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EXTENSION OF LOEWNER'S CAPACITY THEOREM(1)

BY

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ABSTRACT. Analogues of a well-known theorem of Loewner concerning conformal capacity of a space ring are given in the case of an arbitrary domain.

1. Introduction. By a well-known result of Loewner [2], the conformal capacity of a space ring is positive if and only if the boundary components of the ring are nondegenerate. In terms of the moduli of path families, this result can be stated as follows: Given two disjoint connected compact sets F and F^* in the n-dimensional Möbius space $\overline{R}^n$, the modulus of the family of paths joining F and F^* in $\overline{R}^n$ is positive if and only if both F and F^* are nondegenerate. In fact, the following uniform version of this Loewner result is valid (Väisälä [9, §12]): Given a number $r \in (0, 1]$, there exists a number $\delta_n(r) > 0$ such that the modulus of the family of paths joining any two connected compact sets F and F^* in $\overline{R}^n$ is at least $\delta_n(r)$, whenever the spherical diameter of both F and F^* is at least r.

The purpose of this paper is to exhibit analogues of these two results in the case where the domain $\overline{R}^n$ is replaced by an arbitrary domain D and the sets F and F^* are replaced by arbitrary connected sets in D. We show in §§5 and 6 that the Loewner result mentioned above remains true also in this general situation, while the corresponding uniform version holds only if the boundary of the domain in question possesses certain regularity. For a plane domain D with finitely many boundary components the uniform version holds if and only if each boundary point of D has arbitrarily small neighborhoods U such that $U \cap D$ contains only a finite number of components. Our proofs are based on a certain type of comparison principle for moduli of path families, which is introduced in §3. This comparison principle allows us to infer, also, that the modulus of the family of paths, joining, in an arbitrary domain D, two disjoint compact subsets F and F^* of D, is positive if and only if both F and F^* are of positive capacity.

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2. **Preliminaries.** Unless otherwise stated, all point sets considered in this paper are assumed to lie in $\overline{R}^n$, $n \geq 2$, the Möbius space obtained by adding the point ∞ to Euclidean n-space R^n. For each point $x \in R^n$ we let x_i denote the ith coordinate of x, taken with respect to a fixed orthonormal basis $(e_1, \cdots, e_n)$, and x will be treated as a vector with norm $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$. Given a set E, we let ∂E, $\overline{E}$, and $\complement E$ denote the boundary, closure, and complement of E, all taken with respect to $\overline{R}^n$. Furthermore, given two sets E and F, we let $E \backslash F$ denote the difference set $E \cap \complement F$ and let $d(E, F)$ and $q(E, F)$ denote the Euclidean and spherical distances between E and F. The Euclidean and spherical diameters of a set E are denoted by $d(E)$ and $q(E)$, respectively. Given a point $x \in R^n$ and a number $r > 0$, we let $B^n(x, r)$ denote the n-dimensional ball $\{y: |y - x| < r\}$ and $S^{n-1}(x, r)$ its $(n-1)$-dimensional boundary sphere $\{y: |y - x| = r\}$. We will also employ the abbreviations $B^n(r) = B^n(0, r)$ and $S^{n-1}(r) = S^{n-1}(0, r)$.

A neighborhood of a point or a set is an open set containing it. A domain is an open connected nonempty set. A continuum is a compact connected set containing at least two points. A path in a set G is a continuous nonconstant mapping $\gamma: \Delta \longrightarrow G$, where Δ is a closed line interval. The point set $\{\gamma(t): t \in \Delta\}$ will be denoted by $|\gamma|$. Given three sets E, F, and G, we let $\Delta(E, F : G)$ denote the family of all paths joining E and F in G. Let Γ be a family of paths in $\overline{R}^n$ and let $F(\Gamma)$ be the family of all functions ζ which are nonnegative and Borel-measurable in $\overline{R}^n$ and for which

$$\int_\gamma \zeta \, ds \geq 1$$

for each rectifiable path $\gamma \in \Gamma$. The *modulus* of Γ is defined as

$$M(\Gamma) = \inf_{\zeta \in F(\Gamma)} \int_{R^n} \zeta^n \, dm.$$

A homeomorphism f of a domain D is said to be *quasiconformal* if it satisfies the double inequality

$$(1/K)\, M(\Gamma) \leq M(f\Gamma) \leq K M(\Gamma)$$

for some $K \in [1, \infty)$ and for each path family Γ in D.

We state, as a lemma, the uniform version of Loewner's capacity theorem, mentioned in the introduction.

