ON THE ARENS PRODUCTS AND CERTAIN BANACH ALGEBRAS

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ABSTRACT. In this paper, we study several problems in Banach algebras concerned with the Arens products.

1. Introduction. Let $A$ be a Banach algebra, $A^{**}$ its second conjugate space and $\pi_A$ the canonical embedding of $A$ into $A^{**}$. Arens has defined two natural extensions of the product on $A$ to $A^{**}$. Under either Arens product, $A^{**}$ becomes a Banach algebra. In §3, we show that if $A$ is a semisimple Banach algebra which is a dense two-sided ideal of a semisimple annihilator Banach algebra $B$, then $\pi_A(A)$ is a two-sided ideal of $A^{**}$ (with the Arens product). In particular, a semisimple annihilator Banach algebra has such property. This result greatly generalizes some recent results obtained by the author (see [12, p. 82] and [13, p. 830]).

In §4, we study the radical $R^{**}$ of $A^{**}$, where $A$ is a semisimple annihilator Banach algebra. We show that, under either Arens product, $R^{**}$ remains the same and it is the right annihilator of $A^{**}$. A similar result was obtained by Civin and Yood [5] for the group algebra of a compact abelian group.

§5 is devoted to the study of semisimple dual Banach algebras which are two-sided ideals of a $B^*$-algebra. Let $A$ be a semisimple dual Banach algebra which is a dense subalgebra of a $B^*$-algebra $B$ such that $\|\cdot\|$ majorizes $\|\cdot\|_1$ on $A$. We show that $A$ is a two-sided ideal of $B$ if and only if, for any orthogonal family of hermitian minimal idempotents $\{e_\lambda : \lambda \in \Lambda\}$ of $B$ and $x \in A$, $\sum e_\lambda x$ and $\sum e_\lambda x \pi_A$ are summable in the norm of $A$. This result was proved by Ogasawara and Yoshinaga [9] for weakly complete commutative dual $A^*$-algebras. Finally, by using the above result as well as the result in §4, we answer a question of the author affirmatively: if $A$ is a semisimple dual Banach algebra which is a dense two-sided ideal of a $B^*$-algebra, then $A$ is Arens regular and $A^{**}/R^{**}$ is a semisimple Banach algebra which is a dense two-sided ideal of some $B^*$-algebra.
2. Notation and preliminaries. Definitions not explicitly given are taken from Rickart's book [10].

Let \( A \) be a Banach algebra. For each element \( a \in A \), let \( \text{Sp}_A(a) \) denote the spectrum of \( a \) in \( A \). If \( A \) is commutative, \( M_A \) will denote the carrier space of \( A \) and \( C_0(M_A) \) the algebra of all complex-valued functions on \( M_A \), which vanishes at infinity. If \( A \) is a commutative \( B^* \)-algebra, then \( \hat{A} = C_0(M_A) \).

Let \( A \) be a Banach algebra which is a subalgebra of a Banach algebra \( B \). For each subset \( E \) of \( A \), \( \text{cl}(E) \) (resp. \( \text{cl}_A(E) \)) will denote the closure of \( E \) in \( B \) (resp. \( A \)). We write \( \| - \| \) for the norm on \( A \) and \( | - | \) for the norm on \( B \).

For any set \( E \) in a Banach algebra \( A \), let \( \ell_A(E) \) and \( \ell_A(E) \) denote the left and right annihilators of \( E \) respectively. A Banach algebra \( A \) is called an annihilator algebra if \( \ell_A(A) = \ell_A(A) = \{0\} \) and if for every proper closed right ideal \( I \) and every proper closed left ideal \( J \), \( \ell_A(I) \neq \{0\} \) and \( \ell_A(J) \neq \{0\} \). If, in addition, \( \ell_A(I) = I \) and \( \ell_A(J) = J \), then \( A \) is called a dual algebra.

An idempotent \( e \) in a Banach algebra \( A \) is said to be minimal if \( eAe \) is a division algebra. In case \( A \) is semisimple, this is equivalent to saying that \( Ae (eA) \) is a minimal left (right) ideal of \( A \).

In this paper, all algebras and linear spaces under consideration are over the field \( C \) of complex numbers.

3. The Arens products and annihilator algebras. Let \( A \) be a Banach algebra, \( A^* \) and \( A^{**} \) the conjugate and second conjugate spaces of \( A \), respectively. The two Arens products on \( A^{**} \) are defined in stages according to the following rules (see [1]). Let \( x, y \in A, f \in A^*, F, G \in A^{**} \).

(a) Define \( f \circ x \) by \( (f \circ x)(y) = f(xy) \). Then \( f \circ x \in A^* \).
(b) Define \( G \circ f \) by \( (G \circ f)(x) = G(f \circ x) \). Then \( G \circ f \in A^* \).
(c) Define \( F \circ G \) by \( (F \circ G)(f) = F(G \circ f) \). Then \( F \circ G \in A^{**} \).

\( A^{**} \) with the Arens product \( \circ \) denoted by \( (A^{**}, \circ) \).

(a') Define \( x \circ' f \) by \( (x \circ' f)(y) = f(yx) \). Then \( x \circ' f \in A^* \).
(b') Define \( f \circ' F \) by \( (f \circ' F)(x) = F(x \circ' f) \). Then \( f \circ' F \in A^* \).
(c') Define \( F \circ' G \) by \( (F \circ' G)(f) = G(f \circ' F) \). Then \( F \circ' G \in A^{**} \).

