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SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS WITH SUPPORT ON LEAVES OF ASSOCIATED FOLIATIONS

BY

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ABSTRACT. Suppose that the linear partial differential operator $P(x, D)$ has analytic coefficients and that it can be written in the form $P(x, D) = R(x, D)S(x, D)$ where $S(x, D)$ is a polynomial in the homogeneous first order operators $A_1(x, D), \cdots, A_r(x, D)$. Then in a neighborhood of any point x^0 at which the principal part of $S(x, D)$ does not vanish identically, there is a solution of $P(x, D)u = 0$ with support the leaf through x^0 of the foliation induced by the Lie algebra generated by $A_1(x, D), \cdots, A_r(x, D)$. This result yields necessary conditions for hypoellipticity and uniqueness in the Cauchy problem. An application to second order degenerate elliptic operators is also given.

1. **Introduction.** Let $P(x, D)$ be a linear partial differential operator of order m with complex valued coefficients defined and analytic in an open connected set Ω in R^n,

$$P(x, D) = \sum_{|\alpha| \le m} a^{\alpha}(x) D^{\alpha} \tag{1}$$

where $x = (x_1, \cdots, x_n)$, $\alpha = (\alpha_1, \cdots, \alpha_n)$ is an n-tuple of nonnegative integers with $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $D^{\alpha} = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ with $D_j = \partial/\partial x_j$. The principal part $P_m(x, D)$ is the homogeneous part of $P(x, D)$ of order m.

In this note we suppose that in Ω the partial differential operator $P(x, D)$ can be written in the form

$$\begin{aligned} P(x, D) &= R(x, D)S(x, D), \\ S(x, D) &= Q(x; A_1(x, D), \cdots, A_r(x, D)) \end{aligned} \tag{2}$$

where $R(x, D)$ and $S(x, D)$ are linear operators of order $m - l$ and l respectively $(0 \le l \le m)$ and $Q(x; \eta_1, \cdots, \eta_r)$ is a polynomial of order l in $\eta_1, \cdots \eta_r$ with coefficients which are complex valued and analytic functions of x in Ω, and

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(3) $$A_j(x, D) = \sum_{i=1}^{n} a_j^i(x) D_i, \qquad j = 1, \cdots, r,$$

where $a_j^i(x)$ $(i = 1, \cdots, n;\ j = 1, \cdots r)$ are real valued and analytic functions of x in Ω.

We denote by $\mathfrak{L}(A_1, \cdots, A_r)$ the Lie algebra generated by the vector fields $A_1, \cdots, A_r$, i.e. the smallest set of analytic vector fields in Ω which is closed under the operations of taking brackets and linear combinations with analytic coefficients. According to a theorem of Nagano [3], which is an extension of the classical theorem of Frobenius, the Lie algebra $\mathfrak{L}(A_1, \cdots, A_r)$ defines a unique partition of Ω into maximal integral manifolds of $\mathfrak{L}(A_1, \cdots, A_r)$, that is, Ω is the disjoint union of maximal integral manifolds of $\mathfrak{L}(A_1, \cdots, A_r)$. This partition is called a foliation and each maximal integral manifold is called a leaf of the foliation.

The main result of this note is that if $P(x, D)$ can be written in the form (2) in Ω, and if x^0 is any point of Ω such that $S_l(x^0, \xi) \neq 0$ for some $\xi \in R^n$, then there is a neighborhood $U \subset \Omega$ of x^0 and a distribution u in U such that $P(x, D)u = 0$ in U and the support of u is equal to $L^{x^0}(A_1, \cdots, A_r) \cap U$, where $L^{x^0}(A_1, \cdots, A_r)$ is the leaf through x^0 of the foliation of Ω induced by $\mathfrak{L}(A_1, \cdots, A_r)$. Moreover if, in addition, $\dim \mathfrak{L}(A_1, \cdots, A_r)(x^0) = \dim L^{x^0}(A_1, \cdots, A_r) < n$, then the singular support of u is equal to its support and, as a corollary, $P(x, D)$ is not hypoelliptic in Ω. Another corollary gives a necessary condition for uniqueness in the Cauchy problem for $P(x, D)$ (see Corollary 2).

