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INVERSION FORMULAE FOR THE PROBABILITY MEASURES ON BANACH SPACES

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ABSTRACT. Let B be a real separable Banach space, and let μ be a probability measure on $\mathfrak{B}(B)$, the Borel sets of B. The characteristic functional (Fourier transform) ϕ of μ, defined by $\phi(y) = \int_B \exp\{i(y, x)\} d\mu(x)$ for $y \in B^*$ (the topological dual of B), uniquely determines μ.

In order to determine μ on $\mathfrak{B}(B)$, it suffices to obtain the value of $\int_B G(s)\, d\mu(s)$ for every real-valued bounded continuous function G on B. Hence an inversion formula for μ in terms of ϕ is obtained if one can uniquely determine the value of $\int_B G(s)\, d\mu(s)$ for all real-valued bounded continuous functions G on B in terms of ϕ and G. The main efforts of this paper will be to prove such inversion formulae of various types.

For the Orlicz space E_α of real sequences we establish inversion formulae (Main Theorem II) which properly generalize the work of L. Gross and derive as a corollary the extension of the Main Theorem of L. Gross to E_α spaces (Corollary 2.2.12).

In Part I we prove a theorem (Main Theorem I) which is Banach space generalization of the Main Theorem of L. Gross by reinterpreting his necessary and sufficient conditions in terms of convergence of Gaussian measures.

Finally, in Part III we assume our Banach space to have a shrinking Schauder basis to prove inversion formulae (Main Theorem III) which express the measure directly in terms of ϕ and G without the use of extension of ϕ as required in the Main Theorems I and II. Furthermore this can be achieved without using the Lévy Continuity Theorem and we hope that one can use this theorem to obtain a direct proof of the Lévy Continuity Theorem.

0. Introduction. Let $(B, \|\cdot\|_B)$ be a real Banach space with Schauder basis $\{b_n\}$. Let $\mathfrak{B}(B)$ denote the Borel sets of B, that is, the σ-field generated by the open sets. The characteristic functional (Fourier transform) ϕ of a probability measure μ on $\mathfrak{B}(B)$ defined by $\phi(y) = \int_B \exp\{i(y, x)\} d\mu(x)$ for $y \in B^*$ (the topological dual space of B) uniquely determines μ.

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In order to determine μ on $\mathfrak{B}(B)$, it suffices to obtain the value of $\int_B G(s)\,d\mu(s)$ for every real-valued bounded continuous function G on B. Hence an inversion formula for μ in terms of ϕ is obtained if one can uniquely determine the value of $\int_B G(s)\,d\mu(s)$ for all real-valued bounded continuous functions G on B in terms of ϕ and G. The main effort of this paper will be to prove such inversion formulae of various type for different Banach spaces B. The Main Theorems I, II, III give inversion formulae which express $\int_B G(s)\,d\mu(s)$ in terms of ϕ and G.

In case B is a real separable Hilbert space the following Inversion Formula was proved by L. Gross [7, Theorem 4].

Theorem. *Let A be a Hilbert-Schmidt operator with dense range on a real separable Hilbert space H. Let μ be a probability measure on H and $f(t)$, a positive admissible function on $(0, \infty)$. Let $h(t)$ be a positive function on $(0, \infty)$ and denote by ν the measure $n \circ A^{-1}$ where n is the canonical normal distribution on H. Let ϕ be the characteristic functional of μ and denote by C_t the positive square root of $I + t^2 f(t)^2 AA^*$. Let E_n denote expectation with respect to the canonical normal distribution. In order for the inversion formula*

$$
(0.1)\qquad \begin{aligned} &\int_H G(s)\,d\mu(s) \\ &= \lim_{t\to\infty} h(t)(\det C_t)E_n\left\{\phi(ty)^{\sim}\left(\int_H G(f(t)x)\exp[-itf(t)(x, y)]\,d\nu(x)\right)^{\sim}\right\} \end{aligned}
$$

to hold for all real-valued bounded continuous functions G the following two conditions are necessary and sufficient:

$$
(0.2)\qquad f(t)^2 \operatorname{trace}(C_t^{-2}AA^*) \longrightarrow 0 \quad \text{as } t \longrightarrow \infty.
$$

(0.3) *The measures $h(t)\exp[-t^2\|C_t^{-1}s\|^2/2]\,d\mu(s)$ converge weakly to μ as $t \to \infty$.*

Furthermore if (0.2) and (0.3) hold then (0.1) also holds for any bounded measurable function G which is strongly continuous almost everywhere with respect to μ.

The condition (0.3) of L. Gross although valid for Hilbert space seems to depend heavily on the symmetry structure of the space. We reinterpret this condition for a general Banach space in terms of convergence of certain Gaussian measures. In terms of this reinterpretation the theorem can then be extended to a Banach space with Schauder basis as follows. Using the fact that B has a Schauder basis, we can, following ideas of J. Kuelbs [12], imbed B measurably in a real separable Hilbert space H_λ, whose norm is weaker than the Banach norm $\|\cdot\|_B$. We then treat the probability measure μ on B as a probability measure on H_λ. This enables us to get the necessary and sufficient conditions for the inversion of μ regarded as a measure on H_λ using essentially ideas of L. Gross [7]. However this method allows one to obtain such a formula only for G bounded and continuous on H_λ, which is a proper subclass of the G's required. To circumvent the problem we have to use essentially the notion of λ-family introduced by J. Kuelbs and V. Mandrekar [13], which exhibits the detailed structure of the probability measure μ on H_λ which is actually supported on B. In Part I, such an inversion formula is obtained for any Banach space with Schauder basis.

Our initial objective in Part II is to prove a theorem which generalizes Theorem 4 of L. Gross [7] in his form. For this purpose we will need to restrict ourselves to Orlicz space E_α of real sequences since in this case the form of the characteristic functional of a Gaussian measure is known (see [13]). We further assume that the function $\alpha(\cdot)$ associated with E_α possesses a particular property relative to one-dimensional Gaussian measures to obtain an analytic condition for the inversion formulae and we also give a *precise generalization* of the main inversion formulae of L. Gross [7, Theorem 4] to Orlicz spaces of real sequences.

Finally, in the third part we let $(B, \|\cdot\|_B)$ be real Banach space with *shrinking Schauder* basis $\{b_n\}$. Since $\{b_n\}$ is shrinking, the coordinate functionals on B form a basis for B^*, and hence we may consider B^* as a Borel measurable subset of l, the vector space of all sequences of real numbers with topology of coordinatewise convergence. Also we shall let n be the canonical normal distribution on H_λ so that, for each $x \in H_\lambda$, $n(x)$ is a random variable on B^*, and let P_λ be the countably additive cylinder set measure on B^* induced by the above family. Then we shall prove a theorem (Main Theorem III) which gives a class of inversion formulae different from that of the Main Theorem I. In Main Theorem I we have an extension of the characteristic functional to l, whereas in the Main Theorem III we have no extension of the characteristic functional. Furthermore, since $\{b_n\}$ is shrinking, it is possible to prove the theorem without using the Lévy Continuity Theorem and we hope that one might be able to use this theorem to obtain a proof for the Lévy Continuity Theorem.

