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# ON THE ARENS PRODUCTS AND CERTAIN BANACH ALGEBRAS

BY

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ABSTRACT. In this paper, we study several problems in Banach algebras concerned with the Arens products.

**1. Introduction.** Let $A$ be a Banach algebra, $A^{**}$ its second conjugate space and $\pi_A$ the canonical embedding of $A$ into $A^{**}$. Arens has defined two natural extensions of the product on $A$ to $A^{**}$. Under either Arens product, $A^{**}$ becomes a Banach algebra. In §3, we show that if $A$ is a semisimple Banach algebra which is a dense two-sided ideal of a semisimple annihilator Banach algebra $B$, then $\pi_A(A)$ is a two-sided ideal of $A^{**}$ (with the Arens product). In particular, a semisimple annihilator Banach algebra has such property. This result greatly generalizes some recent results obtained by the author (see [12, p. 82] and [13, p. 830]).

In §4, we study the radical $R^{**}$ of $A^{**}$, where $A$ is a semisimple annihilator Banach algebra. We show that, under either Arens product, $R^{**}$ remains the same and it is the right annihilator of $A^{**}$. A similar result was obtained by Civin and Yood [5] for the group algebra of a compact abelian group.

§ 5 is devoted to the study of semisimple dual Banach algebras which are two-sided ideals of a $B^*$-algebra. Let $A$ be a semisimple dual Banach algebra which is a dense subalgebra of a $B^*$-algebra $B$ such that $\|\cdot\|$ majorizes $|\cdot|$ on $A$. We show that $A$ is a two-sided ideal of $B$ if and only if, for any orthogonal family of hermitian minimal idempotents $\{e_\lambda : \lambda \in \Lambda\}$ of $B$ and $x \in A$, $\sum_\lambda xe_\lambda$ and $\sum_\lambda e_\lambda x$ are summable in the norm of $A$. This result was proved by Ogasawara and Yoshinaga [9] for weakly complete commutative dual $A^*$-algebras. Finally, by using the above result as well as the result in §4, we answer a question of the author affirmatively: if $A$ is a semisimple dual Banach algebra which is a dense two-sided ideal of a $B^*$-algebra, then $A$ is Arens regular and $A^{**}/R^{**}$ is a semisimple Banach algebra which is a dense two-sided ideal of some $B^*$-algebra.

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2. **Notation and preliminaries.** Definitions not explicitly given are taken from Rickart's book [10].

Let $A$ be a Banach algebra. For each element $a \in A$, let $\mathrm{Sp}_A(a)$ denote the spectrum of $a$ in $A$. If $A$ is commutative, $M_A$ will denote the carrier space of $A$ and $C_0(M_A)$ the algebra of all complex-valued functions on $M_A$, which vanishes at infinity. If $A$ is a commutative $B^*$-algebra, then $\hat{A} = C_0(M_A)$.

Let $A$ be a Banach algebra which is a subalgebra of a Banach algebra $B$. For each subset $E$ of $A$, $\mathrm{cl}(E)$ (resp. $\mathrm{cl}_A(E)$) will denote the closure of $E$ in $B$ (resp. $A$). We write $\|\cdot\|$ for the norm on $A$ and $|\cdot|$ for the norm on $B$.

For any set $E$ in a Banach algebra $A$, let $l_A(E)$ and $r_A(E)$ denote the left and right annihilators of $E$ respectively. A Banach algebra $A$ is called an annihilator algebra if $l_A(A) = r_A(A) = (0)$ and if for every proper closed right ideal $I$ and every proper closed left ideal $J$, $l_A(I) \neq (0)$ and $r_A(J) \neq (0)$. If, in addition, $r_A(l_A(I)) = I$ and $l_A(r_A(J)) = J$, then $A$ is called a dual algebra.

An idempotent $e$ in a Banach algebra $A$ is said to be minimal if $eAe$ is a division algebra. In case $A$ is semisimple, this is equivalent to saying that $Ae$ $(eA)$ is a minimal left (right) ideal of $A$.

In this paper, all algebras and linear spaces under consideration are over the field $C$ of complex numbers.

3. **The Arens products and annihilator algebras.** Let $A$ be a Banach algebra, $A^*$ and $A^{**}$ the conjugate and second conjugate spaces of $A$, respectively. The two Arens products on $A^{**}$ are defined in stages according to the following rules (see [1]). Let $x, y \in A$, $f \in A^*$, $F, G \in A^{**}$.

(a) Define $f \circ x$ by $(f \circ x)(y) = f(xy)$. Then $f \circ x \in A^*$.

(b) Define $G \circ f$ by $(G \circ f)(x) = G(f \circ x)$. Then $G \circ f \in A^*$.

(c) Define $F \circ G$ by $(F \circ G)(f) = F(G \circ f)$. Then $F \circ G \in A^{**}$.

$A^{**}$ with the Arens product $\circ$ denoted by $(A^{**}, \circ)$.

(a$'$) Define $x \circ' f$ by $(x \circ' f)(y) = f(yx)$. Then $x \circ' f \in A^*$.

(b$'$) Define $f \circ' F$ by $(f \circ' F)(x) = F(x \circ' f)$. Then $f \circ' F \in A^*$.

(c$'$) Define $F \circ' G$ by $(F \circ' G)(f) = G(f \circ' F)$. Then $F \circ G \in A^{**}$.

$A^{**}$ with the Arens product $\circ'$ denoted by $(A^{**}, \circ')$.

Each of these products extends the original multiplication on $A$ when $A$ is canonically embedded in $A^{**}$. In general, $\circ$ and $\circ'$ are distinct on $A^{**}$. If they coincide on $A^{**}$, then $A$ is called Arens regular.