Clearly, every linear partial differential operator can be written in the form (2). Indeed, we can simply take $r = n$ and $A_1 = D_1, \cdots, A_n = D_n$, but for this representation of $P(x, D)$ the result of this paper is nothing more than an immediate application of the Cauchy-Kovalevsky theorem since $\dim \mathfrak{L}(D_1, \cdots, D_n) = n$ everywhere and the foliation induced by $\mathfrak{L}(D_1, \cdots, D_n)$ consists of the single leaf Ω. The result of this paper is interesting only when $P(x, D)$ can be written in the form (2) with $A_1, \cdots, A_r$ such that the foliation of Ω induced by $\mathfrak{L}(A_1, \cdots, A_r)$ is finer than Ω. Among the more well-known operators for which this is possible are first order operators

(4) $$P(x, D) = A_1(x, D) + iA_2(x, D) + c(x) \qquad (i = \sqrt{-1}),$$

and second order degenerate elliptic operators of the form

(5) $$P(x, D) = A_1^2(x, D) + \cdots + A_r^2(x, D) + A_0(x, D) + c(x)$$

where $A_0(x, D)$ is also of the form (3).

The result of this paper is a generalization of the result obtained in [4] for first order operators (4). In [4] we also obtained the result that for the first order operator (4), if u is a solution of $P(x, D)u = 0$ in Ω and if $x^0 \in \Omega$ belongs to the support of u, then the leaf $L^{x^0}(A_1, A_2)$ also belongs to the support of u. This result was generalized in [5] as follows: Suppose the operator $P(x, D)$ has analytic coefficients in Ω and suppose there are r analytic vector fields $A_1, \cdots, A_r$ as given by (3), such that for every $x \in \Omega$,

$$(6) \qquad P_m(x, \xi) = 0, \qquad \xi \in R^n \Rightarrow \sum_{i=1}^{n} a_j^i(x)\xi_i = 0, \qquad j = 1, \cdots, r.$$

Then if u is a solution of $P(x, D)u = 0$ in Ω and if $x^0 \in \Omega$ belongs to the support of u then the leaf $L^{x^0}(A_1, \cdots, A_r)$ also belongs to the support of u. Note that in geometrical language, condition (6) means that at every point of Ω the characteristic cone of $P(x, D)$ is orthogonal to the vector fields $A_1, \cdots, A_r$.

At the end of § 3 we illustrate the results of [5] and of the present paper, by applying them to the question of uniqueness in the Cauchy problem for the second order degenerate elliptic operators (5).

2. **Some results on foliations.** Let A and B be two vector fields, $A = \sum_{i=1}^n a^i D_i$, $B = \sum_{i=1}^n b^i D_i$. The bracket $[A, B]$ is the commutator

$$[A, B] = AB - BA = \sum_{i=1}^{n} \left[\sum_{k=1}^{n} a^k(x) D_k b^i(x) - \sum_{k=1}^{n} b^k(x) D_k a^i(x) \right] D_i.$$

If A and B have analytic coefficients in Ω, then $[A, B]$ is also a vector field with analytic coefficients in Ω. The (real) vector space of all real analytic vector fields in Ω equipped with the bracket operation is a Lie algebra denoted by $\mathfrak{L}(\Omega)$. It is also a module over the ring of real analytic functions in Ω. A vector subspace of $\mathfrak{L}(\Omega)$ which is closed under the bracket operation is a Lie subalgebra of $\mathfrak{L}(\Omega)$. $\mathfrak{L}(A_1, \cdots, A_r)$ is the smallest $\mathfrak{A}$-submodule and Lie subalgebra of $\mathfrak{L}(\Omega)$ containing $A_1, \cdots, A_r$.