\( A^{**} \) with the Arens product \( \circ' \) denoted by \( (A^{**}, \circ') \).

Each of these products extends the original multiplication on \( A \) when \( A \) is canonically embedded in \( A^{**} \). In general, \( \circ \) and \( \circ' \) are distinct on \( A^{**} \). If they coincide on \( A^{**} \), then \( A \) is called Arens regular.

Notation. Let \( A \) be a Banach algebra. The mapping \( \pi_A \) will denote the canonical embedding of \( A \) into \( A^{**} \).

The left multiplication in \( (A^{**}, \circ) \) is weakly continuous and the right multiplication in \( (A^{**}, \circ') \) is weakly continuous (see [1, p. 842]). If \( x \in A \) and \( F \in A^{**} \), then \( \pi_A(x) \circ F = \pi_A(x) \circ' F \) and \( F \circ \pi_A(x) = F \circ' \pi_A(x) \) (see [1, p. 843]).
The following result is useful throughout the paper.

**Theorem 3.1.** Let $A$ be a semisimple Banach algebra which is a dense two-sided ideal of a semisimple annihilator Banach algebra $B$. Then $\pi_A(A)$ is a two-sided ideal of $(A^{**}, \circ)$. In particular, $\pi_B(B)$ is a two-sided ideal of $B^{**}$ (with the Arens product).

**Proof.** By [2, p. 3, Proposition 2.2], there exists a constant $k > 0$ such that $k\|a\| \geq |a|$ on $A$ and hence by [2, p. 3, Theorem 2.3], there exists a constant $M$ such that

$$
\|ab\| \leq M\|a\| \|b\| \quad \text{and} \quad \|ba\| \leq M\|a\| \|b\|
$$

for all $a \in A$, $b \in B$. Let $e$ be a minimal idempotent of $B$. Since $eAe = eBe = Ce$, it follows that $e \in A$. Also if $e$ is a minimal idempotent of $A$, then $e$ is a minimal idempotent of $B$. Therefore $A$ and $B$ have the same minimal idempotents. Let $e$ be a minimal idempotent. Since $Ae = Be$, it is easy to see that the norms $\|\cdot\|$ and $|\cdot|$ are equivalent on $Ae$. Since $B$ is an annihilator algebra, it follows immediately from [10, p. 101, Lemma (2.8.20)] and [10, p. 104, Theorem (2.8.23)] that $Be$ is a reflexive Banach space and hence $Ae$ is also reflexive. Let $F \in A^{**}$. We show that $F \circ \pi_A(e) \in \pi_A(A)$. Clearly we can assume that $\|F\| = 1$.

Then by Goldstine’s theorem [6, p. 424, Theorem 5] there exists a net $\{x_\alpha\}$ in $A$ such that $\|x_\alpha\| \leq 1$ for all $\alpha$ and $\pi_A(x_\alpha) \to F$ weakly in $A^{**}$. Hence it follows from the weak continuity of left multiplication that $\pi_A(x_\alpha e) \to F \circ \pi_A(e)$ weakly. Since $\|x_\alpha e\| \leq \|e\|$, by [6, p. 425, Theorem 7] we can assume that there exists some $y \in Ae$ such that $g(x_\alpha e) \to g(y)$ for all $g \in (Ae)^*$. Now for each $f \in A^*$, let $f^\prime$ be the restriction of $f$ to $Ae$. Then we have

$$
\pi_A(y)(f) = \lim_{\alpha} f^\prime(x_\alpha e) = \lim_{\alpha} \pi_A(x_\alpha e)(f) = (F \circ \pi_A(e))(f).
$$

Therefore, we get

$$
F \circ \pi_A(e) = \pi_A(y) \in \pi_A(A).
$$

Let $x \in A$. Since the socle $S$ of $B$ is dense in $B$ by [10, p. 100, Corollary (2.8.16)], we can write $x = \lim_{n \to \infty} x_n$, where $x_n \in S$ $(n = 1, 2, \ldots)$. Since $S$ is also the socle of $A$, it follows easily from (2) that

$$
F \circ \pi_A(x_n) \in \pi_A(A) \quad (n = 1, 2, \ldots).
$$

Let $f \in A^{**}$. By (1) we obtain $\|a \circ' f\| \leq M||f|| |a|$ for all $a \in A$ and consequently

$$
|(F \circ \pi_A(x_n) - F \circ \pi(x))(f)| = |F((x_n - x) \circ' f)| \leq M||F|| ||f|| |x_n - x|.
$$
Since \( x \to x \) in \( |\cdot| \), we have \( F \circ \pi_A(x) = F \circ \pi_A(x) \) in \( \|\cdot\| \). Hence it follows from (3) that \( F \circ \pi_A(x) \in \pi_A(A) \). Similarly we can show that \( \pi_A(x) \circ F \in \pi_A(A) \). Therefore \( \pi_A(A) \) is a two-sided ideal of \((A^{**}, \circ)\) and this completes the proof.

Remark. The preceding result generalizes a part of [13, p. 830, Theorem 5.2] as well as [12, p. 82, Theorem 3.3].

Corollary 3.2. Let \( A \) be as in Theorem 3.1. Then for every minimal idempotent \( e \in A \), \( A^{**} \circ \pi_A(e) \) and \( \pi_A(e) \circ A^{**} \) are minimal left and right ideals of \((A^{**}, \circ)\).

Proof. This follows immediately from Theorem 3.1 since \( A^{**} \circ \pi_A(e) = \pi_A(Ae) \) and \( \pi_A(e) \circ A^{**} = \pi_A(eA) \).