I. INVERSION FORMULAE OF THE CHARACTERISTIC FUNCTIONAL OF A PROBABILITY MEASURE ON BANACH SPACES WITH A SCHAUDER BASIS

1. In this section we present for the sake of completeness some standard concepts and definitions. For further details the reader is referred to [7], [1] and [16].

1.1.1. **Definition.** (a) Let S be a complete separable metric space and let $\mathfrak{M}$ be the space of positive finite measures defined on the σ-field generated by the open subsets of S. A sequence μ_n of measures in $\mathfrak{M}$ is said to converge weakly to a measure μ in $\mathfrak{M}$ if $\int_S f\,d\mu_n \to \int_S f\,d\mu$ for every bounded continuous function f on S. We will denote this convergence by $\mu_n \xrightarrow{W} \mu$. If $\{\mu_t : t \in (0, \infty)\}$ is a family of measures in $\mathfrak{M}$, then we say $\mu_t \overset{W}{\Rightarrow} \mu$ as $t \to \infty$, if for any sequence $\{t_n\}$ approaching infinity, $\mu_n \overset{W}{\Rightarrow} \mu$.

(b) A sequence μ_n of measures in $\mathfrak{M}$ is said to be conditionally compact (tight), if for every $\epsilon > 0$ there exists a compact set K^ϵ in S such that $\mu_n(K^\epsilon) > 1 - \epsilon$ for all n.

1.1.2. **Definition.** Let $(\Omega, \mathcal{F}, P)$ be a probability space and let X be a random variable on Ω taking values in S. Then X is said to be distributed as ν if $\nu = P \circ X^{-1}$. A family of S-valued random variables $\{X_t: t \in (0, \infty)\}$ is said to converge in distribution to an S-valued random variable X as $t \to \infty$ if $P \circ X_t^{-1} \overset{W}{\Rightarrow} P \circ X^{-1}$ as $t \to \infty$. We will denote this convergence by $X_t \overset{\mathfrak{D}}{\to} X$ as $t \to \infty$.

The following definitions are due to I. Segal and are taken here from [7].

1.1.3. **Definition.** A weak distribution on a topological linear space L is an equivalence class of linear mappings F on the (topological) dual space L^* to real-valued random variables on a probability space (depending on F) where two such mappings F_1 and F_2 are equivalent if for every finite set of vectors $y_1, \cdots, y_k$ in L^* the sets $\{F_i(y_1), \cdots, F_i(y_k)\}$ have the same distribution in Euclidean space R^k for $i = 1$ or 2.

1.1.4. **Definition.** (a) A weak distribution m on a Banach space B is said to be continuous if, for any sequence(3) $\{y_k\} \subseteq B^*$, $\|y_k\|_{B^*} \to 0$ implies $m(y_k)$ converges to zero in probability.

(b) A weak distribution m on a topological linear space L is a measure if there exists a probability measure μ defined on the σ-field $\mathcal{S}$ generated by weakly open subsets of L such that the identity map on L^* is a representative of m.

1.1.5. **Definition.** If m is a weak distribution on a locally convex topological linear space L and A is a continuous linear operator on L with adjoint A^*, then the weak distribution $y \to m(A^*y)$ will be denoted by $m \circ A^{-1}$.

1.1.6. **Definition.** A measure μ on a locally convex topological linear space L is defined to be Gaussian if, for every continuous linear functional T on L, $T(x)$ has a Gaussian distribution. μ is called Gaussian with mean zero if, in addition, $T(x)$ has mean zero for every T.

1.1.7. **Remarks.** (a) One special example of a weak distribution on a real separable Hilbert space H is the canonical normal distribution (with variance parameter one). This weak distribution is that unique weak distribution which assigns to each vector y in H^* a normally distributed random variable with mean zero and variance $\|y\|^2$. It follows from the preceding property that the canonical normal distribution carries orthogonal vectors into independent random variables [7, p. 4]. It is known that some of the theory of integration with respect to a

(3) For $y \in B^*$, $\|y\|_{B^*} = \operatorname{Sup}_{\|x\|_B \le 1} |y(x)|$ (see, e.g., [18, p. 160]).

measure can also be carried out with respect to a weak distribution. For details we refer the reader to [9] and the bibliography given there.

(b) By Corollary 3.2 of [7], if m is a continuous weak distribution on a real separable Hilbert space H and A is a Hilbert-Schmidt operator, then $m \circ A^{-1}$ is a measure on H. Hence $n \circ A^{-1}$ is a measure on H if n is the canonical normal distribution on H, since in this case n is clearly a continuous weak distribution on H. We will use the same notation, namely, $n \circ A^{-1}$ for the weak distribution $n \circ A^{-1}$ and its corresponding probability measure μ (see Definition 1.1.4).

1.1.8. **Definition.** The characteristic functional (Fourier transform) of a probability measure μ on the Borel subsets of a linear topological space L is the function $\phi(\cdot)$ on L^* (the topological dual of L) given by

$$\phi(y) = \int_L \exp\{i(y, x)\} d\mu(x) \quad \text{for each } y \in L^*.$$

We shall also need the following definition from [20, p. 190].

1.1.9. **Definition.** An operator from a real separable Hilbert space H into H, which is linear, symmetric, nonnegative, compact, and has finite trace is called an S-operator.

If T is an S-operator on H, then it is well known that T has the representation

$$Tx = \sum_{n=1}^{\infty} \lambda_n (x, e_n) e_n \tag{1.1.10}$$

where $\{e_n\}$ is some orthonormal subset of H, $\lambda_n \geq 0$, and $\sum_{n=1}^{\infty} \lambda_n < \infty$.

1.1.11. **Definition of τ-topology.** Let Σ be the class of all S-operators. The τ-topology on H is the smallest locally convex topology generated by the family of seminorms $p_T(x) = (Tx, x)^{1/2}$ on H as T varies through Σ [18, p. 172].

1.1.12. **Definition.** A tame function on a real Hilbert space H is a function of the form $f(x) = \Phi(Px)$ where P is a finite-dimensional projection on H and Φ is a Baire function on the finite-dimensional space PH.

For such a function we denote by $f^\sim$ the random variable corresponding to f as is done in [7]. It is clear that a function f on H is a tame function if and only if there is a finite-dimensional projection P on H such that $f(x) = f(Px)$ for all x and such that f restricted to the finite-dimensional space PH is a Baire function. Then we note that its expectation with respect to the canonical normal distribution (with variance parameter one) is given by

$$E_n(f^\sim) = (2\pi)^{-k/2} \int_{PH} f(x) \exp[-\|x\|^2/2]\, dx$$

when the integral exists where k is the dimension of PH and dx is Lebesgue measure on PH. (For further details, see [7].)

