^k is

closed in $\mathfrak{D}_0^k$. This assumption is not special, becausc if G is closed in $\mathfrak{D}_0$, then one can find such G^k for some k.

Notice that $\mathfrak{D}_0^k$ is a topological space of second category, and we see G^k is also a set of second category as a closed subset of $\mathfrak{D}_0^k$. On the other hand consider the original topology for G^k. This is given in a natural manner as the maximal integral submanifold. So, this topology may be stronger than the relative topology for G^k. Since G^k is a Hilbert manifold modelled on $\mathfrak{g}^k$ (by this original topology) and a connected topological group, hence G^k is generated by any neighborhood of the identity, G^k satisfies the second countability axiom, because $\mathfrak{g}^k$ has that property. Notice, therefore we can apply the Baire category theorem.

Namely, considering a closed neighborhood W of 0 in $\Gamma^k(T_M)$ contained in $U^{k_1} \cap \Gamma^k(T_M)$, $\Psi(W \cap \mathfrak{g}^k)$ should be closed in $\mathfrak{D}_0^k$, because Ψ is a coordinate map. Since $\Psi(W \cap \mathfrak{g}^k)$ is a neighborhood of e in G^k with the original topology, the Baire category theorem shows that $\Psi(W \cap \mathfrak{g}^k)$ should contain the identity as an interior point of G^k with the relative topology. This implies that the original topology and the relative topology coincide under the assumption of G^k being closed in $\mathfrak{D}_0^k$.

The above fact implies that there exists a star-shaped neighborhood of 0 in $\Gamma^k(T_M)$ such that $G^k \cap \Psi(W) = \Psi(W \cap \mathfrak{g}^k)$. This property holds for any $s \geq k$, namely, $G^s \cap \Psi(W) = \Psi(W \cap \mathfrak{g}^s)$ (cf. Theorem E).

After this point, the arguments are only routine. Since G^k is a topological group, there exists an open neighbrohood V^k of 0 in $\Gamma^k(T_M)$ such that $\Psi(V^k) = \Psi(V^k)^{-1}$, $\Psi(V^k)^2 \subset \Psi(W)$. Remark that these properties hold for every $s \geq k$, that is, $\Psi(V^k \cap \Gamma^s(T_M)) = \Psi(V^k \cap \Gamma^s(T_M))^{-1}$, $\Psi(V^k \cap \Gamma^s(T_M))^2 \subset \Psi(W \cap \Gamma^s(T_M))$. Define $\tilde{\Psi} : G \times V^k \cap \mathfrak{m} \to \mathfrak{D}_0$ by $\tilde{\Psi}(g,u) = g\Psi(u)$. Obviously, this can be extended to the *continuous* mapping of $G^s \times V^k \cap \mathfrak{m}^s$ into $\mathfrak{D}_0^s$ for every $s \geq k$. This is only continuous, because the group multiplication is not differentiable (cf. Theorem E and ($\mathfrak{D}_0$, 1–7) in §3).

It is easy to see (using the star-shaped property of W) that $\tilde{\Psi} : G^s \times V^k \cap \mathfrak{m}^s \to \mathfrak{D}_0^s$ is injective.

Assume there exists a sequence $(g_n, u_n) \in G^s \times V^k \cap \mathfrak{m}^s$ such that $\tilde{\Psi}(g_n, u_n)$ converges to $\tilde{\Psi}(g,u)$. Then, $g^{-1}g_n\tilde{\Psi}(g_n,u_n)$ converges to

$$\Psi(u) \in \Psi(V^k \cap \Gamma^s(T_M)).$$

Therefore, for sufficiently large n, $g^{-1}g_n\Psi(u_n)$ is contained in $\Psi(V^k \cap \Gamma^s(T_M))$, so that $g^{-1}g_n \in \Psi(W \cap \Gamma^s(T_M))$. $g^{-1}g_n\Psi(u_n)$ should be expressed by $\Psi(y_n + u_n)$, where $y_n \in \mathfrak{g}^s$, because $g^{-1}g_n\Psi(u_n)$ is on a slice containing $\Psi(u_n)$. Since $g^{-1}g_n\Psi(u_n)$ converges to $\Psi(u)$, y_n has to converge to 0 and u_n does to u. This implies that g_n converges to g.