**Notation.** Let $A$ be a Banach algebra. The mapping $\pi_A$ will denote the canonical embedding of $A$ into $A^{**}$.

The left multiplication in $(A^{**}, \circ)$ is weakly continuous and the right multiplication in $(A^{**}, \circ')$ is weakly continuous (see [1, p. 842]). If $x \in A$ and $F \in A^{**}$, then $\pi_A(x) \circ F = \pi_A(x) \circ' F$ and $F \circ \pi_A(x) = F \circ' \pi_A(x)$ (see [1, p. 843]).

The following result is useful throughout the paper.

**Theorem 3.1.** *Let $A$ be a semisimple Banach algebra which is a dense two-sided ideal of a semisimple annihilator Banach algebra $B$. Then $\pi_A(A)$ is a two-sided ideal of $(A^{**}, \circ)$. In particular, $\pi_B(B)$ is a two-sided ideal of $B^{**}$ (with the Arens product).*

**Proof.** By [2, p. 3, Proposition 2.2], there exists a constant $k > 0$ such that $k\|\cdot\| \geq |\cdot|$ on $A$ and hence by [2, p. 3, Theorem 2.3], there exists a constant $M$ such that

$$\|ab\| \leq M\|a\|\,|b| \quad \text{and} \quad \|ba\| \leq M\|a\|\,|b| \tag{1}$$

for all $a \in A$, $b \in B$. Let $e$ be a minimal idempotent of $B$. Since $eAe = eBe = Ce$, it follows that $e \in A$. Also if $e$ is a minimal idempotent of $A$, then $e$ is a minimal idempotent of $B$. Therefore $A$ and $B$ have the same minimal idempotents. Let $e$ be a minimal idempotent. Since $Ae = Be$, it is easy to see that the norms $\|\cdot\|$ and $|\cdot|$ are equivalent on $Ae$. Since $B$ is an annihilator algebra, it follows immediately from [10, p. 101, Lemma (2.8.20)] and [10, p. 104, Theorem (2.8.23)] that $Be$ is a reflexive Banach space and hence $Ae$ is also reflexive. Let $F \in A^{**}$. We show that $F \circ \pi_A(e) \in \pi_A(A)$. Clearly we can assume that $\|F\| = 1$. Then by Goldstine's theorem [6, p. 424, Theorem 5] there exists a net $\{x_\alpha\}$ in $A$ such that $\|x_\alpha\| \leq 1$ for all $\alpha$ and $\pi_A(x_\alpha) \to F$ weakly in $A^{**}$. Hence it follows from the weak continuity of left multiplication that $\pi_A(x_\alpha e) \to F \circ \pi_A(e)$ weakly. Since $\|x_\alpha e\| \leq \|e\|$, by [6, p. 425, Theorem 7] we can assume that there exists some $y \in Ae$ such that $g(x_\alpha e) \to g(y)$ for all $g \in (Ae)^*$. Now for each $f \in A^*$, let $f'$ be the restriction of $f$ to $Ae$. Then we have

$$\pi_A(y)(f) = \lim_\alpha f'(x_\alpha e) = \lim_\alpha \pi_A(x_\alpha e)(f) = (F \circ \pi_A(e))(f).$$

Therefore, we get

$$F \circ \pi_A(e) = \pi_A(y) \in \pi_A(A). \tag{2}$$

Let $x \in A$. Since the socle $S$ of $B$ is dense in $B$ by [10, p. 100, Corollary (2.8.16)], we can write $x = \lim_{n\to\infty} x_n$, where $x_n \in S$ $(n = 1, 2, \cdots)$. Since $S$ is also the socle of $A$, it follows easily from (2) that

$$F \circ \pi_A(x_n) \in \pi_A(A) \qquad (n = 1, 2, \cdots). \tag{3}$$

Let $f \in A^{**}$. By (1) we obtain $\|a \circ' f\| \leq M\|f\|\,|a|$ for all $a \in A$ and consequently

$$|(F \circ \pi_A(x_n) - F \circ \pi(x))(f)| = |F((x_n - x) \circ' f)| \leq M\|F\|\,\|f\|\,|x_n - x|.$$

Since $x_n \to x$ in $|\cdot|$, we have $F \circ \pi_A(x_n) \to F \circ \pi_A(x)$ in $\|\cdot\|$. Hence it follows from (3) that $F \circ \pi_A(x) \in \pi_A(A)$. Similarly we can show that $\pi_A(x) \circ F \in \pi_A(A)$. Therefore $\pi_A(A)$ is a two-sided ideal of $(A^{**}, \circ)$ and this completes the proof.

**Remark.** The preceding result generalizes a part of [13, p. 830, Theorem 5.2] as well as [12, p. 82, Theorem 3.3].

**Corollary 3.2.** *Let $A$ be as in Theorem 3.1. Then for every minimal idempotent $e \in A$, $A^{**} \circ \pi_A(e)$ and $\pi_A(e) \circ A^{**}$ are minimal left and right ideals of $(A^{**}, \circ)$.*

**Proof.** This follows immediately from Theorem 3.1 since $A^{**} \circ \pi_A(e) = \pi_A(Ae)$ and $\pi_A(e) \circ A^{**} = \pi_A(eA)$.

**4. The radical of the algebra $(A^{**}, \circ)$.** This section is devoted to the discussion of the radical of the algebra $(A^{**}, \circ)$. The main result in this section is useful in §5. Civin and Yood [5] had studied this problem for the group algebra of an infinite locally compact abelian group.

