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# MULTIPLIERS AND LINEAR FUNCTIONALS FOR THE CLASS $N^+$

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ABSTRACT. Multipliers for the classes $H^p$ are studied recently by several authors, see Duren's book, *Theory of $H^p$ spaces*, Academic Press, New York, 1970. Here we consider corresponding problems for the class $N^+$ of holomorphic functions in the unit disk such that

$$\lim_{r\to 1} \int_0^{2\pi} \log^+|f(re^{i\theta})|\,d\theta = \int_0^{2\pi} \log^+|f(e^{i\theta})|\,d\theta < \infty.$$

Our results are:

1. $N^+$ is an $F$-space in the sense of Banach with the distance function

$$\rho(f, g) = \frac{1}{2\pi} \int_0^{2\pi} \log(1 + |f(e^{i\theta}) - g(e^{i\theta})|)\,d\theta.$$

2. A complex sequence $\Lambda = \{\lambda_n\}$ is a multiplier for $N^+$ into $H^q$ for a fixed $q$, $0 < q < \infty$, if and only if $\lambda_n = O(\exp[-c\sqrt{n}])$ for a positive constant $c$.

3. A continuous linear functional $\phi$ on the space $N^+$ is represented by a holomorphic function $g(z) = \sum b_n z^n$ which satisfies $b_n = O(\exp[-c\sqrt{n}])$ for a positive constant $c$.

Conversely, such a function $g(z) = \sum b_n z^n$ defines a continuous linear functional on the space $N^+$.

1. **Introduction.** Let $X$ and $Y$ be linear spaces, consisting of complex sequences

$$X = \{\{a_0, a_1, a_2, \cdots\}\}, \qquad Y = \{\{b_0, b_1, b_2, \cdots\}\}.$$

When $X$ or $Y$ is a coefficient space of a class of functions, e.g. $H^p$ etc., we write simply as $H^p$ etc. for $X$ or $Y$.

A sequence of complex numbers $\Lambda = \{\lambda_0, \lambda_1, \lambda_2, \cdots\}$ is called a *multiplier* for $X$ into $Y$, denoted as $\Lambda \in (X, Y)$, if for any sequence $\{a_n\} \in X$ we have $\{\lambda_n a_n\} \in Y$.

Multipliers for $H^p$ spaces are studied by several authors, see [3, p. 99]. We consider here multipliers for the class $N^+$, defined below.

2. **The class $N^+$ as an $F$-space.** Let $D$ be the unit disk $\{|z| < 1\}$. A holomorphic function $f(z)$ in $D$ is said to belong to the *class* $N$ of functions of

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bounded characteristic, if

$$T(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta \tag{2.1}$$

is bounded for $0 \leq r < 1$. A function $f(z) \in N$ is said to belong to the *class* $N^+$ if there holds

$$\lim_{r\to 1} \int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta = \int_0^{2\pi} \log^+ |f(e^{i\theta})|\, d\theta. \tag{2.2}$$

Then, for $0 < p < q < \infty$,

$$H^\infty \subset H^q \subset H^p \subset N^+ \subset N, \tag{2.3}$$

and these inclusion relations are proper [7, p. 82, where $N$ and $N^+$ are denoted as $A$ and $D$, respectively].

The class $H^\infty$ or $H^p$, $1 \leq p < \infty$, can be considered as a Banach space with the norm, respectively,

$$\|f\|_\infty = \sup_{|z|<1} |f(z)|, \quad \text{or} \quad \|f\|_p = \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^p\, d\theta\right\}^{1/p}, \quad 1 \leq p < \infty. \tag{2.4}$$

The class $H^p$, $0 < p < 1$, does not form a Banach space but is a complete metric space with the distance function

$$\|f - g\|_p^p = \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta}) - g(e^{i\theta})|^p\, d\theta. \tag{2.5}$$

Now for the class $N^+$, which obviously forms a linear space, we define a distance function by

$$\rho(f, g) = \frac{1}{2\pi}\int_0^{2\pi} \log(1 + |f(e^{i\theta}) - g(e^{i\theta})|)\, d\theta. \tag{2.6}$$

Since

$$\log^+ x \leq \log(1 + x) \leq \log^+ x + \log 2, \qquad x \geq 0, \tag{2.7}$$

we know that $\rho(f, g)$ is finite for $f, g \in N^+$.

Using the inequalities

$$\begin{aligned} \log(1 + |x + y|) &\leq \log(1 + |x| + |y|) \leq \log(1 + |x|)(1 + |y|) \\ &\leq \log(1 + |x|) + \log(1 + |y|), \end{aligned} \tag{2.8}$$

$\rho(f, g)$ is seen to satisfy the triangle inequality.

$\rho(f, g) = 0$ means $f(e^{i\theta}) = g(e^{i\theta})$ a.e., which implies $f(z) = g(z)$ by the uniqueness theorem of Riesz.

