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# $C^2$-PRESERVING STRONGLY CONTINUOUS MARKOVIAN SEMIGROUPS

BY

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ABSTRACT. Let $X$ be a compact $C^2$-manifold. Let $\| \|$, $\| \|'$ denote the supremum norm and the $C^2$-norm, respectively, and let $\{P^t\}$ be a Markovian semigroup on $C(X)$. The semigroup's infinitesimal generator $A$, with domain $\mathfrak{D}$, is defined by $Af = \lim_{t\to 0} t^{-1}(P^t f - f)$, whenever the limit exists in $\| \|$.

**Theorem.** *Assume that* $\{P^t\}$ *preserves* $C^2$*-functions and that the restriction of* $\{P^t\}$ *to* $C^2(X)$, $\| \|'$ *is strongly continuous. Then* $C^2(X) \subset \mathfrak{D}$ *and* $A$ *is a bounded operator from* $C^2(X)$, $\| \|'$ *to* $C(X)$, $\| \|$.

From the conclusion is obtained a representation of $Af \cdot (x)$ as an integro-differential operator on $C^2(X)$. The representation reduces to that obtained by Hunt [*Semi-groups of measures on Lie groups*, Trans. Amer. Math. Soc. 81 (1956), 264–293] in case $X$ is a Lie group and $P^t$ commutes with translations.

Actually, a stronger result is proved having the above theorem among its corollaries.

1. **Introduction.** G. A. Hunt showed in [3] that all $C^2$-functions lie in the domain of the infinitesimal generator of a translation-invariant strongly continuous Markovian semigroup on $C(X)$, where $X$ is a Lie group. From this Hunt went on to obtain a representation of the infinitesimal generator as an integro-differential operator of a certain type. Hunt then considered the converse question of which integro-differential operators generate semigroups of the class he was considering. He thus characterized all such semigroups by writing down explicitly the general form of their generators.

Translation-invariance means, roughly speaking, that the semigroup sends smooth functions nicely into smooth functions and that the associated stochastic process on $X$ is a homogeneous one. We undertake to obtain results similar to those of Hunt's under less restrictive hypotheses, viz., when there is no group structure on $X$ (and therefore no notion of homogeneity) but only a notion of smooth functions. We take $X$ to be a compact $C^2$-manifold, assume that the Markovian

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semigroup sends $C^2(X)$ into itself in a strongly continuous fashion, and prove in Theorem 2 that all $C^2$-functions lie in the domain of the infinitesimal generator, which is an integro-differential operator on $C^2(X)$. We do not consider the converse question of which such operators generate such semigroups.

For related results, see Nelson [5], in addition to [3]. See also Dynkin [1, Theorem 5.7, p. 152].

This paper is a condensed version of the first chapter of the writer's doctoral thesis [6], written under the direction of Edward Nelson.

2. **Main Theorem.** We prove a stronger result than that mentioned in the Introduction, a result which has other applications as well. A *Markovian operator* $P$ is a positivity-preserving endomorphism of $C(X)$ such that $P1 = 1$.

**Theorem 1.** *Let $X$ be a compact $C^2$-manifold and let $P(t)$ be a function from $[0, \delta)$ to the Markovian operators on $C(X)$, with $P(0) = I$. If the domain $\mathfrak{D}$ of the strong derivative $P'(0)$ contains a subset $\mathfrak{D}'$ dense in $C^2(X)$, $\| \|'$, then*

(i) $C^2(X) \subset \mathfrak{D}$,

(ii) *$P'(0)$ is a bounded operator from $C^2(X)$, $\| \|'$ to $C(X)$, $\| \|$,*

(iii) *$P'(0)f \cdot (x)$ can be represented as the following integro-differential operator $(*)$, for $f$ in $C^2(X)$:*

$$(*) \qquad \begin{aligned} &\sum a_{ij}(x) D_i D_j f \cdot (x) + \sum b_i(x) D_i f \cdot (x) \\ &+ \int_{X \setminus \{x\}} f(y) - f(x) - \sum D_i f \cdot (x)(x_i(y) - x_i(x))\, \mu_x(dy), \end{aligned}$$

