TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 180, June 1973

P-COMMUTATIVE BANACH *-ALGEBRAS(1), (2)

BY

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ABSTRACT. Let A be a complex *-algebra. If f is a positive functional on A, let $I_f = \{x \in A: f(x^*x) = 0\}$ be the corresponding left ideal of A. Set $P = \bigcap I_f$, where the intersection is over all positive functionals on A. Then A is called P-commutative if $xy - yx \in P$ for all $x, y \in A$. Every commutative *-algebra is P-commutative and examples are given of noncommutative *-algebras which are P-commutative. Many results are obtained for P-commutative Banach *-algebras which extend results known for commutative Banach *-algebras. Among them are the following: If $A^2 = A$, then every positive functional on A is continuous. If A has an approximate identity, then a nonzero positive functional on A is a pure state if and only if it is multiplicative. If A is symmetric, then the spectral radius in A is a continuous algebra seminorm.

1. **Introduction.** The theory of commutative Banach *-algebras is, generally speaking, much better understood than the corresponding noncommutative theory. The purpose of this paper is to study a new class of noncommutative Banach *-algebras, called P-commutative Banach *-algebras, which properly contains the class of commutative Banach *-algebras; as we shall see, many of the well-known properties of commutative Banach *-algebras remain true for P-commutative algebras.

The structure of the paper is as follows. In §2 we introduce the notation and terminology which we shall use throughout. In §3 we define P-commutative algebras, prove some theorems concerning subalgebras, ideals, and quotients, and then discuss several examples. In §4 we show that if A is a P-commutative Banach *-algebra satisfying $A^2 = A$, then every positive functional on A is automatically continuous. N. Varopoulos [9] proved the same result for commutative Banach *-algebras with continuous involution. I. Murphy [5] gave a different proof which did not require continuity of the involution. We also show that if A is a P-commutative Banach *-algebra with an approximate identity, then the existence of a nonzero positive functional on A implies the existence of a multiplicative linear functional on A; and in fact, a nonzero positive linear functional is a pure state if

Presented to the Society, January 17, 1972; received by the editors February 22, 1972 and, in revised form, July 24, 1972.

AMS (MOS) subject classifications (1970). Primary 46H05, 46J99; Secondary 46K99.

Key words and phrases. Banach *-algebra, positive functional, *-representation, multiplicative linear functional, symmetric *-algebra.

(1) The contents of this paper form a portion of the author's doctoral dissertation at Texas Christian University under the direction of Professor Robert S. Doran.

(2) This research was partially supported by the National Science Foundation.

and only if it is multiplicative. In §5 we study P-commutative Banach *-algebras which are symmetric. We show, among other things, that the spectral radius in such an algebra is a continuous algebra seminorm. An example will show that the assumption of symmetry cannot be dropped. Finally we show that an arbitrary Banach *-algebra with an approximate identity is P-commutative if its spectral radius $\nu(\cdot)$ satisfies $\nu(x^*x) \leq (x)^2$ for all x.

2. **Preliminaries.** We shall consider only associative complex linear algebras. By a *-algebra we will mean an algebra with an involution $x \longrightarrow x^*$; that is, a conjugate-linear anti-automorphism of period two. In normed algebras identities will be assumed to have norm one, and approximate identities will be two-sided and bounded by one. If A is a *-algebra, we denote by A_e the algebra obtained from A by adjoining an identity. We let J (or J_A) denote the Jacobson radical of A, and R (or R_A) the reducing ideal of A; that is, the intersection of the kernels of all irreducible *-representations of A on Hilbert spaces (R is called the *-radical in [7]). For typographical convenience we let J_e denote the radical of A_e (similarly, we write R_e). The words "multiplicative linear functional" will be abbreviated "MLF". If f is a positive functional on A, we set $I_f = \{x \in A: f(x^*x) = 0\}$. A positive functional f on A will be called "pure" if for every positive functional g on A satisfying $g(x^*x) \leq f(x^*x)$ for all $x \in A$, it follows that $g = \lambda f$ for some $0 \leq \lambda \leq 1$.