Let $\mathfrak{L}$ be a vector subspace of $\mathfrak{L}(\Omega)$. For any $x \in \Omega$ we set $\mathfrak{L}(x) = \{A(x): A \in \mathfrak{L}\}$. $\mathfrak{L}(x)$ is a subspace of R^n and is called the integral element of $\mathfrak{L}$ at x. An integral manifold N of $\mathfrak{L}$ is a connected submanifold of Ω such that for every $x \in N$, the tangent space to N at x is equal to $\mathfrak{L}(x)$.

Theorem 1. *If $\mathfrak{L}$ is a Lie subalgebra of $\mathfrak{L}(\Omega)$, then through every point $x \in \Omega$ passes a maximal integral manifold L^x of $\mathfrak{L}$. Any integral manifold of $\mathfrak{L}$ containing x is an open submanifold of L^x.*

According to Theorem 1, $\mathfrak{L}$ defines a unique partition of Ω by integral manifolds of $\mathfrak{L}$ (that is, Ω is the disjoint union of maximal integral manifolds of $\mathfrak{L}$). This partition of Ω will be called the foliation defined by $\mathfrak{L}$ and each maximal integral manifold will be called a leaf of the foliation. Note that for every $x \in \Omega$, the dimension of the leaf L^x containing x is equal to the dimension of the integral element $\mathfrak{L}(x)$.

With the additional assumption that $\dim \mathfrak{L}(x)$ is constant in Ω, Theorem 1 is the classical theorem of Frobenius (see Chevalley [2]). However, the Frobenius theorem is also valid in the C^∞ case. Theorem 1 was proved by Nagano [3] and it is not generally valid in the C^∞ case.

We will apply Theorem 1 to the Lie subalgebra $\mathfrak{L}(A_1, \cdots, A_r)$. The leaf of its foliation through x will be denoted by $L^x(A_1, \cdots, A_r)$.

3. Statement of results and an application.

Theorem 2. *Suppose that $P(x, D)$ can be written in the form (2) in Ω and that x^0 is any point of Ω such that $S_l(x^0, \xi) \neq 0$ for some $\xi \in R^n$. Then there is a neighborhood $U \subset \Omega$ of x^0 and a distribution u in U such that*

$$P(x, D)u = 0 \quad \text{in } U$$

and

$$\operatorname{supp} u = L^{x^0}(A_1, \cdots, A_r) \cap U.$$

The proof of Theorem 2 is presented in § 4. The solution of $P(x, D)u = 0$ constructed in this proof is such that if $\dim \mathfrak{L}(A_1, \cdots, A_r)(x^0) < n$ then sing supp u = supp u. This leads immediately to the following corollary.

Corollary 1. *Suppose that $P(x, D)$ can be written in the form (2) in Ω and that, for some $x^0 \in \Omega$, $S_l(x^0, \xi) \neq 0$ for some $\xi \in R^n$ and $\dim \mathfrak{L}(A_1, \cdots, A_r)(x^0) < n$. Then $P(x, D)$ is not hypoelliptic in Ω.*

Recall that the operator $P(x, D)$ is said to be hypoelliptic in Ω if for any open subset U of Ω, $P(x, D)u \in C^\infty(U)$ implies that $u \in C^\infty(U)$.

Another immediate application of Theorem 2 is on the question of uniqueness in the Cauchy problem.