*where $x_1, x_2, \cdots, x_n$ are local coordinates near $x$, $b_i$ is continuous, $a_{ij}$ is continuous or of Baire class 1, $\{a_{ij}(x)\}$ is a positive semidefinite $n \times n$ matrix, and $\mu_x$ is a positive (possibly unbounded) measure on $X \setminus \{x\}$ for which $\int_X (f(y) - f(x))^2 \mu_x(dy)$ is finite.*

**Proof.** Conclusions (i) and (ii) will follow from the Banach-Steinhaus theorem, once its hypotheses are seen to be met. Conclusion (iii) will then follow by a slight modification of Hunt's argument in [3].

The machinations which follow are designed to obtain the bound required in order to invoke the Banach-Steinhaus theorem. Choose and fix a coordinate neighborhood $U$ in $X$ with coordinate functions $x_1, x_2, \cdots, x_n$. Without loss of generality we may assume that, for $1 \leq i \leq n$, $x_i$ is a $C^2$-function defined on all of $X$ with $\|x_i\| < 4$ and that $x_i(U)$ contains the open interval $(-3, 3)$. Moreover, we may assume that *each of the coordinate functions belongs to $\mathfrak{D}$ as well.* This last follows since $\mathfrak{D}'$ is in any case dense in $C^2(X)$, $\| \|'$ so that we may choose $n$ functions in $\mathfrak{D}'$ sufficiently close in $\| \|'$ to the $n$ coordinate functions respectively that these new functions could serve as coordinates in $U$.

If it were apparent at this stage that the squares of the coordinate functions also belong to $\mathfrak{D}$, then our argument could be considerably shortened. As it is, we must construct functions in $\mathfrak{D}$, each of which behaves like the square of the distance from a fixed point in $X$. For abbreviative purposes we set

$$U_r = \{y \in U: |x_i(y)| < r,\ 1 \leq i \leq n\},$$

and we construct a family of functions $\{\phi_x\}_{x \in U_1}$ that satisfies

(1) $$\phi_x \geq 1 \text{ on } X \setminus U_2,$$

(2) $$\sup_{t>0,\ x \in U_1} |t^{-1} P(t) \phi_x \cdot (x)| < \infty,$$

(3) $$\phi_x(y) \geq (3/4) \sum (x_i(y) - x_i(x))^2 \quad \text{if } y \in U_2.$$

To carry out this construction, choose and fix some $C^2$-function $f$: $X \longrightarrow \mathbf{R}$ satisfying (a) $f(y) = \Sigma\, x_i(y)^2$ if $y \in U_1$, (b) $f(y) > 15n + 4$ if $y \notin U_2$, (c) $D_i D_j f \cdot (y) = 0$ if $i \neq j$ and $y \in U_2$, (d) $D_i D_i f \cdot (y) \geq 2$ if $y \in U_2$. We are free to choose functions in $\mathfrak{D}'$ as close to $f$ in $\| \ \|'$ as we please. Choose $\phi$ in $\mathfrak{D}'$ so close to $f$ that $\|f - \phi\|' < 1$ and, for arbitrary real numbers $\xi_1, \xi_2, \cdots, \xi_n$, we have $y \in U_2$ implies

(4) $$\sum D_i D_j \phi \cdot (y)\, \xi_i\, \xi_j \geq (3/2) \sum \xi_i^2.$$

It is possible to find such a $\phi$ since (c) and (d) guarantee that $f$ satisfies the inequality similar to (4) with "2" replacing "3/2". Now, for $x$ in $U_1$, we define functions $\phi_x = \phi_x(y)$ by

$$\phi_x = \phi - \sum D_i \phi \cdot (x) x_i - \phi(x) + \sum D_i \phi \cdot (x) x_i(x).$$

For brevity we write $\phi_x = \phi - \alpha_x - \beta_x + \gamma_x$, where, of course, $\beta_x$ and $\gamma_x$ are constants, for fixed $x$. Since $\|x_i\| \leq 4$ and $\|D_i f - D_i \phi\| \leq 1$ it follows that for $y \notin U_2$ we have $|\alpha_x(y)| \leq 12n$, $|\beta_x| \leq 2$, $|\gamma_x| \leq 3n$. By (b) we have $\phi(y) > 15n + 3$ if $y \notin U_2$. Hence $\phi_x \cdot (y) \geq 1$ if $y \notin U_2$, proving (1).