Now let A denote a Banach *-algebra. A continuous positive functional f on A will be called a "state" if $\|f\| = 1$. We denote by $\sigma(x)$ (or $\sigma_A(x)$) the spectrum of an element x in A, and by $\nu(x)$ (or $\nu_A(x)$) the spectral radius of x. We remark that if A has an approximate identity, then every positive functional f on A is representable [8]; a positive functional f on A is said to be representable if there exists a *-representation π of A on a Hilbert space H and a cyclic vector $\eta \in H$ such that $f(x) = (\pi(x)\eta \mid \eta)$ for every $x \in A$. In particular, f is Hermitian, continuous, and satisfies $f(R) \equiv 0$. Finally, we remark that any results we use from C. Rickart's book [7] will not require the assumption of local continuity of the involution by virtue of J. Ford's square root lemma [3].

3. **The definition, immediate consequences, and some examples.** Let A be a *-algebra, and let $P = \bigcap I_f$, where the intersection is taken over all positive functionals on A.

Proposition 3.1. *P is a two-sided ideal in A.*

Proof. Since P is the intersection of a family of left ideals, it suffices to show that P is a right ideal. If f is a positive functional on A and $y \in A$, set $f_y(x) = f(y^*xy)$ for each $x \in A$. Then f_y is a positive functional on A, and if

$x \in P$, we have $f[(xy)^*(xy)] = f(y^*x^*xy) = f_y(x^*x) = 0$. Hence $xy \in P$ and P is a right ideal. Q.E.D.

The following definition was suggested to the author by Professor R. S. Doran.

Definition 3.2. A *-algebra A is said to be P-commutative if $xy - yx \in P$ for every $x, y \in A$.

Proposition 3.3. *Let A be a P-commutative *-algebra. Then A_e is P-commutative.*

Proof. It is clear that $P_A \subset P_e$. Now let $(x, \lambda), (y, \alpha) \in A_e$. Then $(x, \lambda)(y, \alpha) - (y, \alpha)(x, \lambda) = (xy - yx, 0) \in P_A$ since A is P-commutative. Therefore, A_e is P-commutative. Q.E.D.

Proposition 3.4. *Let A be a P-commutative *-algebra, B a *-algebra, and ϕ a *-homomorphism of A onto B. Then B is P-commutative.*

Proof. Let f be an arbitrary positive functional on B. Then $f \circ \phi$ is a positive functional on A; so if $x \in P_A$, then $0 = (f \circ \phi)(x^*x) = f[\phi(x)^*\phi(x)]$, and thus, $\phi(x) \in P_B$. Hence, if $\phi(x), \phi(y)$ are arbitrary elements of B, then $\phi(x)\phi(y) - \phi(y)\phi(x) = \phi(xy - yx) \in \phi(P_A) \subset P_B$. Therefore, B is P-commutative. Q.E.D.

It follows from Proposition 3.4 that if I is a *-ideal in A, then A/I is a P-commutative *-algebra.

For the remainder of this paper, unless stated otherwise, A will denote a Banach *-algebra.

Since for every Banach *-algebra A, it is true that $R = \bigcap I_f$, f representable (see [7, 4.6.8, p. 226]), it is true that $P \subset R$. So we see immediately that if A is P-commutative, then A/R is commutative. In particular, if A is P-commutative and reduced (i.e., $R = \{0\}$), then A is commutative. Now if A has an approximate identity, then every positive functional on A is representable, and hence, $P = R$ and A is P-commutative if and only if A/R is commutative.

It is not surprising that a closed *-subalgebra of a P-commutative Banach *-algebra need not be P-commutative. We will give an example later in this section to illustrate this fact. However, we have the following result.