Thus $\rho(f, g)$ is a distance function. We will prove the following

**Theorem 1.** *The class* $N^+$ *can be considered as an F-space in the sense of Banach* [1, *p.* 51]. *That is, the distance function* $\rho(f, g)$ *satisfies the following conditions*:

(i) $\rho(f, g) = \rho(f - g, 0)$.

(ii) *Suppose* $f, f_n \in N^+$ *and* $\rho(f_n, f) \longrightarrow 0$, *then for each complex number* $\alpha$,

$$\rho(\alpha f_n, \alpha f) \longrightarrow 0. \tag{2.9}$$

(iii) *Suppose* $\alpha, \alpha_n$ *be complex numbers and* $\alpha_n \longrightarrow \alpha$, *then for each* $f(z) \in N^+$,

$$\rho(\alpha_n f, \alpha f) \longrightarrow 0. \tag{2.10}$$

(iv) $N^+$ *is complete with respect to this metric.*

In the sequel we shall write sometimes $f(\theta)$ for $f(e^{i\theta})$

**Proof.** (i) is obvious.

(ii) (2.9) is obvious if $|\alpha| \le 1$. We suppose $|\alpha| > 1$. We can assume $\alpha > 1$. If $\rho(f_n, f) \longrightarrow 0$, it is easily seen that $f_n(\theta) \longrightarrow f(\theta)$ in measure. We can choose a subsequence $\{f_{n_k}\}$ such that $f_{n_k}(\theta) \longrightarrow f(\theta)$ a.e. We write $g_k$ for $f_{n_k}$.

There is a closed set $E \subset [0, 2\pi)$ such that meas $(E) > 2\pi - \epsilon$ and $g_k(\theta)$ converges uniformly to $f(\theta)$ on $E$ as $k \longrightarrow \infty$. Then $2\pi\rho(\alpha f, \alpha g_k) = \int_E + \int_{E^C}$, where

$$\int_{E^C} \log(1 + \alpha|g_k(\theta) - f(\theta)|)\, d\theta \le \int_{E^C} \log(\alpha + \alpha|g_k(\theta) - f(\theta)|)\, d\theta$$

$$\le \int_{E^C} \log \alpha\, d\theta + \int_{E^C} \log(1 + |g_k(\theta) - f(\theta)|)\, d\theta$$

$$\le \epsilon \log \alpha + 2\pi\rho(g_k, f),$$

hence we have $\rho(\alpha f, \alpha g_k) \longrightarrow 0$. By the same arguments we know that every subsequence $\{g_m\}$ of $\{f_n\}$ contains a subsequence $\{g_{m_h}\}$ such that $\rho(\alpha f, \alpha g_{m_h}) \longrightarrow 0$ as $h \longrightarrow \infty$. Thus the sequence $\{f_n\}$ itself has the property (2.9): $\rho(\alpha f, \alpha f_n) \longrightarrow 0$ as $n \longrightarrow \infty$.

(iii) If $\alpha_n \longrightarrow \alpha$, we have

$$\rho(\alpha_n f, \alpha f) = \frac{1}{2\pi}\int_0^{2\pi} \log(1 + |(\alpha_n - \alpha)f|)\, d\theta = \int_E + \int_{E^C},$$

where $E$ is a closed subset of $[0, 2\pi)$ such that $f(\theta)$ is continuous on $E$, and meas $(E) > 2\pi - \delta$. We choose $\delta$ small enough so that $\int_{E^C} \log^+|f(\theta)|\, d\theta < \epsilon$. Then

$$\int_{E^C} \log(1 + |(\alpha_n - \alpha)f(\theta)|)\, d\theta \le \int_{E^C} \log 2\, d\theta + \int_{E^C} \log^+|(\alpha_n - \alpha)f|\, d\theta$$

$$\le \delta \log 2 + \int_{E^C} \log^+|f(\theta)|\, d\theta \le \delta \log 2 + \epsilon,$$

which shows that (2.10) holds: $\rho(\alpha_n f, \alpha f) \longrightarrow 0$ as $\alpha_n \longrightarrow \alpha$.

(iv) Suppose

$$\rho(f_n, f_m) \to 0 \quad \text{as } n, m \to \infty. \tag{2.11}$$

Then $\{f_n(\theta)\}$ converges in measure on $[0, 2\pi)$. Further, it is obvious that

$$\int_0^{2\pi} \log^+ |f_n(\theta)|\, d\theta \le \rho(f_n, 0) \le K \tag{2.12}$$

for a constant $K$, independent of $n$. Thus, since $f_n(z) \in N^+$,

$$\int_0^{2\pi} \log^+ |f_n(re^{i\theta})|\, d\theta \le K. \tag{2.13}$$

Applying the theorem of Khintchine-Ostrowskii [7, p. 83], we have by (2.13) that $f_n(z)$ converges uniformly to a holomorphic function $f(z)$ on each closed disk $|z| \le r < 1$, $\int \log^+|f(re^{i\theta})|\, d\theta \le K$, and that $f_n(\theta)$ converges to $f(\theta)$, boundary values of $f(z)$, on $[0, 2\pi)$ in measure. We will show that