4. The radical of the algebra \((A^{**}, \circ)\). This section is devoted to the discussion of the radical of the algebra \((A^{**}, \circ)\). The main result in this section is useful in \( \S 5 \). Civi and Yood [5] had studied this problem for the group algebra of an infinite locally compact abelian group.

Throughout this section, unless otherwise stated, \( A \) will be a semisimple annihilator Banach algebra. Let \( R_1^{**} \) (resp. \( R_2^{**} \)) denote the radical of \((A^{**}, \circ)\) (resp. \((A^{**}, \circ')\)); \( R_1^{**} \) and \( R_2^{**} \) may not be zero (see [5, p. 857, Theorem 3.14] and [13, p. 831, Theorem 5.5]). By Theorem 3.1, \( \pi_A(A) \) is a two-sided ideal of \((A^{**}, \circ)\).

Theorem 4.1. Let \( A \) be a semisimple annihilator Banach algebra. Then the following statements hold:

(i) \( R_1^{**} \) is weakly closed.
(ii) \( R_1^{**} = \{ F \in A^{**} : A^{**} \circ F = (0) \} = \{ F \in A^{**} : F \circ' A^{**} = (0) \} \).
(iii) \( R_1^{**} \) coincides with \( R_2^{**} \).

Proof. Let \( E_A \) be the set of all minimal idempotents of \( A \). For each \( e \in E_A \), let \( M = (1 - \pi_A(e)) \circ A^{**} \). We show that \( M \) is a maximal modular right ideal of \((A^{**}, \circ)\). In fact, suppose there exists a right ideal \( M' \) of \((A^{**}, \circ)\) properly containing \( M \). Let \( F \in M' \) be such that \( F \notin M \). Then \( \pi_A(e) \circ F = F - (1 - \pi_A(e)) \circ F \in M' \) and \( \pi_A(e) \circ F \neq (0) \). Hence \((\pi_A(e) \circ A^{**}) \cap M' \neq (0) \) and consequently by Corollary 3.2 \( M' \supseteq \pi_A(e) \circ A^{**} \). Hence \( M' = A^{**} \). Therefore \( M \) is maximal.

Let \( \{ G_a \} \) be a net in \( M \) such that \( G_a \to G \) weakly for some \( G \in A^{**} \). Since \( \pi_A(e) \circ G_a = 0 \) for each \( a \), it follows that \( \pi_A(e) \circ G = 0 \) and hence \( G \in M \). Therefore \( M \) is weakly closed. Let \( R = \bigcap \{ (1 - \pi_A(e)) \circ A^{**} : e \in E_A \} \) and \( T = \{ F \in A^{**} : A^{**} \circ F = (0) \} \).

Then \( R \) is weakly closed and \( T \subseteq R_1^{**} \subseteq R \). Let \( F \in R \). Then \( \pi_A(e) \circ F = 0 \) for all \( e \in E_A \). Since the socle of \( A \) is dense in \( A \), we have \( \pi_A(A) \circ F = (0) \). Since
\(\pi_A(A)\) is weakly dense in \((A^{**}, \circ)\), it follows that \(A^{**} \circ F = (0)\) and so \(F \in T\). Consequently \(R_1^{**} = R = T\). Similarly by using maximal modular left ideals, we can show that \(R_2^{**} = \{F \in A^{**}: F \circ' A^{**} = (0)\}\). Let \(F \in R_1^{**}, G \in A^{**}\) and \(x_a \in A\) such that \(\pi_A(x_a) \rightarrow G\) weakly. Then \(F \circ \pi_A(x_a) = F \circ' \pi_A(x_a) \rightarrow F \circ' G\) weakly. Since by Theorem 3.1 \(F \circ \pi_A(x_a) \in R_1^{**} \cap \pi_A(A) = (0)\), we have \(F \circ' G = 0\) and so \(F \in R_2^{**}\). Hence \(R_1^{**} \subset R_2^{**}\). Similarly we can show that \(R_2^{**} \subset R_1^{**}\). Therefore they are equal and this completes the proof of the theorem.

**Remark 1.** Theorem 4.1 (ii) is a generalization of [5, p. 857, Theorem 3.15 (i)].

**Remark 2.** In general, \(R_1^{**} \neq \{F \in A^{**}: F \circ A^{**} = (0)\}\). In fact, let \(A\) be the group algebra of an infinite compact abelian group. Then by [5, p. 857, Theorem 3.12] \(R_1^{**} \neq (0)\). By [5, p. 855, Lemma 3.8], \(A^{**}\) has a right identity. Hence it follows that \(\{F \in A^{**}: F \circ A^{**} = (0)\} = (0) \neq R_1^{**}\).

**Notation.** In the rest of this paper, let \(R^{**} = R_1^{**} = R_2^{**}\).

**Corollary 4.2.** Suppose \(A\) is a semisimple commutative annihilator Banach algebra and \(M_A\) its carrier space. Let \(Q\) be the closed subspace of \(A^*\) spanned by \(M_A\) and let \(Q^\perp = \{F \in A^{**}: F(Q) = (0)\}\). Then \(Q^\perp = R^{**}\).

**Proof.** It is well known that \(M_A\) is discrete. For each \(b \in M_A\), let \(e_b\) be the minimal idempotent of \(A\) corresponding to the characteristic function of \(b\) ([10, p. 168, Theorem (3.6.3)]). For each \(b \in M_A\) and \(x \in A\), we have \(xe_b = e_bxe_b = b(x)e_b\). Therefore \((f \circ e_b)(x) = f(e_b)b(x)\) for all \(f \in A^*\). Hence \(f \circ e_b = f(e_b)b\).