Throughout this section, unless otherwise stated, $A$ will be a semisimple annihilator Banach algebra. Let $R_1^{**}$ (resp. $R_2^{**}$) denote the radical of $(A^{**}, \circ)$ (resp. $(A^{**}, \circ')$); $R_1^{**}$ and $R_2^{**}$ may not be zero (see [5, p. 857, Theorem 3.14] and [13, p. 831, Theorem 5.5]). By Theorem 3.1, $\pi_A(A)$ is a two-sided ideal of $(A^{**}, \circ)$.

**Theorem 4.1.** *Let $A$ be a semisimple annihilator Banach algebra. Then the following statements hold:*

(i) *$R_1^{**}$ is weakly closed.*

(ii) $R_1^{**} = \{F \in A^{**}: A^{**} \circ F = (0)\} = \{F \in A^{**}: F \circ' A^{**} = (0)\}$.

(iii) *$R_1^{**}$ coincides with $R_2^{**}$.*

**Proof.** Let $E_A$ be the set of all minimal idempotents of $A$. For each $e \in E_A$, let $M = (1 - \pi_A(e)) \circ A^{**}$. We show that $M$ is a maximal modular right ideal of $(A^{**}, \circ)$. In fact, suppose there exists a right ideal $M'$ of $(A^{**}, \circ)$ properly containing $M$. Let $F \in M'$ be such that $F \notin M$. Then $\pi_A(e) \circ F = F - (1 - \pi_A(e)) \circ F \in M'$ and $\pi_A(e) \circ F \neq 0$. Hence $(\pi_A(e) \circ A^{**}) \cap M' \neq (0)$ and consequently by Corollary 3.2 $M' \supset \pi_A(e) \circ A^{**}$. Hence $M' = A^{**}$. Therefore $M$ is maximal. Let $\{G_\alpha\}$ be a net in $M$ such that $G_\alpha \to G$ weakly for some $G \in A^{**}$. Since $\pi_A(e) \circ G_\alpha = 0$ for each $\alpha$, it follows that $\pi_A(e) \circ G = 0$ and hence $G \in M$. Therefore $M$ is weakly closed. Let

$$R = \bigcap \{(1 - \pi_A(e)) \circ A^{**}: e \in E_A\} \text{ and } T = \{F \in A^{**}: A^{**} \circ F = (0)\}.$$

Then $R$ is weakly closed and $T \subset R_1^{**} \subset R$. Let $F \in R$. Then $\pi_A(e) \circ F = 0$ for all $e \in E_A$. Since the socle of $A$ is dense in $A$, we have $\pi_A(A) \circ F = (0)$. Since

$\pi_A(A)$ is weakly dense in $(A^{**}, \circ)$, it follows that $A^{**} \circ F = (0)$ and so $F \in T$. Consequently $R_1^{**} = R = T$. Similarly by using maximal modular left ideals, we can show that $R_2^{**} = \{F \in A^{**}: F \circ' A^{**} = (0)\}$. Let $F \in R_1^{**}$, $G \in A^{**}$ and $\{x_\alpha\} \subset A$ such that $\pi_A(x_\alpha) \to G$ weakly. Then $F \circ \pi_A(x_\alpha) = F \circ' \pi_A(x_\alpha) \to F \circ' G$ weakly. Since by Theorem 3.1 $F \circ \pi_A(x_\alpha) \in R_1^{**} \cap \pi_A(A) = (0)$, we have $F \circ' G = 0$ and so $F \in R_2^{**}$. Hence $R_1^{**} \subset R_2^{**}$. Similarly we can show that $R_2^{**} \subset R_1^{**}$. Therefore they are equal and this completes the proof of the theorem.

**Remark 1.** Theorem 4.1 (ii) is a generalization of [5, p. 857, Theorem 3.15 (i)].

**Remark 2.** In general, $R_1^{**} \neq \{F \in A^{**}: F \circ A^{**} = (0)\}$. In fact, let $A$ be the group algebra of an infinite compact abelian group. Then by [5, p. 857, Theorem 3.12] $R_1^{**} \neq (0)$. By [5, p. 855, Lemma 3.8], $A^{**}$ has a right identity. Hence it follows that $\{F \in A^{**}: F \circ A^{**} = (0)\} = (0) \neq R_1^{**}$.

**Notation.** In the rest of this paper, let $R^{**} = R_1^{**} = R_2^{**}$.

**Corollary 4.2.** *Suppose $A$ is a semisimple commutative annihilator Banach algebra and $M_A$ its carrier space. Let $Q$ be the closed subspace of $A^*$ spanned by $M_A$ and let $Q^\perp = \{F \in A^{**}: F(Q) = (0)\}$. Then $Q^\perp = R^{**}$.*

**Proof.** It is well known that $M_A$ is discrete. For each $h \in M_A$, let $e_h$ be the minimal idempotent of $A$ corresponding to the characteristic function of $h$ ([10, p. 168, Theorem (3.6.3)]). For each $h \in M_A$ and $x \in A$, we have $xe_h = e_h x e_h = h(x)e_h$. Therefore $(f \circ e_h)(x) = f(e_h)h(x)$ for all $f \in A^*$. Hence $f \circ e_h = f(e_h)h$. Let $F \in A^{**}$. Then $(\pi_A(e_h) \circ F)(f) = F(f \circ e_h) = f(e_h)F(h)$ for all $f \in A^*$. Hence it follows easily that $Q^\perp = \{F \in A^{**}: A^{**} \circ F = (0)\}$. Therefore by Theorem 4.1, $Q^\perp = R^{**}$.