$$f(z) \in N^+ \tag{2.14}$$

and

$$\rho(f_n, f) \to 0. \tag{2.15}$$

We choose a subsequence $\{f_{n_k}(\theta)\}$ of $\{f_n(\theta)\}$ such that $f_{n_k}(\theta) \to f(\theta)$ a.e. Then

$$\begin{aligned}
\rho(f, f_n) &= \frac{1}{2\pi}\int_0^{2\pi} \log(1 + |f(\theta) - f_n(\theta)|)\, d\theta \\
&\le \varliminf_{k\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \log(1 + |f_{n_k}(\theta) - f_n(\theta)|)\, d\theta \\
&\le \rho(f_{n_k}, f_n) + \epsilon, \quad \text{if } k \text{ is sufficiently large.}
\end{aligned}$$

Thus, if $n_k$ and $n$ are sufficiently large, we have from (2.11), $\rho(f, f_n) < 2\epsilon$, hence $\rho(f, f_n) \to 0$, which proves (2.15).

Next, since $\{f_{n_k}(\theta)\}$ converges to $f(\theta)$ a.e. on $[0, 2\pi)$, there is a closed set $E$ such that meas $(E) > 2\pi - \epsilon$ and $f_{n_k}(\theta)$ converges to $f(\theta)$ uniformly on $E$. We have $\int_0^{2\pi} \log^+|f_{n_k}(\theta)|\, d\theta = \int_E + \int_{E^C}$. Obviously, $\int_E \log^+|f_{n_k}(\theta)|\, d\theta \le \int_E \log^+|f(\theta)|\, d\theta + \epsilon$, if $k$ is large. Further,

$$\begin{aligned}
&\int_{E^C} \log^+|f_{n_k}(\theta)|\, d\theta \\
&\le \int_{E^C} \log^+|f_{n_k}(\theta) - f(\theta)|\, d\theta + \int_{E^C} \log^+|f(\theta)|\, d\theta + \int_{E^C} \log 2\, d\theta \\
&\le \rho(f_{n_k}, f) + \int_{E^C} \log^+|f(\theta)|\, d\theta + \epsilon \log 2 \\
&\le \epsilon + \int_{E^C} \log^+|f(\theta)|\, d\theta + \epsilon \log 2, \quad \text{if } k \text{ is large.}
\end{aligned}$$

Thus, we have

$$\int_0^{2\pi} \log^+ |f_{n_k}(\theta)|\, d\theta \le \int_0^{2\pi} \log^+ |f(\theta)|\, d\theta + (2 + \log 2)\epsilon.$$

Since $f_{n_k} \in N_*^+$, we get

$$\int_0^{2\pi} \log^+ |f_{n_k}(re^{i\theta})|\, d\theta \le \int_0^{2\pi} \log^+ |f_{n_k}(\theta)|\, d\theta$$

$$\le \int_0^{2\pi} \log^+ |f(\theta)|\, d\theta + (2 + \log 2)\epsilon.$$

Letting $k \longrightarrow \infty$ on the left-hand side, we obtain

$$\int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta \le \int_0^{2\pi} \log^+ |f(\theta)|\, d\theta + (2 + \log 2)\epsilon,$$

for any $\epsilon > 0$. This shows that

$$\lim_{r \to 1} \int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta = \int_0^{2\pi} \log^+ |f(\theta)|\, d\theta,$$

which proves (2.14), and our proof of Theorem 1 is completed.

**Remark 1.** We note that the above properties (i), (ii) and (iii) hold for the class $N$ also. But the completeness (iv) relative to the metric (2.6) is not tenable for $N$. In fact, if we put $f_n(z) = (z/n) \exp[n(1 + z)/(1 - z)]$, then $f_n(z) \in N$ and $|f_n(e^{i\theta})| = 1/n$ a.e., hence

$$\rho(f_n, f_m) \le \log(1 + 1/n + 1/m) \to 0 \quad \text{as } n, m \longrightarrow \infty,$$

but $f_n(0) = 0$ and $f_n(\frac{1}{2}) = \exp[3n]/2n \longrightarrow \infty$ as $n \longrightarrow \infty$.

3. **Multiplier as a closed operator.** From the proof of Theorem 1(iv) we know that $\rho(f_n, f) \longrightarrow 0$ implies the uniform convergence of $f_n(z)$ to $f(z)$ on each closed disk $|z| \le r < 1$. Hence, if $f_n(z) = \sum a_k^{(n)} z^k$ and $f(z) = \sum a_k z^k$, we have

$$\text{(3.1)} \qquad a_k^{(n)} \to a_k \quad (k = 0, 1, \cdots) \quad \text{if } \rho(f_n, f) \to 0 \text{ as } n \longrightarrow \infty.$$

Let $Y$ be an $F$-space consisting of complex sequences $\{b_k\}_0^\infty$ such that

$$\text{(3.2)} \quad \beta^{(n)} = \{b_k^{(n)}\} \to \beta = \{b_k\} \text{ in } Y \text{ implies } b_k^{(n)} \to b_k, \qquad k = 0, 1, \cdots.$$

We suppose that addition and scalar multiplication in the space $Y$ are defined in the usual way.