1.1.13. **Remark.** Let H be a real separable Hilbert space and let $\{P_j\}$ be any sequence of finite-dimensional projections converging strongly to the identity operator. If a complex-valued function f on H is uniformly continuous in the topology τ then $\lim_{j\to\infty}$ in Prob$(f \circ P_j)^\sim$ exists with respect to the canonical normal distribution and equals $f^\sim$ [7, Theorem, p. 5]. Now if, for each j, $E_n[(f \circ P_j)^\sim]$ exists and if $\lim_{j\to\infty} E_n[(f \circ P_j)^\sim]$ exists then, following L. Gross [8], we say that f is integrable with respect to the canonical normal distribution. That is, for a uniformly τ-continuous function f on H which is integrable we denote

$$E_n(f^\sim) = \lim_{j\to\infty} E[(f \circ P_j)^\sim]. \tag{1.1.14}$$

2. We shall study here Banach spaces with Schauder basis. We need some preliminary results for measures on such spaces. We start with the following definition from [3, p. 67].

1.2.1. **Definition.** Let B be a Banach space. A Schauder basis $\{b_i\}$ in B is a sequence of elements of B such that for each x in B there is a unique sequence of real numbers $\{a_i\}$, depending on x, such that

$$\lim_{n\to\infty} \left\| x - \sum_{i=1}^{n} a_i b_i \right\|_B = 0;$$

the series $\sum_{i=1}^\infty a_i b_i$ is called the expansion of x in the basis $\{b_i\}$, and the coefficient $a_i = \beta_i(x)$ is the ith coordinate of x in the basis $\{b_i\}$.

Throughout this part B will denote a real Banach space with Schauder basis $\{b_n\}$ such that without loss of generality $\|b_n\|_B = 1$ [3, p. 68]. We will write the expansion of x as $\sum_{n=1}^\infty \beta_n(x) b_n$ and this emphasizes that the coefficients generate coordinate functionals on B. It is clear that these coordinate functionals are linear and it is well known that they are continuous as well [3, p. 68]. Further it is possible to assume without loss of generality [3, Theorem 1, p. 67] that

$$\|x\|_B = \operatorname*{Sup}_k \left\| \sum_{n=1}^{k} \beta_n(x) b_n \right\|_B . \tag{1.2.2}$$

1.2.3. **Definition.** Let B be a Banach space with Schauder basis. Then we say that $\lambda \in l_\infty^+$ is *adequate* for B if

(i) for all $x \in B$, $\sum_{i=1}^\infty \lambda_i \beta_i^2(x)$ is finite;

(ii) for all x there exists a nonnegative constant C_λ such that

$$\sum_{i=1}^{\infty} \lambda_i \beta_i^2(x) \le C_\lambda \|x\|_B^2.$$

Here l_∞ denotes the space of bounded sequences. We note at the outset that if B is a Hilbert space then $\lambda = (1, 1, 1, \cdots)$ is adequate and then $C_\lambda = 1$.

1.2.4. **Remarks.** (a) For every Banach space with a Schauder basis, $\lambda \in l_1^+$ is adequate, since $\sup_n |\beta_n(x)| \le K\|x\|_B$ [3, p. 68] and $C_\lambda = \Sigma_{i=1}^\infty \lambda_i$.

(b) If $B = l_p$ (the set of sequences $x = (x_1, x_2, \cdots)$, with $\Sigma|x_i|^p < \infty$) ($p \ge 2$) then $\lambda \in l^*_{p/2}$ is adequate with $C_\lambda = \|\lambda\|_{l^*_{p/2}}$.

If $\lambda \in l_\infty^+$ is adequate for B then for that λ we define the inner product

$$(x, y) = \sum_{i=1}^{\infty} \lambda_i \beta_i(x)\, \beta_i(y) \qquad \text{for all } x, y \text{ in } B. \tag{1.2.5}$$

Let $\|x\|_\lambda = (x, x)^{1/2}$ then $\|x\|_\lambda \le C_\lambda \|x\|_B$. Let H_λ be the Hilbert space obtained by the completion of B under the norm $\| \ \|_\lambda$.

Upon replacing y by b_k in (1.2.5) we get $(x, b_k) = \lambda_k \beta_k(x)$. Since $\lambda_k > 0$, $\beta_k(x)$ is uniformly continuous in x in $\|\cdot\|_\lambda$-topology on B, and since B is dense in H_λ, $\beta_k(\cdot)$ can be extended uniquely to a continuous linear functional $\hat{\beta}_k(\cdot)$ on H_λ. Furthermore it can easily be seen that for $x, y \in H_\lambda$,

$$(x, y) = \sum_{n=1}^{\infty} \lambda_n \hat{\beta}_n(x) \hat{\beta}_n(y).$$

From (1.2.2) and the fact that β_n is a $\|\cdot\|_\lambda$-continuous function on B, it follows that $\|x\|_B$ is a measurable function in $\|\cdot\|_\lambda$-topology, and hence B is a $\|\cdot\|_\lambda$-measurable subset of H_λ. Therefore if μ is a measure on B, it can be regarded as a measure on H_λ via $\mu(A) = \mu(A \cap B)$ for all $A \in \mathcal{B}(H_\lambda)$.

Let ν be a Gaussian measure with mean zero on $(B, \mathcal{B}(B))$. Then by argument similar to Lemma 2.2 of [12], ν is a Gaussian measure on H_λ, and therefore there exists a nonnegative, symmetric trace class operator (that is, an S-operator) T_ν on H_λ such that

$$(T_\nu x, x)_{H_\lambda} = \int_{H_\lambda} (y, x)^2\, d\nu(y)$$

for $x \in H_\lambda$, and that ν is uniquely determined on H_λ by the operator T_ν. These results are well known and appear, for example, in [20]. Furthermore, T_ν has the representation (1.1.10), that is,

$$T_\nu(\cdot) = \sum_{k=1}^{\infty} \eta_k(\cdot, g_k) g_k \tag{1.2.6}$$

on H_λ where $\{g_k\}$ is an orthonormal sequence in H_λ and $\eta_k \ge 0$, $\Sigma_k \eta_k < \infty$.

1.2.7. **Remarks.** (a) Since B is separable, $\mathcal{B}(B)$ is the same as σ-field generated by the weakly open sets and the latter one is the same as σ-field generated by the field of the cylinder sets.

(b) Since T_ν is an S-operator, $T_\nu^{1/2}$ is a Hilbert-Schmidt operator on H_λ, and hence $n \circ T_\nu^{-1/2}$ is a measure on H_λ. But T_ν uniquely determines ν, so by

Definition 1.1.4, ν is the probability measure on H_ν corresponding to the weak distribution $n \circ T_\nu^{-1/2}$.

(c) Since $n \circ T_\nu^{-1/2}(B) = 1$ and $\| \|_B$ is $\| \|_\lambda$ measurable, we get that $n \circ T_\nu^{-1/2}$ is a countably additive probability measure on B.

The following ideas are given in [13].

1.2.8. **Definition.** If $\lambda \in l_\infty^+$ and $\{\mu_t: t \in A\}$ is a family of probability measures on B such that

$$\mu_t\left\{x \in B: \sum_{n=1}^{\infty} \lambda_n \beta_n^2(x) < \infty\right\} = 1$$

for each $t \in A$ we say λ is sufficient for the family $\{\mu_t: t \in A\}$.