2.1. **Lemma.** *The number*

$$\delta_n(r) = \inf M(\Delta(F, F^* : \overline{R}^n)),$$

where the infimum is taken over all continua F and F^* in $\overline{R}^n$ with $q(F) \geq r$ and $q(F^*) \geq r$, is positive for each $r > 0$ and zero for $r = 0$.

3. **A comparison principle for moduli.** We begin by introducing a method for estimating the modulus of a path family in terms of the moduli of certain auxiliary path families (cf. [3, 3.11]).

3.1. Theorem. *Let* F_1, F_2, F_3 *be three sets in a domain* D *and let* $\Gamma_{ij} = \Delta(F_i, F_j : D)$, $1 \leq i, j \leq 3$. *Then*

$$M(\Gamma_{12}) \geq 3^{-n} \min\{M(\Gamma_{13}), M(\Gamma_{23}), \inf M(\Delta(|\gamma_{13}|, |\gamma_{23}| : D))\},$$

where the infimum is taken over all rectifiable paths $\gamma_{13} \in \Gamma_{13}$, $\gamma_{23} \in \Gamma_{23}$.

Proof. We may assume that F_1, F_2, F_3 are nonempty sets, for otherwise there is nothing to prove. Choose $\zeta \in F(\Gamma_{12})$. If

$$\int_{\gamma_{13}} \zeta\, ds \geq \frac{1}{3} \tag{3.1.1}$$

for every rectifiable path $\gamma_{13} \in \Gamma_{13}$ or if

$$\int_{\gamma_{23}} \zeta\, ds \geq \frac{1}{3} \tag{3.1.2}$$

for every rectifiable path $\gamma_{23} \in \Gamma_{23}$, then $3\zeta \in F(\Gamma_{13})$ or $3\zeta \in F(\Gamma_{23})$, which implies

$$\int_{R^n} \zeta^n\, dm \geq 3^{-n} \min\{M(\Gamma_{13}), M(\Gamma_{23})\}. \tag{3.1.3}$$

If neither (3.1.1) nor (3.1.2) is true for some rectifiable paths $\gamma_{13} \in \Gamma_{13}$, $\gamma_{23} \in \Gamma_{23}$, then

$$\int_{\alpha} \zeta\, ds > \frac{1}{3} \tag{3.1.4}$$

for every rectifiable path $\alpha \in \Delta(|\gamma_{13}|, |\gamma_{23}| : D)$. Thus $3\zeta \in F(\Delta(|\gamma_{13}|, |\gamma_{23}| : D))$, which implies

$$\int_{R^n} \zeta^n\, dm \geq 3^{-n} M(\Delta(|\gamma_{13}|, |\gamma_{23}| : D)). \tag{3.1.5}$$

Since $\zeta \in F(\Gamma_{12})$ was arbitrary and since either (3.1.3) or (3.1.5) must be true, the assertion follows.

By special choice of the domain D or the sets F_1, F_2, F_3, one can remove the inf-factor in Theorem 3.1.

3.2. Corollary. *Let* F_1, F_2, F_3 *be three sets in* $\overline{R}^n$ *and let* $\Gamma_{ij} = \Delta(F_i, F_j : \overline{R}^n)$, $1 \leq i, j \leq 3$. *Then*

$$M(\Gamma_{12}) \geq 3^{-n} \min\{M(\Gamma_{13}), M(\Gamma_{23}), \delta_n(r)\},$$

where $r = \min\{q(F_1, F_3), q(F_2, F_3)\}$ *and* $\delta_n(r)$ *is as defined in* 2.1.

3.3. Theorem. *Let* F_1, F_2, F_3 *be three sets in a domain* D, *let* D *contain the spherical ring* $B^n(b) \setminus \overline{B}^n(a)$, $0 < a < b < \infty$, *let* F_3 *lie in* $\overline{B}^n(a)$, *and let* Γ_{ij} *be as in Theorem* 3.1. *If one of the three conditions*

(1) F_i *lies in* $\underline{C}B^n(b)$, $i = 1, 2$,

(2) F_1 *lies in* $\complement B^n(b)$ *and* F_2 *is connected with* $d(F_2) \geq 2b$,

(3) F_i *is connected with* $d(F_i) \geq 2b$, $i = 1, 2$,

is satisfied, then

$$M(\Gamma_{12}) \geq 3^{-n} \min\{M(\Gamma_{13}), M(\Gamma_{23}), c_n \log(b/a)\},$$

where c_n *is a positive constant depending only on* n.

Proof. We may assume that F_1, F_2, F_3 are nonempty sets. If (1) is satisfied, then the assertion follows directly from Theorem 3.1 and from Väisälä [9, 10.12].