Let \(F \in A^{**}\). Then \((\pi_A(e_b) \circ F)(f) = F(f \circ e_b) = f(e_b)F(b)\) for all \(f \in A^*\). Hence it follows easily that \(Q^\perp = \{F \in A^{**}: A^{**} \circ F = (0)\}\). Therefore by Theorem 4.1, \(Q^\perp = R^{**}\).

**Remark.** The above result is a generalization of [5, p. 857, Theorem 3.15 (ii)].

**Corollary 4.3.** Let \(M\) be a maximal modular right ideal of \((A^{**}, \circ)\). Then either \((l(M))^2 = (0)\) or there exists a minimal idempotent \(e\) of \(A\) such that \(M = (1 - \pi_A(e)) \circ A^{**}\). In the latter case, \(M\) is weakly closed. A similar result holds for left ideals.

**Proof.** If \(l(M) \subset R^{**}\), then by Theorem 4.1 \((l(M))^2 = (0)\). Suppose \(l(M) \not\subset R^{**}\). We claim that \(l(M) \cap \pi_A(A) \neq (0)\). Assume this is not so. Then \(\pi_A(A) \circ l(M) \subset \pi_A(A) \cap l(M) = (0)\). Hence \(A^{**} \circ l(M) = (0)\) and so by Theorem 4.1, \(l(M) \subset R^{**}\). This contradiction shows that \(l(M) \cap \pi_A(A) \neq (0)\). Therefore by [10, p. 98, Lemma (2.8.6)], \(l(M) \cap \pi_A(A)\) contains a minimal idempotent \(\pi_A(e)\) of \(\pi_A(A)\). By the maximality of \(M\), we have \(M = (1 - \pi_A(e)) \circ A^{**}\). Also \(M\) is weakly closed by the proof of Theorem 4.1 and this completes the proof.
We remark that a similar result for left ideals has been obtained by Civin for the group algebra of an infinite locally compact abelian group (see [3]).

5. Banach algebras which are ideals in a $B^*$-algebra. In this section, we study semisimple dual Banach algebras which are two-sided ideals in a $B^*$-algebra. There are many examples having such properties in analysis. The algebras $C_p$ discussed in [8] and the proper $H^*$-algebras are such examples. Unless otherwise stated, $A$ will be a semisimple dual Banach algebra which is a dense subalgebra of a $B^*$-algebra $B$ such that $\|\cdot\|$ majorizes $|\cdot|$ on $A$. It is well known that $B$ is also a dual algebra (see [12, p. 81]).

The following result is contained in Lemma 5.1 in [7].

**Lemma 5.1.** $A$ and $B$ have the same minimal idempotents and the same socle.

**Proof.** Let $e$ be a minimal idempotent of $A$. Then it is clear that $e$ is a minimal idempotent of $B$. By the proof of [12, p. 82, Lemma 3.2] $\|\cdot\|$ and $|\cdot|$ are equivalent on $Ae$ and $Be = Ae$, $eA = eB$. Therefore the socle $S$ of $A$ is a dense two-sided ideal of $B$. Let $f$ be a minimal idempotent of $B$. Then $Sf \subseteq Bf \subseteq S$ and so $Bf \subseteq S \subseteq A$. Therefore $f$ is a minimal idempotent of $A$. Now it is clear that $S$ is also the socle of $B$.

We shall now give a characterization for $A$ to be a two-sided ideal of $B$.

**Theorem 5.2.** Let $A$ be a semisimple dual Banach algebra which is a dense subalgebra of a $B^*$-algebra $B$ such that $\|\cdot\|$ majorizes $|\cdot|$ on $A$. Then the following statements are equivalent:

(i) $A$ is a two-sided ideal of $B$.

(ii) There exists a constant $M > 0$ such that $\|\sum_{k=1}^{n} e_k x_k\| \leq M\|x\|$ and $\|\sum_{k=1}^{n} x_k e_k\| \leq M\|x\|$, where $x \in A$ and $e_1, e_2, \ldots, e_n$ are any mutually orthogonal hermitian minimal idempotents of $B$.

(iii) For any orthogonal family of hermitian minimal idempotents $\{e_\lambda : \lambda \in \Lambda\}$ of $B$ and $x \in A$, $\sum_{\lambda} x e_\lambda$ and $\sum_{\lambda} e_\lambda x$ are summable in the norm of $A$ and especially when $\{e_\lambda : \lambda \in \Lambda\}$ is a maximal family, $x = \sum_{\lambda} x e_\lambda = \sum_{\lambda} e_\lambda x$ in $A$.

**Proof.** We know that $B$ is a dual algebra and $A$ and $B$ have the same minimal idempotents and the same socle by Lemma 5.1.

(i) $\Rightarrow$ (ii). Suppose (i) holds. Then by [2, p. 3, Theorem 2.3] there exists a constant $M$ such that $\|\sum_{k=1}^{n} e_k x_k\| \leq M\|\sum_{k=1}^{n} e_k\| \|x\| = M\|x\|$. Similarly, $\|\sum_{k=1}^{n} x_k e_k\| \leq M\|x\|$ and this proves (ii).