Since $\phi_x(x) = 0$ we have

$$\begin{aligned} P(t)\phi_x \cdot (x) &= (P(t)\phi_x - \phi_x) \cdot (x) \\ &= (P(t)\phi - \phi) \cdot (x) - \sum D_i \phi \cdot (x)\, (P(t) x_i - x_i) \cdot (x). \end{aligned}$$

Dividing by $t > 0$ and taking the limit as $t \longrightarrow 0$ shows that $\lim t^{-1} P(t) \phi_x \cdot (x) = P'(0)\phi \cdot (x) - \Sigma\, D_i \phi \cdot (x) P'(0) x_i \cdot (x)$, and (2) follows.

To verify (3), note that $\phi_x$ and its gradient vanish at $x$. Also note that $D_i D_j \phi_x \cdot (y) = D_i D_j \phi \cdot (y)$ if $y \in U$. These facts and Taylor's theorem imply that for $z \in U_2$ there is some $y \in U_2$ such that

$$\phi_x(z) = 2^{-1} \sum D_i D_j \phi \cdot (y)\, (x_i(z) - x_i(x))\, (x_j(z) - x_j(x)).$$

This and (4) imply (3).

With the aid of the functions just constructed, we may obtain the bound we seek. Let $g$ be a $C^2$-function, let $x$ be in $U_1$ and let $g_x = g - g(x) - \Sigma D_i g \cdot (x)(x_i - x_i(x))$. Since $g_x$ vanishes to second order at $x$, there is some $N$ such that $y \in U_2$ implies

$$-N \sum (x_i(y) - x_i(x))^2 \leq g_x(y) \leq N \sum (x_i(y) - x_i(x))^2.$$

We cannot expect this inequality to hold for all $y$ in $X$, since the $x_i$'s are only *local* coordinates. However, this local inequality and (3) imply that on $U_2$ we have the functional inequality

$$-(4/3)\, N\phi_x \leq g - g(x) - \sum D_i g \cdot (x)\,(x_i - x_i(x)) \leq (4/3)\, N\phi_x.$$

The function in the middle here is bounded on $X$ (independently of $x$ in $U_1$), and $\phi_x \geq 1$ outside $U_2$, so by adjusting $N$ appropriately we may regard the preceding inequality as a *global* one, holding for each $x$ in $U_1$, with $N$ independent of $x$. This inequality is then preserved if we apply $P(t)$ to each member, divide by $t > 0$, and evaluate the terms of the resulting inequality at $x$, from which it follows that

$$\begin{aligned} |t^{-1}(P(t)g - g)(x) - \sum D_i g \cdot (x)\, t^{-1}(P(t)x_i - x_i) \cdot (x)| \\ \leq (4/3)\, Nt^{-1} P(t)\phi_x \cdot (x). \end{aligned}$$

Now (2) shows that the expression inside the absolute value sign is bounded independently of $x$ in $U_1$ and $t > 0$. So is $\Sigma D_i g \cdot (x) t^{-1}(P(t)x_i - x_i) \cdot (x)$ since $x_i \in \mathfrak{D}$ and $\|D_i g\| < \infty$. It follows that $g \in C^2(X)$ implies

(5) $$\sup_{t>0,\, x \in U_1} |t^{-1}(P(t)g - g) \cdot (x)| < \infty.$$

In (5), "$U_1$" may be replaced by "$X$", since $X$ is compact. By the principle of uniform boundedness there is some $M$ such that

$$\sup_{t>0,\, x \in X} |t^{-1}(P(t)g - g) \cdot (x)| \leq M \|g\|'.$$

Thus $M$ is a uniform bound for the norms of the continuous linear maps $A(t) = t^{-1}(P(t) - I)$ from $C^2(X)$, $\| \|'$ to $C(X)$, $\| \|$. Since $A(t)g \rightarrow P'(0)g$ for each $g$ in a dense subset of $C^2(X)$, the Banach-Steinhaus theorem [2, p. 41] implies conclusions (i) and (ii) of Theorem 1.