Every λ adequate for B in l_∞^+ is sufficient for any family of probability measures on B.

1.2.9. **Definition.** A family of probability measures $\{\mu_t: t \in A\}$ on B is a λ-family for some $\lambda \in l_\infty^+$ if λ is sufficient for $\{\mu_t: t \in A\}$ and for every $\epsilon, \delta > 0$ there is a sequence $\{\epsilon_N\}$ such that

$$\mu_t\left\{x \in B: \sum_{n=N+1}^{\infty} \lambda_n \beta_n^2(x) < \delta\right\} > 1 - \epsilon$$

implies

$$\mu_t\left\{x \in B: \left\| \sum_{n=N+1}^{\infty} \beta_n(x) b_n \right\|_B < k(\delta)\right\} > 1 - (\epsilon + \epsilon_N) \quad \text{for all } t$$

where $\lim_N \epsilon_N = 0$ and k is a strictly increasing continuous function on $[0, \infty)$ with $k(0) = 0$.

A family of probability measures $\{\mu_t: t \in (0, \infty)\}$ on B is said to be a λ-family as $t \to \infty$, if for any sequence $\{t_n\}$ approaching infinity, the family $\{\mu_{t_n}: n = 1, 2, \cdots\}$ is a λ-family.

It is quite clear that any family of probability measures on a real separable Hilbert space is a λ-family with $\lambda = (1, 1, \cdots)$ and $k(\delta) = \delta^{1/2}$.

For $x \in B$, $N = 1, 2, \cdots$, we define

$$P_N x = \sum_{k=1}^{N} \beta_k(x) b_k \qquad Q_N x = \sum_{k=N+1}^{\infty} \beta_k(x) b_k$$

and for $y \in B^*$, $N = 1, 2, \cdots$, we define

$$P_N y(\cdot) = \sum_{k=1}^{N} \beta_k(\cdot) y(b_k).$$

1.2.10. **Definition.** If $\{\mu_t: t \in A\}$ is a family of probability measures on B and $\lambda \in l_\infty^+$ then we say the family of c.f.'s $\{\phi_{\mu_t}(\cdot): t \in A\}$ is λ-continuous at zero in B^* if

(i) for every integer N the family $\{\phi_{\mu_t}(\cdot): t \in A\}$ are equicontinuous at zero in $P_N(B^*)$, and

(ii) $\lim_N \operatorname{Sup}_t \lim_k J_{N,k}[1 - \operatorname{Re} \phi_{\mu_t}(\cdot)] = 0$ where

$$J_{N,k}[\cdots] = \int_{(P_{N+k}-P_N)B^*} [\cdots]\xi_\lambda(N, k, dy)$$

and $\xi_\lambda(N, k, \cdot)$ is the Gaussian product measure on $(P_{N+k} - P_N)B^*$ with each coordinate $y(b_i)$, $N + 1 \leq i \leq N + k$, Gaussian with mean zero and variance λ_i.

1.2.11. Lemma. *Let B be a Banach space with a Schauder basis and suppose*

(i) $\lambda \in l_\infty^+$, *and is adequate for* B;

(ii) $\{\mu_t: t \in (0, \infty)\}$ *is a λ-family as $t \to \infty$ of probability measures on* B;

(iii) $\mu_t \overset{W}{\Rightarrow} \mu$ *on H_λ as $t \to \infty$ where μ is a probability measure on B.*

Then $\mu_t \overset{W}{\Rightarrow} \mu$ on B as $t \to \infty$.

Proof. Let $\{t_n\}$ be a sequence in $(0, \infty)$ such that $t_n \to \infty$ as $n \to \infty$. Then (iii) implies that $\mu_{t_n} \overset{W}{\Rightarrow} \mu$ on H_λ, that is, $\{\mu_{t_n}: n = 1, 2, \cdots\}$ is compact on H_λ. Since $\{\mu_{t_n}: n = 1, 2, \cdots\}$ is compact on H_λ and P_N is continuous it follows that $\{\mu_{t_n} \circ P_N^{-1}: n = 1, 2, \cdots\}$ is compact on H_λ for all $N = 1, 2, \cdots$, which is equivalent to saying $\{\mu_{t_n} \circ P_N^{-1}: n = 1, 2, \cdots\}$ is compact on B for all $N = 1, 2, \cdots$. Thus the c.f.'s $\{\phi_{\mu_{t_n} \circ P_N^{-1}}(\cdot): n = 1, 2, \cdots\}$ are equicontinuous at zero on $P_N(B^*)$ [15, Corollary 2, p. 193] and since

$$\phi_{\mu_{t_n} \circ P_N^{-1}}(y) = \phi_{\mu_{t_n}}(P_N y)$$

the equicontinuity at zero on $P_N(B^*)$ of $\{\phi_{\mu_{t_n}}(\cdot): n = 1, 2, \cdots\}$ follows. Hence condition (i) in Definition 1.2.10 is satisfied. Now let ϵ be an arbitrary positive number and choose a compact set K_ϵ in H_λ such that $\mu_{t_n}(K_\epsilon) > 1 - \epsilon/2$ for $n = 1, 2, \cdots$. With R^k denoting k-dimensional Euclidean space, we get

$$\begin{aligned} J_{N,k}[1 - \text{Real } \phi_{t_n}(y)] &= \int_{R^k}\int_{H_\lambda} (1 - \cos(y, x))\mu_{t_n}(dx)\xi_\lambda(N, k, dy) \\ &\leq \int_{R^k}\int_{K_\epsilon} (1 - \cos(y, x))\mu_{t_n}(dx)\xi_\lambda(N, k, dy) + \frac{\epsilon}{2} \\ &\leq \frac{1}{2} \int_{R^k}\int_{K_\epsilon} (y, x)^2\mu_{t_N}(dx)\xi_\lambda(N, k, dy) + \frac{\epsilon}{2}. \end{aligned}$$

Also, Fubini's Theorem and the evaluation of the integral with respect to $\xi_\lambda(N, k, \cdot)$ give

$$\int_{R^k}\int_{K_\epsilon} (y, x)^2\mu_{t_n}(dx) = \int_{K_\epsilon} \sum_{N+1}^{N+k} \lambda_i x_i^2 \mu_{t_n}(dx).$$

Thus we have

$$\lim_k J_{N,k}[1 - \text{Real } \phi_{t_n}(y)] \leq \frac{1}{2} \sum_{N+1}^{\infty} \int_{K_\epsilon} \lambda_i x_i^2 \mu_{t_n}(dx) + \frac{\epsilon}{2}.$$