**Remark.** The above result is a generalization of [5, p. 857, Theorem 3.15 (ii)].

**Corollary 4.3.** *Let $M$ be a maximal modular right ideal of $(A^{**}, \circ)$. Then either $(l(M))^2 = (0)$ or there exists a minimal idempotent $e$ of $A$ such that $M = (1 - \pi_A(e)) \circ A^{**}$. In the latter case, $M$ is weakly closed. A similar result holds for left ideals.*

**Proof.** If $l(M) \subset R^{**}$, then by Theorem 4.1 $(l(M))^2 = (0)$. Suppose $l(M) \not\subset R^{**}$. We claim that $l(M) \cap \pi_A(A) \neq (0)$. Assume this is not so. Then $\pi_A(A) \circ l(M) \subset \pi_A(A) \cap l(M) = (0)$. Hence $A^{**} \circ l(M) = (0)$ and so by Theorem 4.1, $l(M) \subset R^{**}$. This contradiction shows that $l(M) \cap \pi_A(A) \neq (0)$. Therefore by [10, p. 98, Lemma (2.8.6)], $l(M) \cap \pi_A(A)$ contains a minimal idempotent $\pi_A(e)$ of $\pi_A(A)$. By the maximality of $M$, we have $M = (1 - \pi_A(e)) \circ A^{**}$. Also $M$ is weakly closed by the proof of Theorem 4.1 and this completes the proof.

We remark that a similar result for left ideals has been obtained by Civin for the group algebra of an infinite locally compact abelian group (see [3]).

5. **Banach algebras which are ideals in a $B^*$-algebra.** In this section, we study semisimple dual Banach algebras which are two-sided ideals in a $B^*$-algebra. There are many examples having such properties in analysis. The algebras $C_p$ discussed in [8] and the proper $H^*$-algebras are such examples. Unless otherwise stated, $A$ will be a semisimple dual Banach algebra which is a dense subalgebra of a $B^*$-algebra $B$ such that $\|\cdot\|$ majorizes $|\cdot|$ on $A$. It is well known that $B$ is also a dual algebra (see [12, p. 81]).

The following result is contained in Lemma 5.1 in [7].

**Lemma 5.1.** *$A$ and $B$ have the same minimal idempotents and the same socle.*

**Proof.** Let $e$ be a minimal idempotent of $A$. Then it is clear that $e$ is a minimal idempotent of $B$. By the proof of [12, p. 82, Lemma 3.2] $\|\cdot\|$ and $|\cdot|$ are equivalent on $Ae$ and $Be = Ae$, $eA = eB$. Therefore the socle $S$ of $A$ is a dense two-sided ideal of $B$. Let $f$ be a minimal idempotent of $B$. Then $Sf \subset Bf \cap S$ and so $Bf \subset S \subset A$. Therefore $f$ is a minimal idempotent of $A$. Now it is clear that $S$ is also the socle of $B$.

We shall now give a characterization for $A$ to be a two-sided ideal of $B$.

**Theorem 5.2.** *Let $A$ be a semisimple dual Banach algebra which is a dense subalgebra of a $B^*$-algebra $B$ such that $\|\cdot\|$ majorizes $|\cdot|$ on $A$. Then the following statements are equivalent:*

(i) *$A$ is a two-sided ideal of $B$.*

(ii) *There exists a constant $M > 0$ such that $\|\Sigma_{k=1}^n e_k x\| \le M\|x\|$ and $\|\Sigma_{k=1}^n x e_k\| \le M\|x\|$, where $x \in A$ and $e_1, e_2, \cdots, e_n$ are any mutually orthogonal hermitian minimal idempotents of $B$.*

(iii) *For any orthogonal family of hermitian minimal idempotents $\{e_\lambda : \lambda \in \Lambda\}$ of $B$ and $x \in A$, $\Sigma_\lambda x e_\lambda$ and $\Sigma_\lambda e_\lambda x$ are summable in the norm of $A$ and especially when $\{e_\lambda : \lambda \in \Lambda\}$ is a maximal family, $x = \Sigma_\lambda x e_\lambda = \Sigma_\lambda e_\lambda x$ in $A$.*

**Proof.** We know that $B$ is a dual algebra and $A$ and $B$ have the same minimal idempotents and the same socle by Lemma 5.1.

(i) $\Rightarrow$ (ii). Suppose (i) holds. Then by [2, p. 3, Theorem 2.3] there exists a constant $M$ such that $\|\Sigma_{k=1}^n e_k x\| \le M|\Sigma_{k=1}^n e_k|\,\|x\| = M\|x\|$. Similarly, $\|\Sigma_{k=1}^n x e_k\| \le M\|x\|$ and this proves (ii).

(ii) $\Rightarrow$ (iii). Suppose (ii) holds. Let $\{e_\lambda : \lambda \in \Lambda\}$ be an orthogonal family of hermitian minimal idempotents of $B$ and $x \in A$. Let $\{E_\gamma : \gamma \in \Gamma\}$ be the direct set of all finite sums $e_{\lambda_1} + e_{\lambda_2} + \cdots + e_{\lambda_n}$ ($\lambda_k \in \Lambda$ and $n = 1, 2, \cdots$). Since $\|xE_\gamma\| < M\|x\|$ by (ii), it follows from the Alaoglu theorem that $\{\pi_A(xE_\gamma)\}$ has weak limit points in $A^{**}$. Let $F \in A^{**}$ be a weak limit point of $\{\pi_A(xE_\gamma)\}$. Then for any $y \in A$, $\pi_A(y) \circ F$ is a weak limit point of $\pi_A(yxE_\gamma)$. Since $A$ is a dual algebra, by Theorem 3.1 $\pi_A(y) \circ F \in \pi_A(A)$. Let $\{e_\alpha: \alpha \in \Delta\}$ be a maximal orthogonal family of hermitian minimal idempotents of $B$ containing $\{e_\lambda: \lambda \in \Lambda\}$. Then it is easy to see that $\pi_A(y) \circ F \circ \pi_A(e_\alpha) = \pi_A(yxe_\alpha)$ $(\alpha \in \Delta)$. Since $\{e_\alpha: \alpha \in \Delta\}$ is maximal, it follows that $\pi_A(y) \circ F = \pi_A(yx)$ (see [9, p. 21]). Hence $\{yxE_\gamma\}$ converges weakly to $yx$ and so by the Orlicz-Banach theorem [6, p. 93], $\sum_\lambda yxe_\lambda$ is summable in the norm of $A$. Since $A$ is a dual algebra by [10, p. 91, Corollary (2.8.3)] $x \in \mathrm{cl}_A(Ax)$. Hence, for any given $\epsilon > 0$, there exists some $z \in A$ such that $\|x - zx\| < \epsilon$. Now by (ii) we have $\|xE_\gamma\| \le M\|x - zx\| + \|zxE_\gamma\| < M\epsilon + \|zxE_\gamma\|$. Since $\sum_\lambda zxe_\lambda$ is summable in $\|\cdot\|$ and $\epsilon$ is arbitrary, it follows that $\sum_\lambda xe_\lambda$ is summable in $\|\cdot\|$. If $\{e_\lambda: \lambda \in \Lambda\}$ is a maximal family, then it is easy to see that $x = \sum_\lambda xe_\lambda$. Similarly we can show that $x = \sum_\lambda e_\lambda x$ and this proves (iii).