To prove (iii) we follow Hunt [3]. Let $\psi_x$ be some $C^2$-function bounded away from zero on the complement of $U$ and agreeing with $\Sigma (x_i - x_i(x))^2$ on $U$. The mapping which sends $f$ to $P'(0)(f\psi_x) \cdot (x)$ is readily seen to be defined on $C^2(X)$ and to be a positive, hence bounded, linear functional on $C^2(X)$, $\| \|$. It therefore admits a unique extension to $C(X)$ since $C^2(X)$ is dense in $C(X)$.

By the Riesz-Markoff theorem this extended mapping is implemented by some finite positive regular Borel measure $\nu_x$ on $X$ so that

$$P'(0)(f\psi_x)\cdot(x) = \int_X f(y)\,\nu_x(dy).$$

We define a positive (possibly unbounded) measure $\mu_x$ on $X\setminus\{x\}$ by $\mu_x = \psi_x^{-1}\nu_x$.

For $f$ in $C^2(X)$, let $f_x$ be the Taylor polynomial with terms up to and including order two of $f$, expanded at $x$. Then $f - f_x = h\psi_x$ for some $h$ in $C(X)$ with $h(x) = 0$ so that

$$P'(0)(f-f_x)\cdot(x) = P'(0)(h\psi_x)\cdot(x) = \int_{X\setminus\{x\}} h(y)\,\nu_x(dy).$$

Hence

$$\begin{aligned} P'(0)f\cdot(x) &= P'(0)f_x\cdot(x) + P'(0)(f-f_x)\cdot(x) \\ &= \sum c_{ij}(x)\,D_iD_jf\cdot(x) + \sum b_i(x)\,D_if\cdot(x) + \int_{X\setminus\{x\}}(f(y)-f_x(y))\,\mu_x(dy) \end{aligned}$$

where

$$b_i(x) = P'(0)x_i\cdot(x)$$

and

$$2c_{ij}(x) = P'(0)\,[(x_i - x_i(x))\,(x_j - x_j(x))]\cdot(x).$$

By (ii) of Theorem 1, $b_i$ and $c_{ij}$ are continuous. When $f_x(y)$ is written out explicitly in the integrand, one sees that the second order term in it is integrable with respect to $\mu_x$, and this term can therefore be deleted, provided we adjust the coefficient functions $c_{ij}$ appropriately. (*) results. The new coefficient functions $a_{ij}$ are such that $\{a_{ij}(x)\}$ is a positive semidefinite $n\times n$ matrix, as may be shown just as in [3].

By now the only assertion in Theorem 1 which is not yet obvious is that $a_{ij}$ is the pointwise limit of a sequence of continuous functions. To see this, let $r_x = (x_i - x_i(x))(x_j - x_j(x))$ so that $r_x$ is in $C^2(X)$. From (*) it follows that $2a_{ij}(x) = \lim_{n\to\infty} q_n(x)$, where $q_n(x) = P'(0)[r_x\exp(-n\psi_x)]\cdot(x)$. We can easily arrange to choose the functions $\{\psi_x\}$ in such a way that $\|\psi_x - \psi_y\|' \to 0$ as $y\to x$. Since $\|r_x - r_y\|' \to 0$ as $y\to x$ it follows that

$$\|r_x\exp(-n\psi_x) - r_y\exp(-n\psi_y)\|' \to 0$$

as $y\to x$. By (ii) of Theorem 1 we then have

$$\|P'(0)\,[r_x\exp(-n\psi_x)] - P'(0)\,[r_y\exp(-n\psi_y)]\| \to 0$$

as $y\to x$, from which it follows that $q_n(y)\to q_n(x)$ as $y\to x$. Hence $a_{ij}$ is a pointwise limit of a sequence of continuous functions. (In case

$i = j$, the sequence $\{q_n\}$ is monotonically decreasing. Hence $a_{ii}$ is upper semicontinuous.) This completes the proof of Theorem 1.