Since K_ϵ is compact in the sequence space H_λ, for each $\epsilon > 0$ there exists an $N \geq N_0$ such that $\sup_{x \in K_\epsilon} (\Sigma_N^\infty \lambda_i x_i^2) < \epsilon$. This and the above inequality imply that given $\epsilon > 0$ there exists N_0 such that for $N \geq N_0$

$$\sup_n \lim_k J_{N,k}[1 - \text{Real } \phi_{t_n}(y)] < \epsilon.$$

Thus condition (ii) of Definition 1.2.10 is satisfied, and hence $\{\phi_{\mu_{t_n}}(\cdot): n = 1, 2, \cdots\}$ is λ-continuous for $\lambda \in l_\infty^+$ adequate for B. This together with the assumption that $\{\mu_t: t \in (0, \infty)\}$ is a λ-family as $t \to \infty$ imply [13, Lemmas 2.3, 2.4] that $\{\mu_{t_n}: n = 1, 2, \cdots\}$ is conditionally compact on B. Hence $\mu_{t_n} \overset{W}{\Rightarrow} \mu$ on B since $\{\mu_{t_n}: n = 1, 2, \cdots\}$ is conditionally compact on B and $\mu_{t_n} \circ P_N^{-1} \overset{W}{\Rightarrow} \mu \circ P_N^{-1}$ for all $N = 1, 2, \cdots$ [1, p. 35]. Since this is true for any sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ we get $\mu_t \overset{W}{\Rightarrow} \mu$ on B as $t \to \infty$.

3. Throughout this section we assume $\lambda \in l_\infty^+$ and adequate for B, a Banach space with a Schauder basis. Suppose l is the space of all real sequences with the topology of coordinatewise convergence and $P_\lambda(\cdot)$ is the product probability measure on l such that the ith coordinate is Gaussian with mean zero and variance λ_i where $\lambda = \{\lambda_i\} \in l_\infty^+$ as above. If μ is a probability measure on B then λ adequate for B is sufficient for μ. For $y \in l$ we define the "stochastic linear functional" on B in the following manner:

$$(y, x)^{\approx} = \lim_N \sum_{i=1}^{N} \hat{\beta}_i(x) y_i = \lim_N \sum_{i=1}^{N} \beta_i(x) y_i.$$

Then as in [13, p. 119],

1.3.1. **Lemma.** *The stochastic linear functional.*

$$(y, x)^{\approx} = \lim_N \sum_{i=1}^{N} \hat{\beta}_i(x) y_i \tag{1.3.2}$$

is Borel measureable on $l \times B$ *and if* $F = \{(y, x): (y, x)^{\approx}$ *exists and is finite*$\}$, *then* $P_\lambda \times \mu(F) = 1$.

1.3.3. **Definition.** If μ is a probability measure on the Borel subsets of B with the c.f. ϕ, we define the *extended characteristic functional* $\phi^{\approx}(\cdot)$ on l as follows:

$$\phi^{\approx}(y) = \int_B \exp\{i(y, x)^{\approx}\} d\mu(x) \qquad (y \in l)$$

following [13, p. 119].

Then $\overset{\approx}{\phi}(\cdot)$ is a Borel measurable function on l which is defined almost everywhere with respect to the measure P_λ. Furthermore, since each $y \in B^*$ generates the unique sequence of real numbers $\overset{\approx}{y} = \{y(b_1), \cdots, y(b_k), \cdots\}$ we may consider B^* as a linear subset of l under the J. Kampé de Fériet map [11, pp. 123–127], and hence the terminology extended c.f. since, for $y \in B^*$ and $x \in B$, $\overset{\approx}{(y, x)} = (y, x)$ which implies that $\phi(y) = \overset{\approx}{\phi}(\overset{\approx}{y})$.

1.3.4. **Remark.** Let μ be a probability measure on B with c.f. ϕ; then as was shown earlier, μ can be regarded as a probability measure on H_λ. Let ψ be the c.f. of μ when μ is regarded as a measure on H_λ; then

$$\psi(y) = \int_{H_\lambda} \exp\{i(y, x)\} d\mu(x) \quad \text{for all } y \in H_\lambda^* \subseteq B^* \subseteq l.$$

By Theorem 1 of [7, p. 7], ψ is uniformly τ-continuous on H_λ^*, and hence by Theorem of [7, p. 5], the random variable $\tilde{\psi}$ (that is, the Gross extension of ψ) is well defined with respect to the canonical normal distribution n on H_λ. Finally from (1.3.2), the fact that $\mu(B) = 1$, and Lemma 4.3 of [14] modified for this situation, it follows that $\widetilde{\psi(y)} = \overset{\approx}{\phi}(y)$ almost everywhere with respect to P_λ.

4. Let ν be a Gaussian measure on $(B, \mathfrak{B}(B))$ with mean zero and $\lambda \in l_\infty^+$ be adequate for B. Let T_ν be as described in Remark 1.2.7. Let $f(t)$ be a real-valued function defined on $(0, \infty)$ and denote by C_t the positive square root of $I + t^2 f(t)^2 T_\nu$ on H_λ. Let μ be a probability measure on $(B, \mathfrak{B}(B))$, and define

$$\mu_t(A) = \frac{1}{a_t} \int_A h(t) \exp[-t^2 \|C_t^{-1} s\|_\lambda^2 / 2] \, d\mu(s), \qquad A \in \mathfrak{B}(B),$$

where $h(t)$ is a positive function on $(0, \infty)$, and

$$a_t = \int_B h(t) \exp[-t^2 \|C_t^{-1} s\|_\lambda^2 / 2] \, d\mu(s). \tag{1.4.1}$$

Following Gross [7], a real-valued function $f(t)$ defined on $(0, \infty)$ will be called *admissible* if $t f(t) \to \infty$ as $t \to \infty$. We are now ready to state and prove the Main Theorem I.

1.4.2. **Main Theorem I.** *Let*

(i) *B be a Banach space with Schauder basis $\{b_i\}$ and $\lambda \in l_\infty^+$ be adequate for B (fixed);*

(ii) *μ be a probability measure on $(B, \mathfrak{B}(B))$ with the c.f. ϕ;*

(iii) *$f(t)$ be a positive admissible function on $(0, \infty)$, and $h(t)$ be a positive function on $(0, \infty)$;*

(iv) *X be a B-valued random variable distributed as ν where ν is a Gaussian measure on $(B, \mathfrak{B}(B))$ with mean zero and the property that T_ν is positive-definite;*

(v) *Y_t and Y be B-valued random variables distributed as μ_t and μ respectively;*

(vi) *E_{P_λ} denote the integral with respect to P_λ on l.*

Then for all real-valued, bounded, $\|\cdot\|_B$-continuous functions G on B the following are equivalent:

$$(1.4.3)\quad \begin{cases} \text{(a)}\quad \int_B G(s)\,d\mu(s) = \lim_{t\to\infty} h(t)(\det C_t)E_{P_\lambda} \\ \qquad\qquad \cdot \left\{\phi^{\approx}(ty)\left(\int_B G(f(t)x)\exp[-itf(t)(y,x)^{\approx}]\,d\nu(x)\right)\right\} \\ \text{(b)}\quad \{\mu_t : t\in(0,\infty)\} \text{ is a } \lambda\text{-family as } t\to\infty, \end{cases}$$

$$(1.4.4)\quad \begin{cases} \text{(a)}\quad f(t)C_t^{-1}X \xrightarrow{\mathfrak{D}} 0 \quad \text{as } t\to\infty \\ \text{(b)}\quad Y_t \xrightarrow{\mathfrak{D}} Y \qquad\quad \text{as } t\to\infty. \end{cases}$$