Definition 1. Let $P(x, D)$ be a partial differential operator defined in an open set $\Omega \subset R^n$ and let $x^0 \in \Omega$ be a boundary point of a closed subset F of Ω. We say that there is uniqueness in the Cauchy problem for the system (P, x^0, F) if for every open neighborhood $U \subset \Omega$ of x^0 there is an open neighborhood $V \subset U$ of x^0 such that for every distribution u in U,

$$\left.\begin{array}{l} P(x, D)u = 0 \quad \text{in } U \\ \operatorname{supp} u \subset F \cap U \end{array}\right\} \Longrightarrow u = 0 \quad \text{in } V.$$

Corollary 2. *Let $P(x, D)$, x^0 and F be as in Definition 1 and suppose that $P(x, D)$ has analytic coefficients and it can be written in the form (2) in Ω. Suppose also that $S_l(x^0, \xi) \neq 0$ for some $\xi \in R^n$. If for some open neighborhood $U \subset \Omega$ of x^0,*

$$L^{x^0}(A_1, \cdots, A_r) \cap (U \sim F) = \varnothing,$$

then there is no uniqueness in the Cauchy problem for the system (P, x^0, F).

We conclude this section with an application to the second order degenerate elliptic operator (5). For this operator and for the vector fields $A_1, \cdots, A_r$, condition (6) is satisfied. According to the result of [5] mentioned at the end of the introduction, if u is a solution of $P(x, D)u = 0$ and if $x^0 \in \operatorname{supp} u$ then $L^{x^0}(A_1, \cdots, A_r) \subset \operatorname{supp} u$. Therefore a sufficient condition for uniqueness in the Cauchy problem for the system (P, x^0, F) is

(7) For every open neighborhood $U \subset \Omega$ of x^0,

$$L^{x^0}(A_1, \cdots, A_r) \cap (U \sim F) \neq \varnothing.$$

On the other hand, operator (5) is in the form (2), i.e. it is a polynomial in the operators $A_1, \cdots, A_r, A_0$. Under the additional assumption that not all of the $A_1, \cdots, A_r$ vanish at x^0 it follows from Corollary 2 that a necessary condition for uniqueness in the Cauchy problem for the system (P, x^0, F) is

(8) For every open neighborhood $U \subset \Omega$ of x^0,

$$L^{x^0}(A_1, \cdots, A_r, A_0) \cap (U \sim F) \neq \varnothing.$$

Thus if not all of the $A_1, \cdots, A_r$ vanish at x^0 and if $L^{x^0}(A_1, \cdots, A_r) = L^{x^0}(A_1, \cdots, A_r, A_0)$, condition (7) or (8) is necessary and sufficient for uniqueness in the Cauchy problem of the system (P, x^0, F). If $\dim \mathfrak{L}(A_1, \cdots, A_r)(x^0) = n$ then $L^{x^0}(A_1, \cdots, A_r) = \Omega$ and there is always uniqueness in the Cauchy problem for the system (P, x^0, F) for any closed set F. This last result was first obtained by Bony [1].

4. **Proof of Theorem 2.** We may take $x^0 = 0$. Let $\dim L^0(A_1, \cdots, A_r) = k$. In a neighborhood of 0 we introduce new coordinates $x = (x_1, \cdots, x_n)$ such that the leaf $L^0(A_1, \cdots, A_r)$ is given by the equations

$$x_{k+1} = \cdots = x_n = 0.$$

Clearly this change in coordinates is analytic. For convenience, let

$$x = (y, z), \qquad y = (x_1, \cdots, x_k), \qquad z = (x_{k+1}, \cdots, x_n)$$

so that $L^0(A_1, \cdots, A_r)$ is given by $z = 0$. In the new coordinates, the operators $A_j(x, D)$, $j = 1, \cdots, r$, have the form

$$A_j(y, z; D) = B_j(y, z; D_y) + C_j(y, z; D_z) \tag{9}$$

where

$$B_j(y, z; D_y) = a_j^1(y, z)D_1 + \cdots + a_j^k(y, z)D_k, \tag{10}$$

$$C_j(y, z; D_z) = a_j^{k+1}(y, z)D_{k+1} + \cdots + a_j^n(y, z)D_n, \tag{11}$$

and, on the leaf $L^0(A_1, \cdots, A_r)$,

$$C_j(y, 0; D_z) = 0 \tag{12}$$

so that

$$A_j(y, 0; D) = B_j(y, 0; D_y). \tag{13}$$

We look for the required solution of $P(x, D)u = 0$ in the form

$$u(x) = v(y)\delta(z) \tag{14}$$

where $v(y)$ is a nonvanishing analytic function of y and $\delta(z) = \delta(x_{n+1}) \cdots \delta(x_n)$. Now, for every $j = 1, \cdots r$ and for every u of the form (14), we have