Proposition 3.5. *Let A be a P-commutative symmetric Banach *-algebra with an approximate identity $\{e_\alpha\}$. If B is a closed *-subalgebra of A containing $\{e_\alpha\}$, then B is P-commutative.*

Proof. We adjoin identities to A and B to obtain A_e and B_1 (we use the notation B_1 in this proposition only so that we may distinguish between the reducing ideals R_e of A_e and R_1 of B_1). Now A_e is P-commutative and symmetric [7, 4.7.9, p. 233], and B_1 is a closed *-subalgebra of A_e. Since both A and

B possess approximate identities, all positive functionals are representable, and it follows that $P_A = R_A = R_e$ and $P_B = R_B = R_1$. So by [7, 4.7.19, p. 237], we have that $P_B = R_1 = B_1 \cap R_e = B_1 \cap R_A = B \cap P_A$; it follows that B is P-commutative. Q.E.D.

A related result is given next.

Proposition 3.6. *Let A be a P-commutative symmetric Banach *-algebra. If I is a closed *-ideal of A having an approximate identity, then I is P-commutative.*

Proof. We have that $P_A \subset R_A = J_A$ and $P_I = R_I = J_I$. Since I is an ideal, it follows that $J_I = I \cap J_A$. Thus, we obtain $P_I \supset I \cap P_A$. The inclusion $P_I \subset I \cap P_A$ is obvious; hence I is P-commutative. Q.E.D.

We now give some examples. We begin by giving a few examples of noncommutative, P-commutative Banach *-algebras.

Example 3.7. Let A be a noncommutative radical Banach *-algebra (i.e., $J_A = R_A = A$). Then, for A_e, we have that $P_e = R_e = A$. Now if $(x, \lambda), (y, \alpha) \in A_e$, then $(x, \lambda)(y, \alpha) - (y, \alpha)(x, \lambda) = (xy - yx, 0) \in A = P_e$ which implies A_e is P-commutative. Obviously, A_e is noncommutative.

We next construct an example which is semisimple.

Example 3.8. Let A' be a semisimple, noncommutative Banach algebra with norm $\|\cdot\|'$ and involution $x \to x'$. Let $A = \{(x, y): x, y \in A'\}$. We denote elements of A by either (x, y) or a, b, etc. Furnish A with pointwise algebraic operations, norm $\|(x, y)\| = \|x\|' + \|y\|'$, and involution $(x, y)^* = (y', x')$. Then A is noncommutative and semisimple. We now show that A is P-commutative. Let f be a positive functional on A, let x be an arbitrary element in A', and set $a = (x, 0)$. Then $a^*a = 0$ implies $f(a^*a) = 0$; hence $a \in I_f$ and since f was arbitrary, $a \in P$. Similarly, every element of A of the form $(0, y)$ is in P. But, since P is an ideal in A, it follows that $(x, y) \in P$ for every $x, y \in A'$; i.e., $A = P$. Therefore, A is P-commutative.

In Example 3.8, if A' has an identity e, then A is an example of a Banach *-algebra having no nonzero positive functionals. Indeed, let f be a positive functional on A, choose an arbitrary $a \in A$, and let $1 = (e, e)$ denote the identity of A. Then $|f(a)|^2 = |f(1 \cdot a)|^2 \leq f(1) f(a^*a) = 0$ and so $f \equiv 0$ on A.

We next use Example 3.8 to show that a closed *-subalgebra of a P-commutative algebra need not be P-commutative.

Example 3.9. Let A' be as in 3.8 above but with the added assumption that it is reduced. Now let $B = \{(x, x): x \in A'\}$. Then B is a closed noncommutative *-subalgebra of A. We show that B is not P-commutative. Let π be a *-representation of A' on some Hilbert space H. Then the map $\pi_B: B \to B(H)$ defined by

$\pi_B[(x, x)) = \pi(x)$ is a *-representation of B on H and $x \in \ker(\pi)$ if and only if $(x, x) \in \ker(\pi_B)$. Since A' is reduced, B is reduced. Therefore, since B is noncommutative, it follows that B cannot be P-commutative.