Assume next that (2) or (3) is satisfied. Choose $\zeta \in F(\Gamma_{12})$. If (3.1.1) holds for every rectifiable path $\gamma_{13} \in \Gamma_{13}$ or if (3.1.2) holds for every rectifiable path $\gamma_{23} \in \Gamma_{23}$, then (3.1.3) holds. If neither (3.1.1) nor (3.1.2) holds for some rectifiable paths $\gamma_{13} \in \Gamma_{13}$, $\gamma_{23} \in \Gamma_{23}$, then (3.1.4) holds for every rectifiable path $\alpha \in \Delta(F_1 \cup |\gamma_{13}|, F_2 \cup |\gamma_{23}| : D)$. Therefore, since $S^{n-1}(t)$ meets both $F_1 \cup |\gamma_{13}|$ and $F_2 \cup |\gamma_{23}|$ for $a < t < b$ and since D contains the spherical ring $B^n(b) \backslash \overline{B}^n(a)$, we obtain

$$\int_{R^n} \zeta^n \, dm \geq 3^{-n} c_n \log \frac{b}{a} \tag{3.3.1}$$

by Väisälä [9, 10.12], where $c_n > 0$ is the n-modulus of the family of all paths joining e_n and $-e_n$ in $S^{n-1}(1)$. Finally, since $\zeta \in F(\Gamma_{12})$ was arbitrary and since either (3.1.3) or (3.3.1) must be true, the assertion follows.

4. **Modulus estimates for collections of path families.** Making use of the comparison principle of §3, we derive implicit positive lower bounds for moduli of certain path families. The ensuing results will be needed in §§5 and 6.

4.1. **Theorem.** *Let* $\mathfrak{F}$ *be a collection of connected sets in a domain* D *and let* $\inf q(F) > 0$, $F \in \mathfrak{F}$. *Then*

$$\inf_{F \in \mathfrak{F}} M(\Delta(A, F : D)) > 0$$

either for each or for no continuum A *in* D.

Proof. Let A and A^* be two continua in D. Assume first that $A \cap A^* = \emptyset$ and, for example, that

$$M(\Delta(A, F : D)) \geq \delta > 0$$

whenever $F \in \mathfrak{F}$. We may further assume that $\infty \notin A$. Choose a number r so that $\overline{B}^n(a, 2r) \subset D$ for each point $a \in A$ and so that $0 < 4r < \min\{\inf q(F), d(A, A^*)\}$. Let $A_1, \cdots, A_p$ be a finite covering of A by closed balls with centers $a_i \in A$, $i = 1, \cdots, p$, and radii r, and let $M(\Delta(A_i, A^* : D)) = \delta_i$. By [4, 1.15], each $\delta_i > 0$. We claim that

$$M(\Delta(A^*, F : D)) \geq 3^{-n} \min\{\delta/p, \delta_1, \cdots, \delta_p, c_n \log 2\}$$

whenever $F \in \mathfrak{F}$, where $c_n > 0$ is as in Theorem 3.3.

For this, let $F \in \mathfrak{F}$. Then, by the subadditivity of the modulus, $M(\Delta(A_i, F\colon D)) \geq \delta/p$ for some i. Fix this i. Since $A^* \cap B^n(a_i, 2r) = \emptyset$ and since $d(F) \geq 4r$, the assertion follows from Theorem 3.3 by setting $F_1 = A^*$, $F_2 = F$, and $F_3 = A_i$.

In the preceding argument we assumed that $A \cap A^* = \emptyset$. Suppose now that $A \cap A^* \neq \emptyset$. If the set $D \setminus (A \cup A^*)$ is nonempty, and therefore contains a continuum A', we may apply the above procedure first to the sets A, A' and then to A', A^*, while if $D \setminus (A \cup A^*) = \emptyset$, and therefore $D = \overline{R}^n$, we may, assuming that $A \neq \overline{R}^n \neq A^*$, choose continua A_1 and A_1^* so that the sets $A \cap A_1$, $A_1 \cap A_1^*$, $A_1^* \cap A^*$ are all empty and apply the above procedure first to the sets A, A_1, then to A_1, A_1^*, and finally to A_1, A^*. This completes the proof of Theorem 4.1.