$$A_j(y, z; D)u(x) = [B_j(y, z; D_y)v(y)]\delta(z) + v(y)[C_j(y, z; D_z)\delta(z)].$$

In view of (12), a simple computation shows that $C_j(y, z; D_z)\delta(z) = a_j(y)\delta(z)$ where $a_j(y) = -(\partial a_j^{k+1}/\partial x_{k+1})(y, 0) - \cdots - (\partial a_j^n/\partial x_n)(y, 0)$. Moreover, if $c(x) = c(y, z)$ is any analytic function of x,

$$c(x)u(x) = c(y, 0)v(y)\delta(z). \tag{15}$$

Hence

$$A_j(y, z; D)u(x) = \{[B_j(y, 0; D_y) + a_j(y)]v(y)\}\delta(z). \tag{16}$$

Now, using (2), (15) and (16) it is easy to see that if $u(x)$ is of the form (14),

$$P(x, D)u(x) = R(x, D)[\tilde{S}(y, D_y)v(y)]\delta(z) \tag{17}$$

where $\tilde{S}(y, D_y)$ is a partial differential operator of order l of the form

$$\begin{aligned} \tilde{S}(y, D_y) &= Q(y, 0; B_1(y, 0; D_y), \cdots, B_r(y, 0; D_y)) \\ &\quad + Q'(y; B_1(y, 0; D_y), \cdots, B_r(y, 0; D_y)) \end{aligned} \tag{18}$$

where $Q'(y; \eta_1, \cdots, \eta_r)$ is a polynomial of degree $< l$ in $\eta_1, \cdots, \eta_r$ with analytic coefficients in y. It follows from (17) that in order to show the existence of a solution u of $P(x, D)u = 0$ as asserted by the theorem, it is enough to show that in a neighborhood of the origin of the y-space, there is a nonvanishing analytic solution $v(y)$ of the equation

$$\tilde{S}(y, D_y)v(y) = 0. \tag{19}$$

That such a $v(y)$ exists would follow from the Cauchy-Kovalevsky theorem provided we can show that there exists a $\xi_y \in R^k$ such that $\tilde{S}_l(0, \xi_y) \neq 0$. For this, we use the assumption of the theorem that there is a $\xi \in R^n$ such that $S_l(0, \xi) \neq 0$. Let $\xi = (\xi_y, \xi_z)$ where $\xi_y = (\xi_1, \cdots, \xi_k)$ and $\xi_z = (\xi_{k+1}, \cdots, \xi_n)$. Then, if Q_l is the homogeneous part of Q consisting of terms of order l, we have

$$\begin{aligned} 0 \neq S_l(0, \xi) &= Q_l(0; A_1(0, \xi), \cdots, A_r(0, \xi)) \\ &= Q_l(0; B_1(0, \xi_y), \cdots, B_r(0, \xi_y)) = \tilde{S}_l(0, \xi_y). \end{aligned}$$

The proof of the theorem is now complete.

Added in proof. Theorem 1 was first stated by R. Hermann (*On the accessibility problem in control theory,* Internat. Sympos. Nonlinear Differential Equations and Nonlinear Mechanics, Academic Press, New York, 1963, pp. 325–332. MR 26 #6891). The definitive results in this area have been recently announced by H. J. Sussmann (*Orbits of families of vector fields and integrability of systems with singularities*, Bull. Amer. Math. Soc. 79 (1973), 197–199). Sussmann gives the details in a paper which is to appear in the Trans. Amer. Math. Soc.

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