(ii) $\Rightarrow$ (iii). Suppose (ii) holds. Let $\{e_\lambda : \lambda \in \Lambda\}$ be an orthogonal family of hermitian minimal idempotents of $B$ and $x \in A$. Let $\{E_\gamma : \gamma \in \Gamma\}$ be the direct set of all finite sums $e_{\lambda_1} + e_{\lambda_2} + \cdots + e_{\lambda_n}$ ($\lambda_k \in \Lambda$ and $n = 1, 2, \ldots$). Since $\|xE_\gamma\| < M\|x\|$ by (ii), it follows from the Alaoglu theorem that $\{\pi_A(xE_\gamma)\}$ has
weak limit points in $A^{**}$. Let $F \in A^{**}$ be a weak limit point of $\{\pi_A(xy)_{\gamma}\}$. Then for any $y \in A$, $\pi_A(y) \circ F$ is a weak limit point of $\pi_A(xy_{\gamma})$. Since $A$ is a dual algebra, by Theorem 3.1 $\pi_A(y) \circ F \in \pi_A(A)$. Let $\{e_\alpha: \alpha \in \Delta\}$ be a maximal orthogonal family of hermitian minimal idempotents of $B$ containing $\{e_\alpha: \alpha \in \Delta\}$. Then it is easy to see that $\pi_A(y) \circ F \circ \pi_A(e_\alpha) = \pi_A(ye_\alpha) (\alpha \in \Delta)$. Since $\{e_\alpha: \alpha \in \Delta\}$ is maximal, it follows that $\pi_A(y) \circ F = \pi_A(xy)$ (see [9, p. 21]). Hence $\{xy_{\gamma}\}$ converges weakly to $yx$ and so by the Orlicz-Banach theorem [6, p. 93], $\sum_{\lambda} xe_\lambda$ is summable in the norm of $A$. Since $A$ is a dual algebra by [10, p. 91, Corollary (2.8.3)] $x \in c_l(A_x)$. Hence, for any given $\epsilon > 0$, there exists some $z \in A$ such that $\|x - zx\| < \epsilon$. Now by (ii) we have $\|xy_{\gamma}\| \leq M\|x - zx\| + \|zx_{\gamma}\| < M\epsilon + \|zx_{\gamma}\|$. Since $\sum_{\lambda} xe_\lambda$ is summable in $\|\cdot\|$ and $\epsilon$ is arbitrary, it follows that $\sum_{\lambda} xe_\lambda$ is summable in $\|\cdot\|$. If $\{e_\lambda: \lambda \in \Lambda\}$ is a maximal family, then it is easy to see that $x = \sum_{\lambda} xe_\lambda$. Similarly we can show that $x = \sum_{\lambda} e_y x$ and this proves (iii).

(iii) $\Rightarrow$ (i). Suppose (iii) holds. Let $x \in A$ and $y \in B$. We shall show that $xy \in B$. Since any element of $B$ is a linear combination of positive elements, we may assume that $y$ is a positive element. We also assume that $x \neq 0$ and $y \neq 0$. Let $E$ be a maximal commutative $*-$subalgebra of $B$ containing $y$. Then the carrier space $M_E$ of $E$ is discrete. For each $\lambda \in M_E$, let $e_\lambda$ be the element of $E$ corresponding to the characteristic function of $\lambda$. Then $\{e_\lambda: \lambda \in M_E\}$ is a maximal orthogonal family of hermitian minimal idempotents in $B$. Since $y \in E$ and $\text{Sp}_E(y) > 0$, we have $ye_\lambda = \beta_\lambda e_\lambda$, where $\beta_\lambda \geq 0$ for all $\lambda$ and $\beta_\lambda \leq |y|$. Since $B$ is a dual $B^*$-algebra, by the proof of (ii) $\Rightarrow$ (iii) (or [9, p. 22, Corollary 1]) $xy = \sum_{\lambda} xe_\lambda$ in $\|\cdot\|$ and so there exists only a countable number of $e_\lambda$ for which $xe_\lambda \neq 0$, say $e_1, e_2, \cdots$. For any two positive integers $m$, $n$ ($m < n$), let $z^n = \sum_{k=m}^n xe_k = \sum_{k=m}^n \beta_k xe_k$. Then $z^n \in A$. We shall show that $\{\sum_{k=1}^n xe_k\}$ is a Cauchy sequence in $A$. Clearly, we can assume that each $z^n_m$ is a nonzero element. Choose $f \in A^*$ such that $\|f\| = 1$ and $\|z^n_m\| = \|z^n_m\|$ by the Hahn-Banach theorem. Then $f(z^n_m) = \sum_{k=m}^n \beta_k f(xe_k)$. Write $f(xe_k) = a_k + ib_k$, where $a_k, b_k$ are real numbers. Then we have

$$\sum_{k=m}^n \beta_k f(xe_k) = \sum_{k=1}^n \beta_k a_k = \|z^n_m\| > 0.$$ 

Since $\beta_k \geq 0$, there exists some $a_k > 0$. Let $\{a_k\} \subset \{a_k\}_{k=m}^n$ such that $a_k, > 0$. Then we have

$$\left\| \sum_{k=m}^n xe_k \right\| = \|z^n_m\| = \sum_{k=m}^n \beta_k a_k \leq \sum_{k'} a_k' \leq |y| \left| \sum_{k'} f(xe_k') \right| \leq |y| \|f\| \left\| \sum_{k'} xe_k' \right\| = |y| \left\| \sum_{k'} xe_k' \right\|.$$ 

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Hence it follows from the assumption that \( \{ \sum_{i=1}^{n} xye_k \} \) is a Cauchy sequence in \( A \). Therefore, there exists an element \( z \in A \) such that \( z = \sum_{k=1}^{\infty} xye_k \) in \( ||\cdot|| \). Also \( xy = \sum_{k=1}^{\infty} xye_k \) in \( ||\cdot|| \). Hence it follows that \( xy = z \in A \). Similarly we can show that \( yx \in A \). Thus \( A \) is a two-sided ideal of \( B \) and this completes the proof of the theorem.