(iii) $\Longrightarrow$ (i). Suppose (iii) holds. Let $x \in A$ and $y \in B$. We shall show that $xy \in B$. Since any element of $B$ is a linear combination of positive elements, we may assume that $y$ is a positive element. We also assume that $x \neq 0$ and $y \neq 0$. Let $E$ be a maximal commutative $*$-subalgebra of $B$ containing $y$. Then the carrier space $M_E$ of $E$ is discrete. For each $\lambda \in M_E$, let $e_\lambda$ be the element of $E$ corresponding to the characteristic function of $\lambda$. Then $\{e_\lambda: \lambda \in M_E\}$ is a maximal orthogonal family of hermitian minimal idempotents in $B$. Since $y \in E$ and $\mathrm{Sp}_E(y) > 0$, we have $ye_\lambda = \beta_\lambda e_\lambda$, where $\beta_\lambda \ge 0$ for all $\lambda$ and $\beta_\lambda \le |y|$. Since $B$ is a dual $B^*$-algebra, by the proof of (ii) $\Longrightarrow$ (iii) (or [9, p. 22, Corollary 1]) $xy = \sum_\lambda xye_\lambda$ in $|\cdot|$ and so there exists only a countable number of $e_\lambda$ for which $xye_\lambda \neq 0$, say $e_1, e_2, \cdots$. For any two positive integers $m, n$ $(m < n)$, let $z_m^n = \sum_{k=m}^n xye_k = \sum_{k=m}^n \beta_k xe_k$. Then $z_m^n \in A$. We shall show that $\{\sum_{k=1}^n xye_k\}$ is Cauchy sequence in $A$. Clearly, we can assume that each $z_m^n$ is a nonzero element. Choose $f \in A^*$ such that $\|f\| = 1$ and $f(z_m^n) = \|z_m^n\|$ by the Hahn-Banach theorem. Then $f(z_m^n) = \sum_{k=m}^n \beta_k f(xe_k)$. Write $f(xe_k) = a_k + ib_k$, where $a_k$, $b_k$ are real numbers. Then we have

$$\sum_{k=m}^{n} \beta_k f(xe_k) = \sum_{k=1}^{n} \beta_k a_k = \|z_m^n\| > 0.$$

Since $\beta_k \ge 0$, there exists some $a_k > 0$. Let $\{a_{k'}\} \subset \{a_k\}_{k=m}^n$ such that $a_{k'} > 0$. Then we have

$$\left\|\sum_{k=m}^{n} xye_k\right\| = \|z_m^n\| = \sum_{k=m}^{n} \beta_k a_k \le \sum_{k'} \beta_{k'} a_{k'}$$
$$\le |y| \sum_{k'} a_{k'} \le |y| \left|\sum_{k'} f(xe_{k'})\right| \le |y|\|f\| \left\|\sum_{k'} xe_{k'}\right\| = |y| \left\|\sum_{k'} xe_{k'}\right\|.$$

Hence it follows from the assumption that $\{\Sigma_{i=1}^n xye_k\}$ is a Cauchy sequence in $A$. Therefore, there exists an element $z \in A$ such that $z = \Sigma_{k=1}^\infty xye_k$ in $\|\cdot\|$. Also $xy = \Sigma_{k=1}^\infty xye_k$ in $|\cdot|$. Hence it follows that $xy = z \in A$. Similarly we can show that $yx \in A$. Thus $A$ is a two-sided ideal of $B$ and this completes the proof of the theorem.

**Remark 1.** (i) $\Rightarrow$ (iii) in the above theorem was obtained by Ogasawara and Yoshinaga for $A^*$-algebras (see [9, p. 30, Theorem 16]). Also (iii) $\Leftrightarrow$ (i) was proved by them for weakly complete commutative $A^*$-algebras (see [9, p. 35, Theorem 2.3]). Some arguments in the proof of (ii) $\Rightarrow$ (iii) of Theorem 5.2 are similar to those in the proof of [9, p. 30, Theorem 16].

**Remark 2.** If $B$ is not a $B^*$-algebra, then Theorem 5.2 is not true. In fact, let $G$ be an infinite compact group and let $A$ be the algebra of all continuous functions on $G$, normed by the maximum of the absolute value. It is well known that $L_2(G)$ is an $A^*$-algebra and $A$ is a dual $A^*$-algebra which is a dense two-sided ideal of $L_2(G)$. However condition (iii) of Theorem 5.2 is not valid for $A$. Since $L_2(G)$ is a proper $H^*$-algebra, condition (iii) holds for $L_2(G)$.