Now let $\Lambda = \{\lambda_k\}$ be a multiplier for $N^+$ into $Y$. From (3.1) and (3.2) we can easily see that $\Lambda$ is a closed operator. Thus, by the closed graph theorem, $\Lambda$ is continuous and hence $\Lambda$ maps bounded sets in $N^+$ to bounded sets in $Y$ [1, p. 54]. As an application, we have

**Theorem 2.** *Let $q$ be a positive number, $0 < q \le \infty$. In order that $\Lambda = \{\lambda_k\}$ be a multiplier for $N^+$ into $H^q$, it is necessary and sufficient that*

$$\lambda_k = O(\exp[-c\sqrt{k}]) \tag{3.3}$$

*for a positive constant* $c$.

**Remark.** Observe that, while the hypothesis of the theorem contains $q$, *the conclusion does not depend on* $q$.

We need some lemmas for the proof of Theorem 2.

**Lemma 1.** *Suppose a complex sequence* $\{\lambda_k\}$ *satisfies*

$$\lambda_k = O(\exp[-c_k\sqrt{k}]) \tag{3.4}$$

*for any positive sequence* $\{c_k\}$, $c_k \downarrow 0$. *Then we have* $\lambda_k = O(\exp[-c\sqrt{k}])$ ((3.3)) *for a positive constant* $c$.

**Proof.** If $\{\lambda_k\}$ satisfies (3.4), we have obviously $\lambda_k \to 0$ and $\overline{\lim}_{k\to\infty}((1/\sqrt{k})\log|\lambda_k|) \leq 0$. In order to obtain (3.3), as is verified immediately, it suffices to show that $\overline{\lim}_{k\to\infty}((1/\sqrt{k})\log|\lambda_k|) < 0$. We will show a contradiction under the assumption that

$$\overline{\lim_{k\to\infty}}\ ((1/\sqrt{k})\log|\lambda_k|) = 0. \tag{3.5}$$

Suppose (3.5) hold. Without loss of generality, we can assume that

$$b_k = -(1/\sqrt{k})\log|\lambda_k| \downarrow 0 \quad \text{as } k \to \infty, \tag{3.5$'$}$$

taking subsequence if necessary. From (3.4), for any sequence $\{c_k\}$, $c_k \downarrow 0$, there is a constant $A = A_{\{c_k\}}$ such that $|\lambda_k| \leq A\exp[-c_k\sqrt{k}]$, i.e.,

$$(c_k - b_k)\sqrt{k} \leq \log A, \tag{3.6}$$

where we note that the constant $A$ depends on the sequence $\{c_k\}$.

Put $c_k^* = \max(2b_k, 1/\sqrt[4]{k})$. Then $c_k^* \downarrow 0$ from (3.5$'$), and we have by (3.6) with $c_k^*$ for $c_k$ and with $A^* = A_{\{c_k^*\}}$ for $A$, $\frac{1}{2}\sqrt[4]{k} \leq \log A^*$, as $k \to \infty$, which is a contradiction, and our Lemma 1 is proved.

**Lemma 2.** *Put*

$$\exp\left[\frac{c}{2}\,\frac{1+z}{1-z}\right] = \sum_{n=0}^{\infty} a_n(c)z^n, \qquad 0 < c \leq 1. \tag{3.7}$$

*Then we have*

$$\log|a_n(c)| \geq \sqrt{cn} + O(\log n) + O(\log c). \tag{3.7$'$}$$

**Proof.** According to the calculations in [7, p. 107–108], we have

$$a_n(c) = \sum_{r=0}^{n}\left(\frac{n!}{r!\,(n-r)!}\;e^{c/2}\;\frac{r}{n}\,\frac{c^r}{r!}\right)$$

and

$$\log |a_n(c)| \geq \log \left( \frac{n!}{p_n!(n-p_n)!} e^{c/2} \frac{p_n}{n} \frac{c^{p_n}}{p_n!} \right), \tag{3.8}$$

where $p_n$ is the largest integer such that

$$p_n \leq \sqrt{\left(\frac{c+1}{2}\right)^2 + cn} - \frac{c+1}{2} = \sqrt{cn}\, \Phi\left(\frac{1}{cn}\left(\frac{c+1}{2}\right)^2\right),$$

in which we denote by $\Phi$ such a function $\Phi(x) = \sqrt{1+x} - \sqrt{x} = 1/(\sqrt{1+x} + \sqrt{x})$. Then $\frac{1}{2} \leq \Phi(x) \leq 1$ for $0 \leq x \leq \frac{1}{2}$, hence