Example 3.10. Let B be any noncommutative, symmetric Banach *-algebra with continuous involution, identity e, and center Z, and assume that R_B is noncommutative. Let A be the closed *-subalgebra of B generated by Z and R_B. Then A is noncommutative and has identity e. We show that A is P-commutative. Since B is symmetric and $e \in A$, we have that $P_A = R_A = A \cap R_B = R_B$. Now let $z_1 + r_1, z_2 + r_2 \in A$. Then $(z_1 + r_1)(z_2 + r_2) - (z_2 + r_2)(z_1 + r_1) = r_1 r_2 - r_2 r_1 \in R_A$. Since R_A is closed in A and since every element of A is the limit of elements of the form $z + r$, where $z \in Z$, $r \in R_B$, it follows that $xy - yx \in R_A$ for every $x, y \in A$, i.e. A is P-commutative.

We next give an example showing that P is not always equal to R. It also shows that, in general, nothing can be said about the relationship between J and P.

Example 3.11. Let A be the set of all formal power series, $\Sigma a_n z^n$, satisfying $\Sigma |a_n|/n! < \infty$ (it will be understood that the sum runs from $n = 1$ to ∞). Give A the usual algebraic operations, norm defined by $\|\Sigma a_n z^n\| = \Sigma |a_n|/n!$, and involution $\Sigma a_n z^n \to \Sigma \bar{a}_n z^n$. Then A is a commutative, radical Banach *-algebra, so $J = R = A$. We show that $P \neq A$. Consider $f: A \to C$ defined by $f(\Sigma a_n z^n) = a_2$. Then f is clearly linear, and f is positive since $f[\Sigma \bar{a}_n z^n)(\Sigma a_n z^n)] = |a_1|^2 \geq 0$. Since there exist power series in A for which $a_1 \neq 0$, it follows that $I_f \neq A$ which implies $P \neq A$.

4. Multiplicative linear functionals and continuity of positive functionals. If A is a Banach *-algebra, we denote by A^2 the set of all finite sums of products of pairs of elements from A. I. Murphy [5] proved the following result:

Theorem. *Assume $A^2 = A$ and let every nonzero positive functional on A dominate a continuous nonzero positive functional. Then every positive functional on A is continuous.*

As a corollary, Murphy proved that if A is commutative and satisfies $A^2 = A$, then every positive functional on A is continuous. We extend this result to P-commutative algebras. The proof follows Murphy's essentially, the main deviation being in showing that $f_u(x^*x) = f_x(u^*u)$.

Theorem 4.1. *If A is a P-commutative Banach *-algebra satisfying $A^2 = A$, then every positive functional on A is continuous.*

Proof. Let f be a nonzero positive functional on A. Then f is Hermitian since every x in A can be written in the form $x = \Sigma_{i=1}^n \lambda_i x_i^* x_i$ (see [5]). For every $u \in A$, let f_u be the positive functional defined in the proof of Proposition 3.1. Then f_u is continuous on A (see [7, 4.5.4, (ii), p. 215]). Now if $f_u \equiv 0$ for every $u \in A$, then $|f(u^*x^*y)|^2 \leq f(u^*x^*xu)f(y^*y) = 0$ for every $u, x, y \in A$; hence, $f(A^3) \equiv 0$ which is false since $A^3 = A$. So we may choose $u \in A$ such that $f_u \not\equiv 0$, and since $f_{\alpha u} = |\alpha|^2 f_u$, we may assume that $\|u^*u\| < 1$. Ford's square root lemma then provides an element $v \in A_e$ such that $v^*v = e - u^*u$. Now we show that $f_u(x^*x) = f_x(u^*u)$ for every $x \in A$. Recall that I_f is a left ideal in A. It is well known that the linear space A/I_f, together with the inner product $(\cdot\,|\,\cdot)$ defined by $(x + I_f | y + I_f) = f(y^*x)$, is a pre-Hilbert space. Furthermore, since A is P-commutative, it follows that $ux + I_f = xu + I_f$ for every $x \in A$. Therefore, we have

$$\begin{aligned} f_u(x^*x) &= f(u^*x^*xu) = (xu + I_f\,|\,xu + I_f) \\ &= (ux + I_f\,|\,ux + I_f) = f(x^*u^*ux) = f_x(u^*u). \end{aligned}$$