On pp. 18–21 of [6] we present an example which shows that $a_{ij}$ need not be continuous.

3. **$C^2$-preserving Markovian semigroups.** The following theorem is known [3] in case $X$ is a Lie group and $P^t$ commutes with translations, in which case hypotheses (a) and (b) are automatically fulfilled.

**Theorem 2.** *Let $X$ be a compact $C^2$-manifold and let $A$, with domain $\mathfrak{D}$, be the infinitesimal generator of a Markovian semigroup $\{P^t\}$ on $C(X)$ satisfying*

(a) $P^t\colon\ C^2(X), \|\ \|' \to C^2(X), \|\ \|'$ *is continuous*, $t \geq 0$,

(b) $\|P^t f - f\|' \to 0$ *for each $f$ in $C^2(X)$.*

*Then*

(i) $C^2(X) \subset \mathfrak{D}$,

(ii) *$A$ is a bounded operator from $C^2(X), \|\ \|'$ to $C(X), \|\ \|$,*

(iii) *$Af \cdot (x)$ may be represented by $(*)$ for $f$ in $C^2(X)$,*

(iv) *$\{P^t\}$ is determined by the restriction of $A$ to $C^2(X)$.*

**Proof.** The infinitesimal generator $A$ is of course just the strong derivative at 0 of the mapping $t \to P^t$. Hypotheses (a) and (b), as is well known [2, p. 307], imply the existence of a dense subspace $\mathfrak{D}'$ of $C^2(X), \|\ \|'$ such that $t^{-1}(P^t f - f)$ converges in $\|\ \|'$–and *a fortiori* in $\|\ \|$–for $f$ in $\mathfrak{D}'$. Hence Theorem 1 implies (i), (ii), (iii). The proof of Theorem 5.4 of [4] proves (iv), since we are assuming $C^2(X)$ invariant under $\{P^t\}$.

4. **Normal derivatives of harmonic extensions.** As a second application of Theorem 1, we give a new proof of a known result.

**Theorem 3.** *Let $S^n$ be the unit sphere in $\mathbf{R}^{n+1}$, and let $f$ be in $C^2(S^n)$. Let $\tilde{f}$ be the continuous function on the closed unit ball that is harmonic in the interior of the ball and agrees with $f$ on the boundary. Then $\partial\tilde{f}/\partial n$, the derivative of $\tilde{f}$ in the direction of the inward normal, exists and is continuous throughout $S^n$.*

**Proof.** For $f$ in $C(S^n)$, $Q$ in $S^n$, let $P(t)$ be defined by

$$P(t) f \cdot (Q) = \tilde{f}\,((1-t)\,Q).$$

Thus $P(0) = I$ and $P(t)$ is a strongly continuous function from $[0, 1)$ to the Markovian operators on $C(S^n)$. The existence of $P'(0)f$ obviously implies the existence and continuity of $\partial\tilde{f}/\partial n$. Hence, to prove the theorem, we need only show that $C^2(S^n) \subset \mathfrak{D}$, where $\mathfrak{D}$ is the domain of $P'(0)$.

Let $\mathfrak{D}'$ be the set of all functions in $C(S^n)$ which are the restrictions to $S^n$ of harmonic functions whose domains are open sets containing the closed unit ball. Obviously we have $\mathfrak{D}' \subset \mathfrak{D}$, and it is easy to see that $P(t)f$ is in $\mathfrak{D}'$ if $t > 0$. Moreover, if $f$ is in $C^2(S^n)$ then $\|P(t)f - f\|' \to 0$ as $t \to 0$, because $P(t)$ is strongly continuous and commutes with each rotation of $S^n$. Therefore $\mathfrak{D}'$ is dense in $C^2(S^n)$, $\| \ \|'$. By Theorem 1 we have $C^2(S^n) \subset \mathfrak{D}$, completing the proof.

The normal derivative of $\tilde{f}$ may be represented by the integro-differential operator (*). However, it is also possible, and more desirable, to represent the normal derivative as a singular integral operator acting on $f$. For an explicit formula in spherical coordinates when $n = 2$, see [6, p. 94].

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