**Corollary 5.3.** *Let $A$ be a reflexive $A^*$-algebra which is a dense subalgebra of a $B^*$-algebra $B$. Then the following statements are equivalent:*

(i) *$A$ is a two-sided ideal of $B$.*

(ii) *$A$ is a dual algebra and, for any orthogonal family of hermitian minimal idempotents $\{e_\lambda: \lambda \in \Lambda\}$ of $B$ and $x \in A$, the set $\{\Sigma_{k=1}^n e_{\lambda_k}x: \lambda_k \in \Lambda\}$ is bounded in $A$.*

**Proof.** (i) $\Rightarrow$ (ii). This follows immediately from [13, p. 831, Theorem 5.4] and Theorem 5.2 (ii).

(ii) $\Rightarrow$ (i). Suppose (ii) holds. Since $A$ is reflexive, $\{\Sigma_{k=1}^n e_{\lambda_k}x: \lambda_k \in \Lambda\}$ has weak limit points in $A$. By the proof of Theorem 5.2, it has a unique weak limit point and so $\Sigma_\lambda e_\lambda x$ is summable in the norm of $A$. Therefore $A$ is a two-sided ideal of $B$ by Theorem 5.2.

It is well known that a reflexive $B^*$-algebra is finite dimensional. The following corollary is a generalization of this result.

**Corollary 5.4.** *Let $A$ be a reflexive $A^*$-algebra which is a dense two-sided ideal of a $B^*$-algebra $B$. If $A$ has an approximate identity, then $A$ is finite dimensional.*

**Proof.** It follows immediately from [5, p. 855, Lemma 3.8] and Corollary 5.3 that $A$ is a dual algebra with an identity. Therefore $A$ is finite dimensional.

It is well known that $B$ is Arens regular if $B$ is a $B^*$-algebra. Let $A$ be a semisimple dual Banach algebra which is a dense two-sided ideal of a $B^*$-algebra $B$. Is $A$ Arens regular? This question was asked in [13, p. 833]. We shall answer this question affirmatively.

**Notation.** In the rest of this section, $B^{**}$ with the Arens product will be denoted by $(B^{**}, *)$.

**Lemma 5.5.** *Suppose $B$ is a dual $B^*$-algebra and $S$ its socle. Let $B'$ be the closed subspace of $B^*$ spanned by $\pi_B(x) * g$, where $x \in S$ and $g \in B^*$. Then $B^*$ coincides with $B'$.*

**Proof.** Suppose this is not true. Then there exists a nonzero linear functional $F \in B^{**}$ such that $F(B') = (0)$. Hence, for all $x \in S$, $(F * \pi_B(x))(g) = F(\pi_B(x) * g) = 0$. Since $S$ is weakly dense in $B^{**}$, it follows that $F * B^{**} = (0)$. Since $B^{**}$ is a $B^*$-algebra, we have $F = 0$, a contradiction. Therefore $B^*$ coincides with $B'$.

In the rest of this section, let $A$ be a semisimple Banach algebra which is a dense two-sided ideal of a $B^*$-algebra $B$. By [2, p. 3, Proposition 2.2], there exists a constant $k$ such that $k\|\cdot\| \geq |\cdot|$ on $A$ and consequently by [2, p. 3, Theorem 2.3] there exists a constant $M$ such that $\|ab\| \leq M\|a\| \, |b|$ and $\|ba\| \leq M\|a\| \, |b|$ for all $a \in A$, $b \in B$. For each $g \in B^*$, let $g_A$ denote the restriction of $g$ to $A$. Then it is easy to see that $g_A \in A^*$. For every element $F \in A^{**}$, let $\tilde{F}$ be the linear functional on $B^*$ defined by $\tilde{F}(g) = F(g_A)$ $(g \in B^*)$. Then $\tilde{F} \in B^{**}$. Let $b \in B$ and $f \in A^*$. Define $(f \circ b)(a) = f(ba)$ $(a \in A)$. Since $|(f \circ b)(a)| \leq M\|f\| \, |b| \, \|a\|$, it follows that $f \circ b \in A^*$.

As before, let $R^{**}$ denote the radical of $(A^{**}, \circ)$.

**Lemma 5.6.** *Suppose $A$ is an annihilator algebra. Then the following statements hold*:

(i) *For each $R \in R^{**}$ and $g \in B^*$, we have $\tilde{R}(g) = 0$.*

(ii) *$R^{**}$ is the left and right annihilator of $(A^{**}, \circ)$.*

**Proof.** (i) Let $g \in B^*$. By Lemma 5.5, we can write $g = \lim_n g_n$ where $g_n = \sum_{i=1}^{m_n} \pi_B(x_i^n) * g_i^n$ with $x_i^n \in S$ (the socle of $B$) and $g_i^n \in B^*$. Clearly $x_i^n \in A$. Then for each $R \in R^{**}$, we have

$$\tilde{R}(g) = \lim_n \sum_{i=1}^{m_n} \tilde{R}(\pi_B(x_i^n) * g_i^n) = \lim_n \sum_{i=1}^{m_n} (R \circ \pi_A(x_i^n))((g_i^n)_A).$$

By Theorem 4.1, we have $R \circ \pi_A(x_k^n) = 0$ and therefore $\tilde{R}(g) = 0$. This proves (i).