Now for every $x \in A$,

$$\begin{aligned} (f - f_u)\,(x^*x) &= f(x^*x) - f_u(x^*x) = f(x^*x) - f_x(u^*u) \\ &= f[x^*(e - u^*u)x] = f(x^*v^*vx) = f[(vx)^*(vx)] \geq 0 \end{aligned}$$

since f is positive on A and A is an ideal in A_e. So f dominates f_u, and by an application of Murphy's theorem, we see that every positive functional on A is continuous. Q.E.D.

The following lemma is well known [2, 2.2.10, p. 28] for the case when A has isometric involution. A slight modification of the proof in [2] shows that it is true for Banach algebras with arbitrary involution. We therefore omit the proof.

Lemma 4.2. *Let A be a Banach *-algebra with approximate identity $\{e_\alpha\}$. Let π be a nondegenerate *-representation of A on a Hilbert space H, and let 1 denote the identity operator on H. Then $\lim \pi(e_\alpha) = 1$, where the limit is in the strong operator topology on $B(H)$.*

It is well known that on a commutative Banach *-algebra a nonzero positive functional is a pure state if and only if it is multiplicative.

Theorem 4.3. *Let A be a P-commutative Banach *-algebra with approximate identity $\{e_\alpha\}$. Then a nonzero positive functional f on A is a pure state if and only if it is multiplicative.*

Proof. Let f be a pure state on A. Since A has an approximate identity, $f(R) \equiv 0$ and so we may define a function $f': A/R \to C$ by $f'(x + R) = f(x)$. Clearly f' is linear and positive, and it is easy to check that it is also pure and that $\|f\| \leq \|f'\|$. Now positive scalar multiples of pure positive functionals are pure positive functionals; hence, there exists a scalar λ, $0 < \lambda \leq 1$, such that $\lambda f'$ is a pure state on A/R. Since A/R is commutative, it follows that $\lambda f'$ is multiplicative which implies that λf is multiplicative on A. Now f is representable, so we may write $f(x) = (\pi(x)\eta \mid \eta)$, where π is a *-representation on the Hilbert space H and $\eta \in H$. Then by Lemma 4.2, we see that $\lim_\alpha f(e_\alpha) = \|\eta\|^2$; and $\|\eta\|^2 \leq \|f\|$ since $\|e_\alpha\| \leq 1$ for every α. Now if we choose $x \in A$ such that $f(x) \neq 0$, then $(\lambda f)(x) = \lim(\lambda f)(xe_\alpha) = (\lambda f)(x) \lim(\lambda f)(e_\alpha)$ which implies $\lim(\lambda f)(e_\alpha) = 1$ or $\lim f(e_\alpha) = 1/\lambda \geq 1$ since $\lambda \leq 1$. But $\lim f(e_\alpha) = \|\eta\|^2 \leq \|f\| = 1$, and hence $1/\lambda = \lim f(e_\alpha) = 1$; thus $\lambda = 1$ and so f is multiplicative on A. This proves the first half of the theorem.

Now assume that f is a nonzero positive MLF on A. Again $f(R) \equiv 0$, so we define f' as before. Then f' is positive and multiplicative on A/R and, hence, is a pure state. Therefore, f is pure. If we choose $x \in A$ such that $f(x) \neq 0$, then $f(x) = \lim f(xe_\alpha) = f(x) \lim f(e_\alpha)$ which implies $\lim f(e_\alpha) = 1$; hence $\|f\| \geq 1$. But $\|f\| \leq \|f'\| = 1$ implies $\|f\| = 1$, and therefore f is a state. Q.E.D.