Proof. We shall first show (1.4.3) implies (1.4.4). Let G be a real-valued, bounded, $\|\cdot\|_B$-continuous function on B, and observe that

$$E_{P_\lambda}\left\{\phi^{\approx}(ty)\left(\int_B G(f(t)x)\exp[-itf(t)(y,x)^{\approx}]\,d\nu(x)\right)\right\}$$
$$= \int_l\left(\int_B e^{it(y,s)^{\approx}}\,d\mu(s)\right)\left(\int_B G(f(t)x)e^{-itf(t)(y,x)^{\approx}}\,d\nu(x)\right)\Bigg\}\,dP_\lambda(y).$$

We note that $e^{it(y,s)^{\approx}}e^{-itf(t)(y,x)^{\approx}}G(f(t)x)$ is jointly measurable since it is a product of a jointly measurable function of s, x and y with a $\|\cdot\|_B$-continuous function of x namely $G(f(t)x)$. Since it is bounded and all the measures are probability measures by Fubini's Theorem [19, p. 140], the Dominated Convergence Theorem, and the definition of P_λ we have

$$E_{P_\lambda}\left\{\phi^{\approx}(ty)\left(\int_B G(f(t)x)\exp[-itf(t)(y,x)^{\approx}]\,d\nu(x)\right)\right\}$$
$$= \int_B\int_B G(f(t)x)\exp[-t^2\|s-f(t)x\|_\lambda^2/2]\,d\nu(x)\,d\mu(s).$$

Now from 1.2.7(c) and the fact that $\mu(B)=1$, it follows that the above is equal to

$$\int_{H_\lambda}\int_{H_\lambda} G(f(t)x)\exp[-t^2\|s-f(t)x\|_\lambda^2/2]\,dn\circ T_\nu^{-1/2}(x)\,d\mu(s).$$

We note that G is $\|\cdot\|_\lambda$-measurable (that is, measurable in the $\|\cdot\|_\lambda$-topology) since the norm $\|x\|_B$ is $\|\cdot\|_\lambda$-measurable and G is $\|\cdot\|_B$-continuous; and $T_\nu^{1/2}$ is a Hilbert-Schmidt operator on H_λ. Hence by Lemma 4.1 of [7] we have that the above equals

$$\frac{1}{(\det C_t)}\int_{H_\lambda}\int_{H_\lambda} G(f(t)C_t^{-1}x+s-C_t^{-2}s)\,dn\circ T_\nu^{-1/2}(x)h(t)\exp[-t^2\|C_t^{-1}s\|_\lambda^2/2]\,d\mu(s).$$

Again from 1.2.7 (c) and the fact that $\mu(B)=1$ we get that this is

$$\frac{1}{(\det C_t)}\int_B\int_B G(f(t)C_t^{-1}x+s-C_t^{-2}s)\,d\nu(x)h(t)\exp[-t^2\|C_t^{-1}s\|_\lambda^2/2]\,d\mu(s).$$

We may now start with the assumption that for all real-valued, bounded, $\|\cdot\|_B$-continuous functions G on B we have

$$(1.4.5)\quad \begin{cases} \text{(a)}\quad \int_B G(s)\,d\mu(s) = \lim_{t\to\infty}\int_B\left[\int_B G(f(t)C_t^{-1}x + s - C_t^{-2}s)\,d\nu(x)\right] \\ \qquad\qquad\qquad\qquad\qquad \cdot\, h(t)\exp[-t^2\|C_t^{-1}s\|_\lambda^2/2]\,d\mu(s) \\ \text{(b)}\quad \{\mu_t\colon t\in(0,\infty)\}\ \text{is a } \lambda\text{-family as } t\to\infty. \end{cases}$$

Putting $G\equiv 1$ in (1.4.5)(a) we get

$$1 = \lim_{t\to\infty}\int_B h(t)\exp[-t^2\|C_t^{-1}s\|_\lambda^2/2]\,d\mu(s).$$

Hence from definition of a_t we have (1.4.6).

$$(1.4.6)\qquad 1 = \lim_{t\to\infty} a_t.$$

Using (1.4.6) we obtain

$$(1.4.7)\qquad \int_B G(s)\,d\mu(s) = \lim_{t\to\infty}\int_B\int_B G(f(t)C_t^{-1}x + (I - C_t^{-2})s)\,d\nu(x)\,d\mu_t(s)$$

for all real-valued, bounded, $\|\cdot\|_B$-continuous functions G on B. From (1.4.7), it follows that

$$(1.4.8)\qquad \nu\circ(f(t)C_t^{-1})^{-1} * \mu_t\circ(I - C_t^{-2})^{-1} \overset{W}{\Longrightarrow} \mu \quad \text{on } B \quad \text{as } t\to\infty.$$

Since G is bounded on B and ν is a probability measure, the measure $G(f(t)x)\,d\nu(x)$ is a measure of bounded variation on B, and hence a measure of bounded variation on H_λ. Therefore by Theorem 1 of [7, p. 7] the Fourier transform of $G(f(t)x)\,d\nu(x)$ is uniformly τ-continuous, and hence the Gross extension of its Fourier transform is well defined [7, Theorem, p. 5]. Similarly the Gross extension of the Fourier transform (c.f.) $\psi(\cdot)$ of μ when regarding μ as a measure on $(H_\lambda, \mathcal{B}(H_\lambda))$ is well defined. Now from Remarks 1.3.4 and 1.2.7(c), it follows that

$$(1.4.9)\qquad \begin{aligned} &E_{P_\lambda}\left\{\phi^{\approx}(ty)\left(\int_B G(f(t)x)\exp[-itf(t)(y,x)^{\approx}]\,d\nu(x)\right)\right\} \\ &= E_{P_\lambda}\left\{\psi(ty)^{\sim}\left(\int_{H_\lambda} G(f(t)x)\exp[-itf(t)(y,x)]\,dn\circ T_\nu^{-1/2}(x)\right)^{\sim}\right\}. \end{aligned}$$

Let $\{P_j\}$ be a sequence of finite-dimensional projections on H_λ converging strongly (that is, in $\|\cdot\|_\lambda$-topology) to the identity operator. Then using the fact that P_j's are continuous together with the Lebesgue Dominated Convergence Theorem we obtain

$$\begin{aligned} &E_{P_\lambda}\left\{\psi(ty)^{\sim}\left(\int_{H_\lambda} G(f(t)x)\exp[-itf(t)(y,x)]\,dn\circ T_\nu^{-1/2}(x)\right)^{\sim}\right\} \\ &= \lim_{j\to\infty} E_{P_\lambda}\left\{\psi(tP_jy)^{\sim}\left(\int_{H_\lambda} G(f(t)x)\exp[-itf(t)(P_jy,x)]\,dn\circ T_\nu^{-1/2}(x)\right)^{\sim}\right\}. \end{aligned}$$

Now using the fact that the integral of a tame function with respect to the product measure P_λ is the same as its integral with respect to the canonical normal distribution n on H_λ, we get

$$E_{P_\lambda}\left\{\psi(ty)^\sim\left(\int_{H_\lambda} G(f(t)x)\exp[-itf(t)(y, x)]\,dn \circ T_\nu^{-1/2}(x)\right)^\sim\right\}$$
$$= \lim_{j\to\infty} E_n\left\{\psi(tP_jy)^\sim\left(\int_{H_\lambda} G(f(t)x)\exp[-itf(t)(P_jy, x)]\,dn \circ T_\nu^{-1/2}(x)\right)^\sim\right\}$$

where by E_n we mean integral with respect to the canonical normal distribution n on H_λ.