$$\tfrac{1}{2} \leq p_n/\sqrt{cn} \leq 1, \qquad p_n^2/n \leq c \leq 1, \tag{3.9}$$

if $(1/cn)((c+1)/2)^2 \leq \frac{1}{2}$. Thus

$$\begin{aligned}
\log a_n(c) &\geq n \times \log n - n + O(\log n) + c/2 + \log p_n + p_n \log c - 2p_n \log p_n \\
&\quad + 2p_n + O(\log p_n) - (n-p_n)\log(n-p_n) + (n-p_n) + O(\log(n-p_n)) - \log n \\
&= n \times \log n - p_n \log(p_n^2/c) + p_n - (n-p_n)(\log n + \log(1 - p_n/n)) \\
&\quad + O(\log n) + O(\log p_n) + c/2 \\
&= p_n - p_n \log(p_n^2/cn) + (n-p_n)p_n/n + O(p_n^2/n) + O(\log n) + O(\log p_n) \\
&\geq 2p_n + O(\log n) + O(\log p_n) \quad \text{(by (3.9))} \\
&\geq \sqrt{cn} + O(\log n) + O(\log cn) = \sqrt{cn} + O(\log n) + O(\log c),
\end{aligned}$$

as required. Q.E.D.

**Remark 3.** Let $\{c_k^*\}$, $c_k^* \downarrow 0$, be a sequence such that

$$1/\sqrt{k} \leq c_k^* \leq 1. \tag{3.10}$$

Then, from (3.7′) we have

$$\log |a_k(c_k^*)| \geq \sqrt{c_k^* k}\,(1 + o(1)). \tag{3.11}$$

**Proof of Theorem 2.** *Necessity.* By Lemma 1, we have only to show that $\{\lambda_k\}$ satisfies (3.4) for any positive sequence $\{c_k\}$, $c_k \downarrow 0$.

Let there be given a sequence $\{c_k\}$, $c_k \downarrow 0$. Put $c_k' = \min(\frac{1}{2}, \max(1/\sqrt[4]{k}, c_k))$. If (3.4) holds for this $\{c_k'\}$, we have (3.4) also for the given sequence $\{c_k\}$. Hence we can suppose

$$1/\sqrt[4]{k} \leq c_k \leq \tfrac{1}{2}. \tag{3.12}$$

Then $c_k^* = 2c_k^2$ satisfies (3.10).

Choose positive sequences $\{\epsilon_k\}$ and $\{\delta_k\}$, $\epsilon_k \downarrow 0$, $\delta_k \downarrow 0$. Let $\{r_k\}$ be a sequence such that

(3.13) $$1 > r_k \geq 1 - 1/k, \quad r_k \uparrow 1,$$

and

(3.14) $$(1 - r^2)/(1 + r^2 - 2r\cos\theta) \leq 1 \quad \text{for } |\theta| \geq \epsilon_k \text{ and } r \geq r_k.$$

Put

(3.15) $$f_k(z) = \exp[c_k^2(1 + r_k z)/(1 - r_k z)] \in N^+.$$

We will show that $\{f_k\}$ is a bounded sequence. Let $V$ be a neighborhood of 0; $V = \{g \in N^+; \rho(g, 0) < \eta\}$. Let $k_0$ be a number such that

(3.16) $$\log(1 + \delta_k) + 2\epsilon_k \log 2 + c_k^2 < \eta,$$

for $k \geq k_0$. Take a number $\alpha$, $0 < \alpha < 1$, such that

(3.17) $$\alpha \exp[(1 + r_{k_0})/(1 - r_{k_0})] \leq \delta_{k_0}, \text{ hence a fortiori } \alpha e \leq \delta_{k_0}.$$

Then, for $k \leq k_0$,

(3.18) $$|\alpha f_k| \leq \delta_{k_0}$$

and

(3.19) $$\rho(\alpha f_k, 0) \leq \log(1 + \delta_{k_0}) < \eta, \quad \text{hence } \alpha f_k \in V.$$

For $k > k_0$,

$$\begin{aligned}\rho(\alpha f_k, 0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \log(1 + |\alpha f_k(e^{i\theta})|)\,d\theta \\ &= \frac{1}{2\pi}\int_{|\theta|\geq\epsilon_k} + \frac{1}{2\pi}\int_{|\theta|<\epsilon_k} \\ &\leq \log(1 + \alpha e) + \frac{1}{2\pi}\int_{|\theta|<\epsilon_k} (\log 2 + \log^+|f_k(e^{i\theta})|)\,d\theta \\ &\leq \log(1 + \delta_{k_0}) + 2\epsilon_k \log 2 + \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+|f_k(e^{i\theta})|\,d\theta \\ &\leq \log(1 + \delta_{k_0}) + 2\epsilon_{k_0}\log 2 + c_k^2 < \eta,\end{aligned}$$

hence $\rho(\alpha f_k, 0) < \eta$, $k = 1, 2, \cdots$, thus $\{\alpha f_k\} \subset V$, if $\alpha$ satisfies (3.17), which shows that $\{f_k\}$ is bounded.