(ii) For each $F \in A^{**}$ and $f \in A^*$, define $\tilde{f}_F(b) = F(f \circ b)$ $(b \in B)$. Then it is easy to see that $\tilde{f}_F \in B^*$ and $(\tilde{f}_F)_A = F \circ f$. Then for all $R \in R^{**}$, we have $(R \circ F)(f) = R(F \circ f) = \tilde{R}(\tilde{f}_F)$. Therefore by (i), $R \circ F = 0$ and so $R^{**} \circ A^{**} = (0)$. By Theorem 4.1, we also have $A^{**} \circ R^{**} = (0)$ and this completes the proof.

Now we are ready to prove the following result:

**Theorem 5.7.** *Let $A$ be a semisimple dual Banach algebra which is a dense two-sided ideal of a $B^*$-algebra. Then the following statements hold:*

(i) *$A$ is Arens regular.*

(ii) *$A^{**}/R^{**}$ is a semisimple Banach algebra which is a dense two-sided ideal of some $B^*$-algebra.*

**Proof.** (i) Let $\{e_\lambda : \lambda \in \Lambda\}$ be a maximal orthogonal family of hermitian minimal idempotents in $B$. Let $\{E_\beta\}$ be the direct set of all finite sums $e_{\lambda_1} + e_{\lambda_2} + \cdots + e_{\lambda_n}$ ($\lambda_n \in \Lambda$, $n = 1, 2, \cdots$). Let $F$ and $G$ be two functionals in $A^{**}$. Since $\|F \circ \pi_A(E_\beta)\| \le M\|F\|\,|E_\beta| = M\|F\|$, it follows from Alaoglu's theorem that $\{F \circ \pi_A(E_\beta)\}$ has weak limit points in $A^{**}$. Let $\{E_\alpha\}$ be a subnet of $\{E_\beta\}$ and $F_1 \in A^{**}$ such that $F \circ \pi_A(E_\alpha) \to F_1$ weakly. By a similar argument, there exists a subnet $\{E_\gamma\}$ of $\{E_\alpha\}$ and $G_1 \in A^{**}$ such that $\pi_A(E_\gamma) \circ G \to G_1$ weakly. Let $a \in A$. Then by Theorem 5.2, $a = \Sigma_\lambda e_\lambda a$ in $\|\cdot\|$. Hence $E_\beta a \to a$ weakly. Thus $E_\gamma a \to a$ weakly. Since $F \circ \pi_A(x) = F \circ' \pi_A(x)$ for all $x \in A$, we have $F_1 \circ \pi_A(a) =$ weak limit $F \circ \pi_A(E_\gamma a) = F \circ \pi_A(a)$. Since $\pi_A(A)$ is weakly dense in $A^{**}$, it follows that $(F - F_1) \circ' A^{**} = (0)$ and so by Theorem 4.1, $F - F_1 \in R^{**}$. Similarly we can show that $G_1 - G \in R^{**}$. Then by Lemma 5.6, we have

$$\begin{aligned} F \circ G &= (F_1 + (F - F_1)) \circ G = F_1 \circ G \\ &= \operatorname*{weak\,lim}_{\gamma} F \circ \pi_A(E_\gamma) \circ G = \operatorname*{weak\,lim}_{\gamma} F \circ' (\pi_A(E_\gamma) \circ G) \\ &= F \circ' G_1 = F \circ' G. \end{aligned}$$

Therefore $A$ is Arens regular by definition and this proves (i).

(ii) Now the algebra $A^{**}/R^{**}$ is a semisimple Banach algebra. For each $a \in A$ and $f \in A^*$, define $(f * a)(b) = f(ab)$ $(b \in B)$. Then $f * a \in B^*$. For each $F \in A^{**}$, we write $\dot{F} = F + R^{**}$ and define a mapping $\Phi$ from $A^{**}/R^{**}$ into $B^{**}$ by $\Phi(\dot{F}) = \tilde{F}$ $(F \in A^{**})$. Suppose $\Phi(\dot{F}) = 0$. Then $\tilde{F}(f * a) = 0$ and therefore $(\pi_A(a) \circ F)(f) = 0$ for all $a \in A$ and $f \in A^*$. Consequently $F \in R^{**}$ and therefore $\dot{F} = R^{**}$. Hence it follows that $\Phi$ is an isomorphism of $A^{**}/R^{**}$ into $B^{**}$. For each $g \in B^*$, we have $\|g_A\| \le k|g|$. Since by Lemma 5.5 (i), $R(g_A) = 0$ for all $R \in R^{**}$, straightforward calculations yield that $k\|F + R\| \ge |\tilde{F}|$ for all $F \in A^{**}$. Hence $k\|\dot{F}\| \ge |\tilde{F}|$ and consequently $\Phi$ is continuous. For each $H \in B^{**}$, define $(H \circ f)(a) = H(f * a)$ $(f \in A^*, a \in A)$. Then $H \circ f \in A^*$. For each $F \in A^{**}$, define $F_H(f) = F((H \circ f))$ $(f \in A^*, F \in A^{**})$. Then $F_H \in A^{**}$. For each $g \in B^*$, we have

$$\tilde{F}_H(g) = F((H \circ g_A)) = F((H * g)_A) = (\tilde{F} * H)(g).$$

Therefore $\tilde{F} * H = \tilde{F}_H$. Consequently $\Phi(A^{**}/R^{**})$ is a two-sided ideal of $B^{**}$. Let $Q$ be the norm closure of $\Phi(A^{**}/R^{**})$ in $B^{**}$. Then $Q$ is a closed two-sided ideal of $B^{**}$. Since $B^{**}$ is a $B^*$-algebra, so is $Q$. This completes the proof of the theorem.

**Remark.** We know that the above result is not true for arbitrary dual $A^*$-algebras (see [13, p. 833, Remark]). Also if $A$ is a dual $A^*$-algebra which is Arens regular, $A$ may not be a two-sided ideal of its completion in an auxiliary norm; in fact, $A$ can be reflexive (see [9, p. 35]).