If A is P-commutative and has an approximate identity, we set

$$M = \{f : f \text{ is a nonzero positive } MLF \text{ on } A\} = \{f : f \text{ is a pure state}\}$$

(M may be empty). Since the image of every irreducible *-representation of A must be commutative, and since the commutant of each image is just the set of scalar operators [7, 4.4.12, p. 211], we see that M is precisely the set of irreducible *-representations of A. Therefore, $R = \bigcap \ker(f)$ $(f \in M)$. If we furnish M with the relative weak *-topology, then, as in the commutative case, M is a locally compact Hausdorff space. If A has an identity, then M is compact. If we define the map $x \to \hat{x}$ of A into $C_0(M)$ (the continuous functions on M which vanish at infinity) by $\hat{x}(f) = f(x)$, then $x \to \hat{x}$ is a norm decreasing *-homomorphism. It is injective if and only if A is reduced.

5. **Symmetric, P-commutative Banach *-algebras.** In this section we study P-commutative Banach *-algebras which are symmetric. Two results for such algebras have already been obtained (see 3.5, 3.6).

Proposition 5.1. *Let A be symmetric and P-commutative. Then a two-sided ideal in A is primitive if and only if it is maximal modular.*

Proof. The "if" part is true for any algebra. Now let I be a primitive ideal in A. Then since A is symmetric $R = J$, and therefore, $R \subset I$. If θ is the quotient map of A onto A/R, then $\theta(I) = I/R$ is a primitive ideal in A/R. But A/R commutative implies I/R is maximal modular which implies $I = \theta^{-1}(I/R)$ is maximal modular in A. Q.E.D.

It is well known that if A is a commutative Banach *-algebra, then A is symmetric if and only if every MLF on A is Hermitian.

Theorem 5.2. *If A is symmetric and P-commutative, then every MLF on A is Hermitian (hence positive).*

Proof. Let f be a MLF on A. Then $R = J \subset \ker(f)$ since $\ker(f)$ is a primitive ideal in A. If we define $f': A/R \to C$ by $f'(x + R) = f(x)$, then f' is a Hermitian MLF on A/R since A/R is commutative. Therefore, f is Hermitian. Q.E.D.

The converse of Theorem 5.2 is not true; that is, there exist nonsymmetric P-commutative algebras on which every MLF is Hermitian.

Example 5.3. Let A be a noncommutative Banach *-algebra having identity e, involution $x \to x'$, and no nonzero MLF's. Let B be a commutative, symmetric, semisimple Banach *-algebra with identity (denoted by e also) and involution $z \to z^*$. Now set $D = \{(x, y, z): x, y \in A, z \in B\}$. Give D pointwise algebra operations, involution $(x, y, z)^* = (y', x', z^*)$, and l_1-norm. Then D is just the product of B with Example 3.8; therefore, D is P-commutative. Now if ϕ is a MLF on D, then $\phi[(x, y, 0)] = 0$ for every $x, y \in A$ since A has no nonzero MLF's. Thus, the functionals $\phi: D \to C$, defined by $\phi[(x, y, z)] = \phi_B(z)$, where ϕ_B is a MLF on B, are the only MLF's on D. But since B is symmetric and commutative, all the ϕ_B's are Hermitian; thus ϕ is Hermitian. It remains to show that D is not symmetric. Let x be an invertible element of A. Set $y = -(x^{-1})'$ and $z \in B$. Then $(e, e, e) + (x, y, z)^*(x, y, z) = (e + y'x, e + x'y, e + z^*z) = (0, 0, e + z^*z)$ which cannot be invertible in D. Thus, D is not symmetric.

We remark that if A is P-commutative and symmetric, then there is a one-to-one correspondence between the set of all MLF's on A and the set of maximal modular two-sided ideals of A. Indeed, let I be such an ideal. Then A symmetric implies $R = J$; therefore $R \subset I$ and so A/I is commutative. The same argument as used for the analogous commutative result now applies (see [6, p. 192]).