Therefore, $\{\Lambda[f_k]\}$ must be a bounded sequence in $H^q$, and we have

(3.20) $$\|\Lambda[f_k]\|_q \leq L \quad \text{with a constant } L.$$

$\Lambda[f_k](z) = \Sigma \lambda_n a_n^{(k)} z^n$ if $f_k(z) = \Sigma a_n^{(k)} z^n$, and by [3, p. 98],

$$(3.21_1) \qquad |\lambda_n a_n^{(k)}| \le C_q \times L \times n^{-1+1/q} \quad \text{if } 0 < q < 1,$$

$$(3.21_2) \qquad \le C_q \times L \quad \text{if } 1 \le q \le \infty,$$

where $C_q$ is a constant depending on $q$.

Using notations of Lemma 2 and Remark 3, $a_n^{(k)} = a_n (2c_k^2) r_k^n = a_n (c_k^*) r_k^n$, whence we have by (3.11) and (3.13) that $|\lambda_k a_k^{(k)}| \ge |\lambda_k|(1 - 1/k)^k \exp[c_k \sqrt{2k}(1 + o(1))]$, and from $(3.21_1)$ or $(3.21_2)$,

$$|\lambda_k| \le C_q' \times L \times k^{-1+1/q} \exp[-c_k \sqrt{2k}(1 + o(1))] = O(\exp[-c_k \sqrt{k}]),$$

where $C_q'$ is a constant. This proves (3.4) and hence (3.3).

*Sufficiency*. Suppose $\Lambda = \{\lambda_k\}$ satisfy (3.3) for a positive constant $c$. If $f(z) = \Sigma a_k z^k \in N^+$, we proved in [8, Theorem 2] that $\log^+|a_k| = o(\sqrt{k})$, hence we can write

$$(3.22) \qquad |a_k| \le A_1 \exp[\eta_k \sqrt{k}] \quad \text{with a sequence } \eta_k \downarrow 0,$$

where $A_1$ is a constant. Let $k_0$ be a number such that $\eta_k < c/2$ for $k \ge k_0$, then we have

$$(3.23) \qquad |\lambda_k a_k| \le A_2 \exp[-c\sqrt{k}/2] \quad \text{for } k \ge k_0$$

with a constant $A_2$. Since $\Sigma \exp[-c\sqrt{k}/2] < \infty$, we obtain that $\Lambda[f](z)$ is continuous on $\bar{D} = \{|z| \le 1\}$, hence $\Lambda[f] \in H^q$, and our Theorem 2 is proved.

**Remark 4.** Let $A$ be the class of functions holomorphic in $D$ and continuous on $\bar{D}$. Then the proof of Theorem 2 shows that for a fixed $q$, $0 < q \le \infty$, *each multiplier for $N^+$ into $H^q$ must carry all elements of $N^+$ into the class $A$.*

**Corollary.** *The space $N^+$ is not locally bounded. That is, every ball $B = B(c) = \{f \in N^+;\ \rho(f, 0) < c\}$ is not bounded.*

**Proof.** Suppose a ball $B(c)$ of radius $c$ were bounded. We choose numbers $\epsilon$, $\delta$, and $c'$ so that

$$(3.24) \qquad 0 < \epsilon(1 + 4 \times \log 2) < c,$$

$$(3.25) \qquad |e^\zeta - 1| < \epsilon \quad \text{if } |\zeta| < \delta,$$

$$(3.26) \qquad \begin{cases} 0 < c'^2 < c - \epsilon(1 + 4 \times \log 2), & c' > 0, \\ 2c'^2/(1 - \cos \epsilon) < \delta. \end{cases}$$

Put

$$(3.27) \qquad g(z) = \exp[c'^2 \times (1 + z)/(1 - z)]$$

and

$$f_r(z) = g(rz) - 1, \qquad 0 < r < 1. \tag{3.28}$$

Then $f_r(z) \in N^+$, and

$$\rho(f_r, 0) = \frac{1}{2\pi}\int_0^{2\pi} \log(1 + |f_r|)\,d\theta = \frac{1}{2\pi}\int_{|\theta|\ge\epsilon} + \frac{1}{2\pi}\int_{|\theta|<\epsilon}$$

$$\le \log(1+\epsilon) + \frac{1}{2\pi}\int_{|\theta|<\epsilon} \log 2\,d\theta + \frac{1}{2\pi}\int_{|\theta|<\epsilon} \log^+|f_r|\,d\theta$$

$$\le \epsilon + 2\epsilon\log 2 + \frac{1}{2\pi}\int_{|\theta|<\epsilon} \log 2\,d\theta + \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+|g(re^{i\theta})|\,d\theta$$