**Remark 1.** (i) \( \Rightarrow \) (iii) in the above theorem was obtained by Ogasawara and Yoshinaga for \( A^* \)-algebras (see [9, p. 30, Theorem 16]). Also (iii) \( \Rightarrow \) (i) was proved by them for weakly complete commutative \( A^* \)-algebras (see [9, p. 35, Theorem 2.3]). Some arguments in the proof of (ii) \( \Rightarrow \) (iii) of Theorem 5.2 are similar to those in the proof of [9, p. 30, Theorem 16].

**Remark 2.** If \( B \) is not a \( B^* \)-algebra, then Theorem 5.2 is not true. In fact, let \( G \) be an infinite compact group and let \( A \) be the algebra of all continuous functions on \( G \), normed by the maximum of the absolute value. It is well known that \( L^2(G) \) is an \( A^* \)-algebra and \( A \) is a dual \( A^* \)-algebra which is a dense two-sided ideal of \( L^2(G) \). However condition (iii) of Theorem 5.2 is not valid for \( A \). Since \( L^2(G) \) is a proper \( H^* \)-algebra, condition (iii) holds for \( L^2(G) \).

**Corollary 5.3.** Let \( A \) be a reflexive \( A^* \)-algebra which is a dense subalgebra of a \( B^* \)-algebra \( B \). Then the following statements are equivalent:

(i) \( A \) is a two-sided ideal of \( B \).

(ii) \( A \) is a dual algebra and, for any orthogonal family of hermitian minimal idempotents \( \{ e_\lambda : \lambda \in \Lambda \} \) of \( B \) and \( x \in A \), the set \( \{ \sum_{k=1}^{n} e_\lambda x : \lambda, k \in \Lambda \} \) is bounded in \( A \).

**Proof.** (i) \( \Rightarrow \) (ii). This follows immediately from [13, p. 831, Theorem 5.4] and Theorem 5.2 (ii).

(ii) \( \Rightarrow \) (i). Suppose (ii) holds. Since \( A \) is reflexive, \( \{ \sum_{k=1}^{n} e_\lambda x : \lambda, k \in \Lambda \} \) has weak limit points in \( A \). By the proof of Theorem 5.2, it has a unique weak limit point and so \( \sum_{\lambda} e_\lambda x \) is summable in the norm of \( A \). Therefore \( A \) is a two-sided ideal of \( B \) by Theorem 5.2.

It is well known that a reflexive \( B^* \)-algebra is finite dimensional. The following corollary is a generalization of this result.

**Corollary 5.4.** Let \( A \) be a reflexive \( A^* \)-algebra which is a dense two-sided ideal of a \( B^* \)-algebra \( B \). If \( A \) has an approximate identity, then \( A \) is finite dimensional.

**Proof.** It follows immediately from [5, p. 855, Lemma 3.8] and Corollary 5.3 that \( A \) is a dual algebra with an identity. Therefore \( A \) is finite dimensional.

It is well known that \( B \) is Arens regular if \( B \) is a \( B^* \)-algebra. Let \( A \) be a semisimple dual Banach algebra which is a dense two-sided ideal of a \( B^* \)-algebra \( B \). Is \( A \) Arens regular? This question was asked in [13, p. 833]. We shall answer this question affirmatively.
Notation. In the rest of this section, $B^{**}$ with the Arens product will be denoted by $(B^{**}, *)$.

Lemma 5.5. Suppose $B$ is a dual $B^*$-algebra and $S$ its socle. Let $B'$ be the closed subspace of $B^*$ spanned by $\pi_B(x) \ast g$, where $x \in S$ and $g \in B^*$. Then $B^*$ coincides with $B'$.

Proof. Suppose this is not true. Then there exists a nonzero linear functional $F \in B^{**}$ such that $F(B') = (0)$. Hence, for all $x \in S$, $(F \ast \pi_B(x))(g) = F(\pi_B(x) \ast g) = 0$. Since $S$ is weakly dense in $B^{**}$, it follows that $F \ast B^{**} = (0)$. Since $B^{**}$ is a $B^*$-algebra, we have $F = 0$, a contradiction. Therefore $B^*$ coincides with $B'$.

In the rest of this section, let $A$ be a semisimple Banach algebra which is a dense two-sided ideal of a $B^*$-algebra $B$. By [2, p. 3, Proposition 2.2], there exists a constant $k$ such that $k\|x\| \geq \|x\|$ on $A$ and consequently by [2, p. 3, Theorem 2.3] there exists a constant $M$ such that $\|ab\| \leq M\|a\|\|b\|$ and $\|ba\| \leq M\|a\|\|b\|$ for all $a \in A, b \in B$. For each $g \in B^*$, let $g_A$ denote the restriction of $g$ to $A$. Then it is easy to see that $g_A \in A^*$. For every element $F \in A^{**}$, let $\tilde{F}$ be the linear functional on $B^*$ defined by $\tilde{F}(g) = F(g_A)$ $(g \in B^*)$. Then $\tilde{F} \in B^{**}$. Let $b \in B$ and $f \in A^*$. Define $(f \circ b)(a) = f(ba)$ $(a \in A)$. Since $|f \circ b(a)| \leq M\|f\|\|b\|\|a\|$, it follows that $f \circ b \in A^*$.