4.2. **Theorem.** *Let $\mathfrak{F}$ be a collection of connected sets in a domain D, let* $\inf q(F) > 0$, $F \in \mathfrak{F}$, *and let* $\inf M(\Delta(A, F\colon D)) > 0$, $F \in \mathfrak{F}$, *for some continuum A in D. Then*

$$\inf_{F, F^* \in \mathfrak{F}} M(\Delta(F, F^* : D)) > 0.$$

Proof. Choose a finite point $b \in D$ and a number r, $0 < 4r < \inf q(F)$, so that $\overline{B}^n(b, 2r) \subset D$. By Theorem 4.1,

$$\inf_{F \in \mathfrak{F}} M(\Delta(\overline{B}^n(b, r), F : D)) = \delta > 0.$$

Thus, by Theorem 3.3,

$$\inf_{F, F^* \in \mathfrak{F}} M(\Delta(F, F^* : D)) \geq 3^{-n} \min\{\delta, c_n \log 2\} > 0.$$

4.3. **Corollary.** *Let $\mathfrak{F}$ be a collection of connected sets in a domain D, let* $\inf q(F) > 0$, $F \in \mathfrak{F}$, *and let $A \cap F \neq \emptyset$ for some continuum $A \subset D$ and for each $F \in \mathfrak{F}$. Then*

$$\inf_{F, F^* \in \mathfrak{F}} M(\Delta(F, F^* : D)) > 0.$$

5. **Two theorems of Loewner type.** Let F be a compact proper subset of $\overline{R}^n$. Then, by Corollary 3.2, $M(\Delta(F, \partial U\colon \overline{R}^n)) > 0$ either for each or for no neighborhood U of F, $\overline{U} \neq \overline{R}^n$. The set F is said to be of *positive capacity* in the first case and of *capacity zero* in the second case.

Rešetnjak [8] uses capacities of condensers for classifying compact sets of positive capacity and compact sets of capacity zero. Both methods lead to the same classification, because $M(\Delta(F, \partial U : \overline{R}^n)) = \operatorname{cap}(F, U)$ by Ziemer [10] (see also Hesse [1]).

We note that, by the subadditivity of the modulus, a compact set $F \neq \overline{R}^n$ is of capacity zero if and only if $M(\Gamma) = 0$, where Γ is the family of all paths with one endpoint in F and the other in $\overline{R}^n \setminus F$.

5.1. Theorem. *Let* F *and* F^* *be two disjoint compact sets in a domain* D. *Then* $M(\Delta(F, F^*:D)) > 0$ *if and only if both* F *and* F^* *are of positive capacity.*

Proof. The necessity part is obvious, while the sufficiency part follows from Theorem 3.1 and Corollary 4.3 by choosing a neighborhood U of $F \cup F^*$ so that $\overline{U} \subset D$ and $\overline{U} \neq \overline{R}^n$, and by setting $F_1 = F$, $F_2 = F^*$, and $F_3 = \partial U$.

Corollary 4.3 gives, furthermore,

5.2. Theorem. *Let* F *and* F^* *be two disjoint connected sets in a domain* D. *Then* $M(\Delta(F, F^*:D)) > 0$ *if and only if both* F *and* F^* *are nondegenerate.*

6. Uniform domains. In this section Lemma 2.1 will be extended to domains more general than $\overline{R}^n$.

6.1. *Uniform domains.* A domain D is called a *uniform domain* if for each $r > 0$ there is a $\delta > 0$ such that $M(\Delta(F, F^*:D)) \geq \delta$ whenever F and F^* are connected subsets of D with $q(F) \geq r$ and $q(F^*) \geq r$.

It is easy to find nonuniform domains. For example, the image of a ball under a quasiconformal mapping is not a uniform domain if the mapping fails to extend to a continuous mapping of the closure of the ball.

Theorem 4.2 reduces the problem of characterizing uniform domains to a problem of considering moduli of those path families that lie close to the boundary.

6.2. Theorem. *A domain* D *is a uniform domain if and only if for each point* $b \in \partial D$ *and for each neighborhood* U *of* b *there is a neighborhood* $V \subset U$ *of* b *and a number* $\delta > 0$ *such that* $M(\Delta(F, F^*:D)) \geq \delta$ *whenever* F *and* F^* *are connected subsets of* D *meeting* ∂U *and* ∂V.

Proof. The necessity part is trivial. To prove the sufficiency part, fix $r > 0$. For each point $b \in \partial D$ choose neighborhoods U_b and V_b of b, a positive number δ_b, and a continuum $A_b \subset D$ so that $q(U_b) < r/2$, $V_b \subset U_b$, and $M(\Delta(A_b, F:D)) \geq \delta_b$ whenever F is a connected set in D meeting ∂U_b and ∂V_b. Cover ∂D by a finite number of the neighborhoods V_b, say $V_1, \cdots, V_p$, choose a continuum $A \subset D$ which contains the sets $D \setminus (V_1 \cup \cdots \cup V_p)$ and $\bigcup A_i$, and let $\delta = \min_{1 \leq i \leq p} \delta_i$. If $\mathfrak{F}$ is the collection of all connected sets F in D for which $q(F) \geq r$, then

$$\inf_{F \in \mathfrak{F}} M(\Delta(A, F:D)) \geq \delta.$$

Hence the assertion follows from Theorem 4.2.