$$\le \epsilon + 4\epsilon\log 2 + c'^2 < c,$$

hence $\{f_r\} \subset B(c)$. Thus every multiplier $\Lambda = \{\lambda_n\}$ should map $\{f_r\}$ into a bounded set in $H^\infty$. Thus, if $f_r(z) = \Sigma a_n r^n z^n$, we have

$$|\lambda_n a_n r^n| \le L = L(\Lambda) \tag{3.29}$$

for every $r$, $0 \le r < 1$, with a constant $L(\Lambda)$ depending on $\Lambda$. We have by (3.7′), using the notation of Lemma 2,

$$|a_n| = |a_n(2c'^2)| \ge \exp[c'\sqrt{2n}(1 + o(1))]. \tag{3.30}$$

From (3.29) and (3.30), it would be concluded that every multiplier $\Lambda = \{\lambda_n\}$ must satisfy

$$\lambda_n = O(\exp[-c'\sqrt{2n}]) \tag{3.31}$$

for the constant $c'$, determined from (3.24)–(3.26). But $\Lambda^* = \{\lambda_n^*\}$ with $\lambda_n^* = \exp[-c'\sqrt{n}]$ is also a multiplier by Theorem 2, which contradicts (3.31). This proves our assertion. Q.E.D.

In this connection, we notice a result of Duren [2, p. 24, Theorem 1]: $\lambda_n = O(n^{1/q-1/p})$ implies $\Lambda = \{\lambda_n\} \in (H^p, H^q)$ for $0 < p \le 2 \le q < \infty$.

**4. Representations of linear functionals on the space $N^+$.** Duren, Romberg, and Shields [4] studied representations of linear functionals on the space $H^p$, $0 < p < 1$. We now investigate corresponding problems on the space $N^+$, following their methods.

**Lemma 3.** *Let $f(z) \in N^+$. Put $f_r(z) = f(rz)$ for $0 < r < 1$. Then $\rho(f_r, f) \to 0$ as $r \to 1$.*

**Proof.** $f_r(\theta)$ tends to $f(\theta)$ a.e. as $r \to 1$. For any $\epsilon \to 0$, there is a closed set $E$ such that meas$(E) > 2\pi - \epsilon$ and $f_r(\theta)$ tends to $f(\theta)$ uniformly on $E$. Then

$$\rho(f, f_r) = \frac{1}{2\pi} \int_0^{2\pi} \log(1 + |f_r(\theta) - f(\theta)|)\, d\theta = \frac{1}{2\pi}\left(\int_E + \int_{E^C}\right),$$

and

$$\int_{E^C} \leq \int_{E^C} \log 2\, d\theta + \int_{E^C} \log^+ |f_r(\theta) - f(\theta)|\, d\theta$$

$$\leq \epsilon \log 2 + \int_0^{2\pi} \log^+ |f(re^{i\theta}) - f(e^{i\theta})|\, d\theta.$$

Since $f(z) \in N^+$, the last integral $\longrightarrow 0$ as $r \longrightarrow 1$, and we get the result.

**Lemma 4.** *Let $f(z) \in N^+$. Put $f_\zeta(z) = f(\zeta z)$ for $|\zeta| < 1$. Then $\{f_\zeta\}$ is a bounded set in $N^+$.*

**Proof.** Let a neighborhood $V = \{g \in N^+;\ \rho(g, 0) < \eta\}$ of $0$ be given. We can choose $\alpha'$, $0 < \alpha' < 1$, such that $\rho(\alpha' f, 0) < \eta/2$. We write $f_{(\theta)}(z) = f(e^{i\theta} z)$, $f_{r(\theta)}(z) = f_r(e^{i\theta} z) = f(re^{i\theta} z)$. Suppose $r_0$ be sufficiently near to 1 such that $\rho(f, f_r) < \eta/2$ for $r_0 \leq r < 1$. Then

$$\rho(\alpha' f_{(\theta)}, 0) = \rho(\alpha' f, 0) < \eta/2,$$

$$\rho(\alpha' f_{r(\theta)}, \alpha' f_{(\theta)}) = \rho(\alpha' f_r, \alpha' f) \leq \rho(f_r, f) < \eta/2.$$

If $\zeta = re^{i\theta}$, we have $f_\zeta = f_{r(\theta)}$. For $r \geq r_0$ we obtain

$$\rho(\alpha' f_\zeta, 0) = \rho(\alpha' f_{r(\theta)}, 0) \leq \rho(\alpha' f_{r(\theta)}, \alpha' f_{(\theta)}) + \rho(\alpha' f_{(\theta)}, 0)$$

$$= \rho(\alpha' f_r, \alpha' f) + \rho(\alpha' f, 0) < \eta.$$

For $0 \leq r \leq r_0$, we can determine $\alpha''$ so small that $\rho(\alpha'' f_\zeta, 0) = \rho(\alpha'' f_r, 0) < \eta$. This, if $\alpha = \min(\alpha', \alpha'')$, we get $\{\alpha f_\zeta\}_{|\zeta|<1} \subset V$. Q.E.D.