As before, let $R^{**}$ denote the radical of $(A^{**}, \circ)$.

Lemma 5.6. Suppose $A$ is an annihilator algebra. Then the following statements hold:

(i) For each $R \in R^{**}$ and $g \in B^*$, we have $\tilde{R}(g) = 0$.

(ii) $R^{**}$ is the left and right annihilator of $(A^{**}, \circ)$.

Proof. (i) Let $g \in B^*$. By Lemma 5.5, we can write $g = \lim_n g_n$ where $g_n = \sum_{i=1}^m \pi_B(x^n_i) \ast g^n_i$ with $x^n_i \in S$ (the socle of $B$) and $g^n_i \in B^*$. Clearly $x^n_i \in A$. Then for each $R \in R^{**}$, we have

$$\tilde{R}(g) = \lim_n \sum_{i=1}^m \tilde{R}(\pi_B(x^n_i) \ast g^n_i) = \lim_n \sum_{i=1}^m (R \circ \pi_A(x^n_i))(g^n_i)_A.$$ 

By Theorem 4.1, we have $R \circ \pi_A(x^n_i) = 0$ and therefore $\tilde{R}(g) = 0$. This proves (i).

(ii) For each $F \in A^{**}$ and $f \in A^*$, define $\tilde{f}_F = F(f \circ b)$ $(b \in B)$. Then it is easy to see that $\tilde{f}_F \in B^*$ and $\tilde{f}_F = F \circ f$. Then for all $R \in R^{**}$, we have $(R \circ F)(f) = R(F \circ f) = \tilde{R}(\tilde{f}_F)$. Therefore by (i), $R \circ F = 0$ and so $R^{**} \circ A^{**} = (0)$. By Theorem 4.1, we also have $A^{**} \circ R^{**} = (0)$ and this completes the proof.

Now we are ready to prove the following result:
Theorem 5.7. Let \( A \) be a semisimple dual Banach algebra which is a dense two-sided ideal of a \( B^* \)-algebra. Then the following statements hold:

(i) \( A \) is Arens regular.

(ii) \( A^{**}/R^{**} \) is a semisimple Banach algebra which is a dense two-sided ideal of some \( B^* \)-algebra.

Proof. (i) Let \( \{e_{\lambda} : \lambda \in \Lambda\} \) be a maximal orthogonal family of hermitian minimal idempotents in \( B \). Let \( \{E_\beta\} \) be the direct set of all finite sums \( e_{\lambda_1} + e_{\lambda_2} + \cdots + e_{\lambda_n} \) \( (\lambda_n \in \Lambda, \ n = 1, 2, \cdots) \). Let \( F \) and \( G \) be two functionals in \( A^{**} \). Since \( \|F \circ \pi_A(E_\beta)\| \leq M\|F\| \|E_\beta\| = M\|F\| \), it follows from Alaoglu’s theorem that \( \{F \circ \pi_A(E_\beta)\} \) has weak limit points in \( A^{**} \). Let \( \{E_a\} \) be a subnet of \( \{E_\beta\} \) and \( F_1 \in A^{**} \) such that \( F \circ \pi_A(E_a) \rightharpoonup F_1 \) weakly. By a similar argument, there exists a subnet \( \{E_\gamma\} \) of \( \{E_a\} \) and \( G_1 \in A^{**} \) such that \( \pi_A(E_\gamma) \circ G \rightharpoonup G_1 \) weakly. Let \( a \in A \). Then by Theorem 5.2, \( a = \sum_\lambda e_\lambda a \in \|\cdot\| \). Hence \( E_\beta a \rightharpoonup a \) weakly. Thus \( E_\gamma a \rightharpoonup a \) weakly. Since \( F \circ \pi_A(x) = F \circ \pi_A(x) \) for all \( x \in A \), we have \( E \circ \pi_A(a) = \text{weak limit} \ F \circ \pi_A(E_\beta a) = F \circ \pi_A(a) \). Since \( \pi_A(A) \) is weakly dense in \( A^{**} \), it follows that \( (F - F_1) \circ A^{**} = (0) \) and so by Theorem 4.1, \( F - F_1 \in R^{**} \). Similarly we can show that \( G_1 - G \in R^{**} \). Then by Lemma 5.6, we have

\[
F \circ G = (F_1 + (F - F_1)) \circ G = F_1 \circ G
\]

\[
= \text{weak lim } F \circ \pi_A(E_\gamma) \circ G = \text{weak lim } F \circ \pi_A(E_\gamma) \circ G
\]

\[
= F \circ G.
\]

Therefore \( A \) is Arens regular by definition and this proves (i).