From Remark 1.1.13, it follows that

$$\lim_{j\to\infty} E_n\left\{\psi(tP_jy)^\sim\left(\int_{H_\lambda} G(f(t)x)\exp[-itf(t)(P_jy, x)]\,dn \circ T_\nu^{-1/2}(x)\right)^\sim\right\}$$
$$= E_n\left\{\psi(ty)^\sim\left(\int_{H_\lambda} G(f(t)x)\exp[-itf(t)(y, x)]\,dn \circ T_\nu^{-1/2}(x)\right)^\sim\right\}.$$

Hence from (1.4.9) we get

$$(1.4.10)\qquad \begin{aligned} &E_{P_\lambda}\left\{\phi^{\approx}(ty)\left(\int_B G(g(t)x)\exp[-itf(t)(y, x)^{\approx}]\,d\nu(x)\right)\right\} \\ &= E_n\left\{\psi(ty)^\sim\left(\int_{H_\lambda} G(f(t)x)\exp[-itf(t)(y, x)]\,dn \circ T_\nu^{-1/2}(x)\right)^\sim\right\}. \end{aligned}$$

Now from (1.4.3)(a) and the fact that $\|\cdot\|_\lambda$-topology is weaker than $\|\cdot\|_B$-topology on B we obtain

$$\int_{H_\lambda} G(s)\,d\mu(s)$$
$$= \lim_{t\to\infty} b(t)(\det C_t)E_n\left\{\psi(ty)^\sim\left(\int_{H_\lambda} G(f(t)x)\exp[-itf(t)(y, x)]\,dn \circ T_\nu^{-1/2}(x)\right)^\sim\right\}$$

for all real-valued, bounded, $\|\cdot\|_\lambda$-continuous functions G on H_λ. Therefore by Theorem 4 of [7] we have

$$(1.4.11)\qquad \begin{cases} \text{(a)} & f(t)^2 \operatorname{trace}(C_t^{-2}T_\nu) \to 0 \text{ as } t \to \infty; \\ \text{(b)} & \text{the measures } b(t)\exp[-t^2\|C_t^{-1}s\|_\lambda^2/2]\,d\mu(s) \text{ converge weakly} \\ & \text{to } \mu \text{ on } H_\lambda \text{ as } t \to \infty. \end{cases}$$

Now (1.4.11)(a) implies [7, Corollary 3.4] that

$$(1.4.12)\qquad f(t)C_t^{-1}X \xrightarrow{\mathfrak{D}} 0 \quad \text{on } H_\lambda \quad \text{as } t \to \infty,$$

and (1.4.11)(b) together with (1.4.6) imply (Definition 1.1.2) that

$$(1.4.13)\qquad Y_t \xrightarrow{\mathfrak{D}} Y \quad \text{on } H_\lambda \quad \text{as } t \to \infty.$$

By the assumption, $\{\mu_t: t \in (0, \infty)\}$ is a λ-family as $t \to \infty$. This, together with (1.4.13), satisfies the hypotheses of Lemma 1.2.11, and hence the conclusion of the lemma which is $Y_t \xrightarrow{\mathfrak{D}} Y$ on B as $t \to \infty$ gives us (1.4.4)(b).

To get (1.4.4)(a) we note that, it is easy to verify using the facts that T_ν is positive-definite (and therefore T_ν has positive eigenvalues) and $tf(t) \to \infty$ as $t \to \infty$, that C_t^{-2} converges strongly to the zero operator as $t \to \infty$. One need only express $\|C_t^{-2}x\|_\lambda^2$ in terms of an orthonormal basis in H_λ which diagonalizes T_ν. Hence $I - C_t^{-2}$ converges strongly to I, and clearly

$$\mu\{x \in B: (I - C_t^{-2})x_n \nrightarrow x \text{ when } x_n \to x\} = 0. \tag{1.4.14}$$

From (1.4.14) and (1.4.4)(b), it follows [1, Theorem 5.5, p. 34] that

$$(I - C_t^{-2})Y_t \xrightarrow{\mathfrak{D}} Y \quad \text{on } B \quad \text{as } t \to \infty. \tag{1.4.15}$$

Let $f(t)C_t^{-1}X$ be distributed as ν_t; then (1.4.8) and (1.4.15) imply [16, Theorem 2.1, p. 58] that for any sequence t_n approaching infinity, $\{\nu_{t_n}: n = 1, 2, \cdots\}$ is conditionally compact on B. Now by Lemma 3.1 of [13], $\{\nu_{t_n}\}$ is a λ-family for any $\lambda \in l_\infty^+$, adequate for B. Since this is true for any sequence $\{t_n\}$ approaching infinity we conclude that $\{\nu_t: t \in (0, \infty)\}$ is a λ-family as $t \to \infty$ of probability measures on $(B, \mathfrak{B}(B))$. From this and (1.4.12) it follows, by Lemma 1.2.11, that

$$f(t)C_t^{-1}X \xrightarrow{\mathfrak{D}} 0 \quad \text{on } B \quad \text{as } t \to \infty,$$

which is (1.4.4)(a).

We now proceed to the proof that (1.4.4) implies (1.4.3). (1.4.4)(b) implies (Definition 1.1.2) that, for any sequence $t_n \to \infty$, $\{\mu_{t_n}: n = 1, 2, \cdots\}$ is compact, and hence it is a λ-family for any $\lambda \in l_\infty^+$, adequate for B [13, Lemma 3.1]. Thus $\{\mu_t: t \in (0, \infty)\}$ is a λ-family as $t \to \infty$, hence (1.4.3)(b) holds.