Openness of the image $\tilde{\Psi}(G^s \times V^k \cap \mathfrak{m}^s)$ is almost trivial.

So the final thing we have to do, is to check

$$\tilde{\Psi}(G^s \times V^k \cap \mathfrak{m}^s) = \tilde{\Psi}(G^k \times V^k \cap \mathfrak{m}^k) \cap \mathfrak{D}_0^s.$$

the $\subset$ direction is trivial. So assume $g\Psi(u) \in \mathfrak{D}_0^s$ for some $g \in G^k$, $u \in V^k \cap \mathfrak{m}^k$. Then, there exists $h \in G^s$ (because G^s is dense in G^k) such that $hg\Psi(u) \in \Psi(V^k)$. So there is $y \in \mathfrak{g}^k$ such that $\Psi(y + u) = hg\Psi(u) \in \mathfrak{D}_0^s$. Therefore, $u \in \Gamma^s(T_M)$, and hence $g \in \mathfrak{D}_0^s$. This implies $u \in \mathfrak{m}^s$, $g \in G^s$. This completes the proof of Theorem F.

Under the assumption of G^k being closed in $\mathfrak{D}_0^k$, Theorem E implies that $G^s \backslash \mathfrak{D}_0^s$ is a smooth Hilbert manifold for every $s \geq k$ and the projection π is a smooth mapping. Moreover $G \backslash \mathfrak{D}_0$ is a smooth Fréchet manifold and the projection π is also smooth.

However, Theorem F implies that $\mathfrak{D}_0^s$ is only a continuous principal bundle over $G^s \backslash \mathfrak{D}_0^s$. This is because Ψ is only a continuous mapping. Probably, if we extend the structure group to the group of diffeomorphisms of G^s, then $\mathfrak{D}_0^s$ has a smooth fibre bundle structure over $G^s \backslash \mathfrak{D}_0^s$, but this is another kind of question.

Since $\tilde{\Psi} : G^{s+l} \times V^k \cap \mathfrak{m}^s \to \mathfrak{D}_0^s$ is a C^l-mapping, $\mathfrak{D}_0$ is a smooth principal bundle over $G \backslash \mathfrak{D}_0$.

Denote by $[h]$ the equivalence class of h. Then, by using ($\mathfrak{D}_0$, 4–7), we have

(1) The mapping $([h], g) \to [hg]$ is a C^l-mapping of $G^{s+l} \backslash \mathfrak{D}_0^{s+l} \times \mathfrak{D}_0^s$ into $G^s \backslash \mathfrak{D}_0^s$.

(2) For a fixed g, the right translation $[h] \to [hg]$ is a smooth mapping of $G^s \backslash \mathfrak{D}_0^s$ onto itself, for every $g \in \mathfrak{D}_0^s$.

(3) The mapping $(g, [h]) \to [gh]$ is a C^l-mapping of $\mathfrak{D}_0^{s+1} \times G^s \backslash \mathfrak{D}_0^s$ into $G^s \backslash \mathfrak{D}_0^s$.

(4) For a fixed $[h] \in G^s \backslash \mathfrak{D}_0^s$, the mapping $g \to [gh]$ is a smooth mapping of $\mathfrak{D}_0^s$ onto $G^s \backslash \mathfrak{D}_0^s$.

Denote $[g]$ by $g_*[e]$. Then, obviously G^s is characterized by $\{g \in \mathfrak{D}_0^s; g_*[e] = [e]\}$. Taking the derivative of this and denoting it by $\mathfrak{l}_X[e]$, the Lie algebra of G is characterized by $\mathfrak{g} = \{X \in \Gamma(T_M); \mathfrak{l}_X[e] = 0\}$. This simple relation will be useful in considering the problem mentioned in the last part of the introduction.

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