(ii) Now the algebra \( A^{**}/R^{**} \) is a semisimple Banach algebra. For each \( a \in A \) and \( f \in A^* \), define \( (f \ast a)(b) = f(ab) \) \( (b \in B) \). Then \( f \ast a \in B^* \). For each \( F \in A^{**} \), we write \( \tilde{F} = F + R^{**} \) and define a mapping \( \Phi \) from \( A^{**}/R^{**} \) into \( B^{**} \) by \( \Phi(\tilde{F}) = \tilde{F} - F \) \( (F \in A^{**}) \). Suppose \( \Phi(\tilde{F}) = 0 \). Then \( \tilde{F}(f \ast a) = 0 \) and therefore \( (\pi_A(a) \circ F)(f) = 0 \) for all \( a \in A \) and \( f \in A^* \). Consequently \( F \in R^{**} \) and therefore \( \tilde{F} = R^{**} \). Hence it follows that \( \Phi \) is an isomorphism of \( A^{**}/R^{**} \) into \( B^{**} \). For each \( g \in B^* \), we have \( \|g_A\| \leq k|g| \). Since by Lemma 5.5 (i), \( R(g_A) = 0 \) for all \( R \in R^{**} \), straightforward calculations yield that \( k\|F + R\| \geq \|\tilde{F}\| \) for all \( F \in A^{**} \). Hence \( k\|\tilde{F}\| \geq \|\tilde{F}\| \) and consequently \( \Phi \) is continuous. For each \( H \in B^{**} \), define \( (H \circ f)(a) = H(f \ast a) \) \( (f \in A^*, \ a \in A) \). Then \( H \circ f \in A^* \). For each \( F \in A^{**} \), define \( F_H(f) = F((H \circ f))(f \in A^*, \ F \in A^{**}) \). Then \( F_H \in A^{**} \). For each \( g \in B^* \), we have

\[
\tilde{F}_H(g) = F((H \circ g_A)) = F((H \ast g)_A) = (\tilde{F} \ast H)(g).
\]

Therefore \( \tilde{F} \ast H = \tilde{F}_H \). Consequently \( \Phi(A^{**}/R^{**}) \) is a two-sided ideal of \( B^{**} \). Let \( Q \) be the norm closure of \( \Phi(A^{**}/R^{**}) \) in \( B^{**} \). Then \( Q \) is a closed two-
sided ideal of $B^{**}$. Since $B^{**}$ is a $B^*$-algebra, so is $Q$. This completes the proof of the theorem.

Remark. We know that the above result is not true for arbitrary dual $A^*$-algebras (see [13, p. 833, Remark]). Also if $A$ is a dual $A^*$-algebra which is Arens regular, $A$ may not be a two-sided ideal of its completion in an auxiliary norm; in fact, $A$ can be reflexive (see [9, p. 35]).

Let $\mathfrak{A} = A^{**}/R^{**}$. Clearly, we can identify $A$ as a closed two-sided ideal of $\mathfrak{A}$.

Corollary 5.8. Let $A$ be as in Theorem 5.7. Then $\mathfrak{A}$ coincides with $A$ if and only if the socle of $\mathfrak{A}$ is dense in $\mathfrak{A}$.

Proof. We use the notation in the proof of Theorem 5.7. Suppose the socle of $\mathfrak{A}$ is dense in $\mathfrak{A}$. Then $Q$ is a dual $B^*$-algebra. For each minimal idempotent $e \in Q$ and $b \in B$, we have $e = ke\pi_B(b)e \in \pi_B(B)$, where $k$ is a constant.

Hence it follows that $Q = B$. Now it is easy to see that $\mathfrak{A}^2 \subset A$. Since the socle of $\mathfrak{A}$ is dense in $\mathfrak{A}$, $\mathfrak{A} \subset A$ and so $\mathfrak{A} = A$. The converse of the corollary is clear and this completes the proof.

If $A$ is reflexive, then it is clear that $A^{**}$ is semisimple. However, in general, $A^{**}$ may not be semisimple as shown in [13, p. 831, Theorem 5.5].

Corollary 5.9. Let $A$ be as in Theorem 5.7. Then $A^{**}$ is semisimple if and only if $A^*$ is spanned by $\pi_A(x) \circ f$, where $f \in A^*$ and $x \in A$.

Proof. Suppose $A^*$ is spanned by $\pi_A(x) \circ f$. Let $F \in R^{**}$. Since $F \circ \pi_A(x) = 0$ for all $x \in A$, it follows that $F(f) = 0$ for all $f \in A^*$. Hence $F = 0$. The converse of the corollary follows immediately from the proof of Lemma 5.5.

Let $A$ be a Banach $*$-algebra. For all $x \in A$, $f \in A^*$ and $F \in A^{**}$, we define

$$f^*(x) = \overline{f(x^*)}$$

and

$$F^*(f) = \overline{F(f^*)},$$

where the bar denotes the complex conjugation. If $A$ is a $B^*$-algebra, then $A^{**}$ is a $B^*$-algebra under the involution $F \rightarrow F^*$ (see [11, p. 192]).

Corollary 5.10. Let $A$ be a dual $A^*$-algebra which is a dense two-sided ideal of a $B^*$-algebra $B$. Then $(A^{**}, \circ)$ is a $*$-algebra and $A^{**}/R^{**}$ is an $A^*$-algebra which is a dense two-sided ideal of a $B^*$-algebra.

Proof. By Theorem 5.7, $A$ is Arens regular and so $A^{**}$ is a $*$-algebra under the involution $F \rightarrow F^*$ by [11, p. 186, Theorem 1]. Clearly $R^{**}$ is a $*$-ideal of $A^{**}$. Now the corollary follows easily from Theorem 5.7.

It was asked in [13, p. 833] whether the algebra $C_p^{**}$ is semisimple. If
$1 < p < \infty$, then $C_p$ is reflexive (see [8, p. 265]) and, therefore, it is semisimple. If $p = 1$, then by [12, p. 831, Theorem 5.5], $C_1^{**}$ is not semisimple unless it is finite dimensional.

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