We denote by $(N^+)^*$ the dual of the space $N^+$. Then

**Theorem 3.** *Let $\phi \in (N^+)^*$. Then there is a unique function $g(z) = \Sigma b_n z^n \in A$ such that*

$$\phi(f) = \lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) g(e^{-i\theta})\, d\theta = \sum_{n=0}^{\infty} a_n b_n \tag{4.1}$$

*for $f(z) = \Sigma a_n z^n \in N^+$, where the series on the right converges absolutely. Further, Taylor coefficients $b_n$ of $g(z)$ satisfy*

$$b_n = O(\exp[-c\sqrt{n}]) \quad \text{for a positive constant } c. \tag{4.2}$$

*Conversely, if a function $g(z) = \Sigma b_n z^n$ satisfies (4.2), $g(z)$ obviously belongs to the class $A$ and defines a functional $\phi(f)$ by (4.1), which is linear and continuous on the space $N^+$: $\phi \in (N^+)^*$.*

We recall that $A$ denotes the class of functions holomorphic in $D$ and continuous on $\overline{D}$, defined in Remark 4.

**Proof.** Uniqueness is obvious. Given a functional $\phi \in (N^+)^*$, let $b_k = \phi(z^k)$, $k = 0, 1, \cdots$. Since $\{z^k\}$ is bounded in $N^+$, $\{b_k\}$ is a bounded sequence and hence

$$(4.3) \qquad g(z) = \sum_{k=0}^{\infty} b_k z^k$$

is well defined and holomorphic in $D$. Suppose $f(z) = \Sigma a_k z^k \in N^+$. Because $f(z)$ is the uniform limit of partial sums on each circle $|z| = r$, we have $\phi(f_r) = \lim_{N\to\infty} \phi(\Sigma_{k=0}^{N} a_k r^k z^k) = \Sigma_{k=0}^{\infty} a_k b_k r^k$. Since $f_r \longrightarrow f$ in $N^+$ by Lemma 3, we get

$$(4.4) \qquad \phi(f) = \lim_{r\to 1} \sum a_k b_k r^k.$$

As $\{f_\zeta\}$ is bounded in $N^+$ by Lemma 4, the functional $\phi$ maps $\{f_\zeta\}$ into a bounded set in the complex plane, hence

$$(4.5) \qquad \phi(f_\zeta) = \lim_{N\to\infty} \phi\left(\sum_{k=0}^{N} a_k \zeta^k z^k\right) = \sum_{k=0}^{\infty} a_k b_k \zeta^k = F(\zeta)$$

is a bounded function $|\zeta| < 1$, thus $\{b_k\}$ is a multiplier for $N^+$ into $H^\infty$, therefore $b_k$ must satisfy the condition (4.2) by Theorem 2. Thus $g(z) = \Sigma b_k z^k \in A$ and (4.1) is deduced from (4.4). Moreover, we know that

$$(4.6) \qquad \sum a_k b_k \text{ converges absolutely}$$

because of (4.2) and the fact that $a_k = O(\exp[o(\sqrt{k})])$ [8, Theorem 2].

Now, suppose $g(z) = \Sigma b_k z^k$ satisfy the condition (4.2). Then for fixed $r$, $0 < r < 1$, a functional $\phi_r(f) = \Sigma_{k=0}^{\infty} a_k b_k r^k$, for $f(z) = \Sigma a_k z^k \in N^+$, is defined. It is easy to see that $\phi_r$ is linear and continuous. But for each fixed $f \in N^+$, $\sup_r |\phi_r(f)| < \infty$. Thus by the principle of uniform boundedness [1, p. 52], $\phi_r(f) \longrightarrow 0$ as $\rho(f, 0) \longrightarrow 0$ holds uniformly for $0 < r < 1$, which implies that

$$(4.7) \qquad \phi(f) = \lim_{r\to 1} \phi_r(f) = \lim_{r\to 1} \sum_{k=0}^{\infty} a_k b_k r^k = \sum_{k=0}^{\infty} a_k b_k$$

is continuous (the series on the last member converges absolutely). Thus the functional $\phi$ defined by (4.1), using the given function $g(z)$, belongs to $(N^+)^*$. This completes the proof. Q.E.D.

**Remark 5.** Professor M. Hasumi pointed out that the metric used by Gamelin-Lumer [5, p. 122], [6, p. 122]; $d(f, g) = |||f - g||| + \int_0^{2\pi} |\log^+|f(\theta)| - \log^+|g(\theta)||\, d\theta$, where $|||f - g||| = \inf_{a>0} [a + \text{meas}(\{\theta;\ |f(\theta) - g(\theta)| \geq a\})]$, defines a topology for $\log^+ L = \{f;\ \log^+|f| \in L^1([0, 2\pi])\}$, which is equivalent to the topology defined with our metric (2.6).

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