Let $\mathfrak{A} = A^{**}/R^{**}$. Clearly, we can identify $A$ as a closed two-sided ideal of $\mathfrak{A}$.

**Corollary 5.8.** *Let $A$ be as in Theorem 5.7. Then $\mathfrak{A}$ coincides with $A$ if and only if the socle of $\mathfrak{A}$ is dense in $\mathfrak{A}$.*

**Proof.** We use the notation in the proof of Theorem 5.7. Suppose the socle of $\mathfrak{A}$ is dense in $\mathfrak{A}$. Then $Q$ is a dual $B^*$-algebra. For each minimal idempotent $e \in Q$ and $b \in B$, we have $e = ke\pi_B(b)e \in \pi_B(B)$, where $k$ is a constant. Hence it follows that $Q = B$. Now it is easy to see that $\mathfrak{A}^2 \subset A$. Since the socle of $\mathfrak{A}$ is dense in $\mathfrak{A}$, $\mathfrak{A} \subset A$ and so $\mathfrak{A} = A$. The converse of the corollary is clear and this completes the proof.

If $A$ is reflexive, then it is clear that $A^{**}$ is semisimple. However, in general, $A^{**}$ may not be semisimple as shown in [13, p. 831, Theorem 5.5].

**Corollary 5.9.** *Let $A$ be as in Theorem 5.7. Then $A^{**}$ is semisimple if and only if $A^*$ is spanned by $\pi_A(x) \circ f$, where $f \in A^*$ and $x \in A$.*

**Proof.** Suppose $A^*$ is spanned by $\pi_A(x) \circ f$. Let $F \in R^{**}$. Since $F \circ \pi_A(x) = 0$ for all $x \in A$, it follows that $F(f) = 0$ for all $f \in A^*$. Hence $F = 0$. The converse of the corollary follows immediately from the proof of Lemma 5.5.

Let $A$ be a Banach $*$-algebra. For all $x \in A$, $f \in A^*$ and $F \in A^{**}$, we define

$$f^*(x) = \overline{f(x^*)} \text{ and } F^*(f) = \overline{F(f^*)},$$

where the bar denotes the complex conjugation. If $A$ is a $B^*$-algebra, then $A^{**}$ is a $B^*$-algebra under the involution $F \to F^*$ (see [11, p. 192]).

**Corollary 5.10.** *Let $A$ be a dual $A^*$-algebra which is a dense two-sided ideal of a $B^*$-algebra $B$. Then $(A^{**}, \circ)$ is a $*$-algebra and $A^{**}/R^{**}$ is an $A^*$-algebra which is a dense two-sided ideal of a $B^*$-algebra.*

**Proof.** By Theorem 5.7, $A$ is Arens regular and so $A^{**}$ is a $*$-algebra under the involution $F \to F^*$ by [11, p. 186, Theorem 1]. Clearly $R^{**}$ is a $*$-ideal of $A^{**}$. Now the corollary follows easily from Theorem 5.7.

It was asked in [13, p. 833] whether the algebra $C_p^{**}$ is semisimple. If

$1 < p < \infty$, then $C_p$ is reflexive (see [8, p. 265]) and, therefore, it is semisimple. If $p = 1$, then by [12, p. 831, Theorem 5.5], $C_1^{**}$ is not semisimple unless it is finite dimensional.

## REFERENCES

1. R. Arens, *The adjoint of a bilinear operation*, Proc. Amer. Math. Soc. 2 (1951), 839–848. MR 13, 659.

2. B. A. Barnes, *Banach algebras which are ideals in a Banach algebra*, Pacific J. Math. 38 (1971), 1–7.

3. P. Civin, *Annihilators in the second conjugate algebra of a group algebra*, Pacific J. Math. 12 (1962), 855–862. MR 26 #2894.

4. ———, *Ideals in the second conjugate algebra of a group algebra*, Math. Scand. 11 (1962), 161–174. MR 27 #5139.

5. P. Civin and B. Yood, *The second conjugate space of a Banach algebra as an algebra*, Pacific J. Math. 11 (1961), 847–870. MR 26 #622.

6. N. Dunford and J. T. Schwartz, *Linear operators*. I: *General theory*, Pure and Appl. Math., vol. 7, Interscience, New York, 1958. MR 22 #8302.

7. T. Husain and P. K. Wong, *Quasi-complemented algebras*, Trans. Amer. Math. Soc. 174 (1972), 141–154.

8. C. A. McCarthy, $c_p$, Israel J. Math. 5 (1967), 249–271. MR 37 #735.

9. T. Ogasawara and K. Yoshinaga, *Weakly completely continuous Banach *-algebras*, J. Sci. Hiroshima Univ. Ser. A 18 (1954), 15–36. MR 16, 1126.

10. C. E. Rickart, *General theory of Banach algebras*, University Series in Higher Math., Van Nostrand, Princeton, N. J., 1960. MR 22 #5903.

11. M. Tomita, *The second dual of a $C^*$-algebra*, Mem. Fac. Kyushu Univ. Ser. A. 21 (1967), 185–193. MR 36 #6955.

12. P. K. Wong, *On the Arens product and annihilator algebras*, Proc. Amer. Math. Soc. 30 (1971), 79–83. MR 43 #6724.

13. ———, *Modular annihilator $A^*$-algebras*, Pacific J. Math. 37 (1971), 825–834.

14. ———, *The Arens product and duality in $B^*$-algebras*. II, Proc Amer. Math. Soc. 27 (1971), 535–538. MR 43 #933.

15. ———, *On the Arens product and commutative Banach algebras*, Proc. Amer. Math. Soc. 37 (1972), 111–113.

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