We next show that uniform domains possess a simple topological property.

6.3. *Finite connectedness.* A domain D is said to be *finitely connected* on the boundary if each point of ∂D has arbitrarily small neighborhoods U such that $U \cap D$ contains only a finite number of components.

6.4. **Theorem.** *A uniform domain is finitely connected on the boundary.*

Proof. Suppose that a domain D is not finitely connected at some boundary point b. Performing a preliminary inversion if necessary, we may assume that $b \neq \infty$. Choose $r > 0$ so that $U \cap D$ has infinitely many components for each neighborhood $U \subset B^n(b, 2r)$ of b. Let V, $\overline{V} \subset B^n(b, r)$, be a neighborhood of b and let (E_k), $k = 1, 2, \cdots$, be a sequence of components of $B^n(b, 2r) \cap D$ such that $E_k \cap \partial V \neq \emptyset$ and $m(E_k) \leq 1/k$. For each k choose a connected set F_k in $E_k \cap \overline{B}^n(b, r)$ meeting ∂V and $S^{n-1}(b, r)$, and for $k = 2, 3, \cdots$, consider the path family $\Gamma_k = \Delta(F_k, F_1 : D)$. The function ζ, defined by setting $\zeta(x) = 1/r$ for $x \in E_k$ and $\zeta(x) = 0$ for $x \in \overline{R}^n \backslash E_k$, belongs to $F(\Gamma_k)$, which implies

$$M(\Gamma_k) \leq m(E_k)/r^n \leq 1/kr^n.$$

Thus $M(\Gamma_k) \longrightarrow 0$ as $k \longrightarrow \infty$. Since $q(F_k) \geq q(\partial V, S^{n-1}(b, r)) > 0$ for each k, the domain D is not a uniform domain.

The converse of Theorem 6.4 is, in general, false. However, in some important special cases the conditions "finitely connected" and "uniform" are equivalent. To give examples, we recall the following concept:

6.5. *Quasiconformal collaredness.* A domain D is said to be *quasiconformally collared* on the boundary if each point of ∂D has arbitrarily small neighborhoods U such that $U \cap D$ can be mapped quasiconformally onto a ball.

By [5, 2.3] and by the proof of Theorem 1.17(3) in [4], a quasiconformally collared domain D satisfies the modulus condition given in Theorem 6.2. Thus we have

6.6. **Corollary.** *Quasiconformally collared domains are uniform domains.*

6.7. **Corollary.** *Let D be a domain which can be mapped quasiconformally onto some quasiconformally collared domain. Then D is a uniform domain if and only if D is finitely connected on the boundary.*

Proof. Finite connectedness implies uniformity for the domain D by [5, 3.2].

6.8. **Corollary.** *A plane domain with finitely many boundary components is a uniform domain if and only if it is finitely connected on the boundary.*

6.9. *Jordan domains.* By Corollary 6.8, each Jordan domain in $\overline{R}^2$ is a uniform domain. The same is not true in higher dimensions. For example, consider domains

$$D = \{x = (x_1, x_2, x_3) \in \overline{R}^3 : |x_2| < g(x_1),\ x_1 > 0\},$$

where the function $g: [0, \infty) \longrightarrow R^1$ satisfies the following conditions:

(i) $g(0) = 0$ and $g(u) > 0$ for $u > 0$.

(ii) g' is continuous and increasing in $(0, \infty)$.

(iii) $\lim_{u \to 0} g'(0) = 0$.

A domain D is called a *wedge* of angle zero. From [4, 5.4] it follows that D is a uniform domain if and only if

$$\int_0^d \frac{du}{g(u)^{1/2}} < \infty$$

for some $d > 0$. In particular, if D is defined by the function $g(u) = u^p$, $p > 1$, then D is a uniform domain if and only if $p < 2$.

6.10. **Remarks.** (1) Uniform domains are used in [5] for the study of the boundary extension of quasiconformal mappings.

(2) Uniform domains are used in [6] for the study of the uniform equicontinuity of quasiconformal mappings.

(3) Palka [7] has used uniform domains for the study of certain convergence problems of quasiconformal mappings. He has also given a somewhat different proof for Theorem 6.2.

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