Theorem 5.4. *Let A be symmetric and P-commutative. Then $R = \{x \in A: \nu(x) = 0\}$.*

Proof. Since $R = J$, it follows that $x \in R$ implies $\nu(x) = 0$. Conversely, let $x \in A$, $\nu(x) = 0$. Since A_e is P-commutative and symmetric, and since $\nu_A(x) = \nu_e(x)$ and $R_A = R_e$, we may assume that A has an identity. Let $f \in M$ (if $M = \emptyset$, the theorem is clear). Then f' (usual definition) is a MLF on A/R. But $\nu(x) = 0$ implies $\nu_{A/R}(x) = 0$; hence $0 = f'(x + R) = f(x)$ since A/R is commutative. Therefore, $x \in \bigcap \ker(f)$ $(f \in M)$ implies $x \in R$ (see the paragraph following Theorem 4.3). Q.E.D.

Lemma 5.5. *Suppose A is symmetric, P-commutative, and has an identity.*

Then for every x in A, $\sigma(x) = \{f(x): f \in M\}$.

Proof. The containment $\sigma(x) \subset \{f(x): f \in M\}$ follows immediately from Theorem 4.3 and [4, Theorem 4.10]. Now suppose that $\lambda = f(x)$ for some $f \in M$. Since $f'(x + R) = f(x) = \lambda$, it follows that $\lambda \in \sigma_{A/R}(x + R)$. Since $\sigma_{A/R}(x + R) \subset \sigma(x)$, the lemma follows. Q.E.D.

Theorem 5.6. *Let A be symmetric and P-commutative. Then for every $x, y \in A$, $\nu(xy) \leq \nu(x)\nu(y)$ and $\nu(x + y) \leq \nu(x) + \nu(y)$.*

Proof. We may assume A has an identity. Then by Lemma 5.5, $\nu(xy) = \sup|f(xy)| \leq (\sup |f(x)|)(\sup|f(y)|) = \nu(x)\,\nu(y)$, where the supremums are taken over all f in M. Similarly, $\nu(x + y) = \sup|f(x + y)| \leq \sup|f(x)| + \sup|f(y)| = \nu(x) + \nu(y)$. Q.E.D.

From Theorem 5.6 we see that $\nu(\cdot)$ is a continuous algebra seminorm in a symmetric, P-commutative Banach *-algebra.

We next give an example to show that, without the assumption of symmetry, $\nu(\cdot)$ need not be an algebra seminorm.

Example 5.7. Let B be a noncommutative Banach *-algebra with an identity and having two elements x and y satisfying $\nu_B(xy) > \nu_B(x)\nu_B(y)$. We may assume that $\nu_B(y) = \nu_B(x)$. Now form the P-commutative algebra A of Example 3.8 using B. Then A is not symmetric (see Example 5.3). It is easy to check that $\sigma_A[(x, y)] = \sigma_B(x) \cup \sigma_B(y)$. It follows that $\nu_A[(x, y)(y, x)] = \nu_A[(xy, yx)] = \nu_B(xy) > \nu_B(x)\nu_B(y) = \nu_A[(x, y)]\,\nu_A[(y, x)]$.

Theorem 5.8. *Let A be a Banach *-algebra with an approximate identity. If $\nu(x^*x) \leq \nu(x)^2$ for every $x \in A$, then A is P-commutative.*

Proof. We first note that $R_A = \bigcap \operatorname{Ker}(f)$, where the intersection is over all positive functionals on A. This is proved in [6, p. 259] for algebras with an identity. It is easily extended to the present situation since $R_A = R_e$ and every positive functional is representable. Now let f be an arbitrary positive functional on A and f' its extension to A_e. Then for every $x \in A$, $|f(x)| \leq f'(e)\nu(x^*x)^{1/2} \leq f'(e)\nu(x)$, and by [1, Corollary 3] $f(xy) = f(yx)$ for every $x, y \in A$. Hence $xy - yx \in \bigcap \operatorname{Ker}(f) = R_A$ for every $x, y \in A$; A is therefore P-commutative. Q.E.D.

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