Furthermore, from (1.4.4)(b), it follows that

$$\lim_{t\to\infty} a_t = 1. \tag{1.4.16}$$

Let G be a real-valued, bounded, $\|\cdot\|_B$-continuous function on B, and let

$$\beta_t = h(t)(\det C_t)E_{P_\lambda}\left\{\phi^{\approx}(ty)\left(\int_B G(f(t)x)\exp[-itf(t)(y, x)^{\approx}]\,d\nu(x)\right)\right\} - \int_B G(s)\,d\mu(s).$$

From (1.4.16) we get

$$\begin{aligned}\lim_{t\to\infty} \beta_t &= \lim_{t\to\infty} \frac{\beta_t}{a_t} \\ &= \lim_{t\to\infty} \frac{1}{a_t} h(t)(\det C_t)E_{P_\lambda}\left\{\phi^{\approx}(ty)\left(\int_B G(f(t)x)\exp[-itf(t)(y, x)^{\approx}\,d\nu(x)\right)\right\} \\ &\quad - \int_B G(s)\,d\mu(s).\end{aligned}$$

By the argument used to obtain (1.4.7) we have

$$\lim_{t\to\infty} \beta_t = \lim_{t\to\infty} \int_B\int_B G(f(t)C_t^{-1}x + (I - C_t^{-2})s)\,d\nu(x)\,d\mu_t(s) - \int_B G(s)\,d\mu(s). \tag{1.4.17}$$

We note that (1.4.15) and (1.4.4)(a) imply [16, Lemma 1.1 and Theorem 1.1, p. 57] $\nu \circ (f(t)C_t^{-1})^{-1} * \mu_t \circ (I - C_t^{-2})^{-1} \overset{W}{\Rightarrow} \mu$ on B as $t \to \infty$, which can equivalently be written as

$$\int_B\int_B G(f(t)C_t^{-1}x + (I - C_t^{-2})s)\,d\nu(x)\,d\mu_t(s) \to \int_B G(s)\,d\mu(s) \quad \text{as } t \to \infty \tag{1.4.18}$$

for all real-valued, bounded, $\|\cdot\|_B$-continuous functions G on B. From (1.4.17) and (1.4.18) we get $\lim_{t\to\infty} \beta_t = 0$ which completes the proof.

1.4.19. **Remark.** We note that in order to obtain the above Inversion Formulae we used the facts that B is a Banach space with a basis for which there exists an adequate $\lambda \in l_\infty^+$. In view of this and condition (1.2.2) we obtain that every measurable subset of B is a measurable subset of the associated Hilbert space H_λ. We shall now consider Inversion Formulae which give exact generalizations of the formulae due to Gross [7]. The spaces for which this can be done are E_α spaces recently studied in [13]. For these spaces with Schauder basis adequacy for $\lambda \in l_\infty^+$ can be obtained easily but (1.2.2) is not valid in general. However, it is known [13, §7] that each measurable subset of E_α is a measurable subset of H_λ.

II. INVERSION FORMULA FOR ORLICZ SPACES OF REAL SEQUENCES

1. We introduce the definition of the Orlicz space E_α following [13, §6].

Let $\alpha(s)$ be a convex function on $[0, \infty)$ such that $\alpha(0) = 0$, $\alpha(s) > 0$ for $s > 0$. Further, assume

$$\alpha(2s) \le M\alpha(s) \tag{2.1.1}$$

for all $s \ge 0$ and some $M < \infty$.

We define E_α as the space of all real sequences satisfying $\sum_{i=1}^\infty \alpha(x_i^2) < \infty$.

Let $\Gamma(s) = \alpha(s^2)$, and for any sequence $x = \{x_i\}$ of real numbers define

$$\|x\|_\Gamma = \sup_y \left\{ \sum_{i=1}^\infty |x_i y_i| : \sum_{i=1}^\infty \Lambda(|y_i|) \le 1 \right\}$$

where Λ is the complementary function of Γ in the sense of Young [21, p. 77]. Then by Theorem 6.3 of [13], the space $(E_\alpha, \|\cdot\|_\Gamma)$ is homeomorphically isometric to Orlicz space S_Γ [21]. It is a Banach space under $\|\cdot\|_\Gamma$ and possesses Schauder basis $\{b_n\}$ where b_n is the vector with one as the nth coordinate and other coordinates zero.

Following [13], we shall denote by E_α the Hilbert space l_2 or an E_α space where $\alpha(\cdot)$ satisfies (2.1.1). We assume that $\alpha_c(\cdot)$, the complementary function of $\alpha(\cdot)$ in the sense of Young, satisfy (2.1.1). Notice that if $E_\alpha = l_2$ then a natural choice for the function α is $\alpha(s) = s$. Hence $\alpha_c(s) = 0$ on $[0, 1]$ but $\alpha_c(s) = \infty$ for $s > 1$. Thus $\alpha_c(\cdot)$ does not satisfy (2.1.1) when $E_\alpha = l_2$ and this is a special case which is easily handled.

We will let S_α, S_{α_c} denote the Orlicz spaces given by $\alpha(\cdot)$ and $\alpha_c(\cdot)$, respectively. Then the dual space of S_α can be identified as S_{α_c}, and since $\alpha_c(\cdot)$ also satisfies (2.1.1), except when $E_\alpha = l_2$, it follows that the dual of S_{α_c} is S_α [21, p. 150]. In view of Young's Inequality we obtain that $\lambda \in S^+_{\alpha_c}$ is adequate for E_α.

For each vector $\lambda = (\lambda_1, \lambda_2, \cdots) \in S^+_{\alpha_c}$ we define the space H_λ as all sequences $x = (x_1, x_2, \cdots)$ such that $\Sigma_{i=1}^\infty \lambda_i x_i^2 < \infty$. Then H_λ is a Hilbert space with $\|x\|_\lambda = (\Sigma_{i=1}^\infty \lambda_i x_i^2)^{1/2}$ and the inner product $(x, y) = \Sigma_{i=1}^\infty \lambda_i x_i y_i$. In the special case $E_\alpha = l_2$ we have $S_{\alpha_c} = l_\infty$ and for simplicity we take $\lambda = (1, 1, \cdots)$. Then $H_\lambda = l_2$ and we shall assume without loss of generality that $\alpha(s) \equiv s$.

The following lemma is proved in [13, p. 136].

2.1.2. **Lemma.** *E_α is a Borel subset of H_λ for each $\lambda \in S^+_{\alpha_c}$. Furthermore, every Borel subset of E_α is a Borel subset of H_λ.*

We note that from Lemma 2.1.2, every probability measure μ on E_α can be regarded as a probability measure on H_λ. Furthermore every countably additive measure on H_λ is countably additive on E_α.

We use the fact that every linear operator on E^*_α into E_α can be represented as an infinite-dimensional matrix to give the following definition.

2.1.3. **Definition.** A linear operator from E^*_α into E_α is an α-operator if the matrix of the operator, $\{t_{ij}\}$, is symmetric, nonnegative-definite with $\Sigma_{i=1}^\infty \alpha(t_{ii}) < \infty$.

2.1.4. **Lemma.** *Let T be an infinite-dimensional matrix $\{t_{ij}\}$ such that T is symmetric, nonnegative-definite and $\Sigma_{i=1}^\infty \alpha(t_{ii}) < \infty$. Then T defined for $y \in E^*_\alpha$, by*

$$(Ty)_i = \sum_{j=1}^\infty t_{ij} y(b_j)$$

is an α-operator on E^*_α into E_α.

Proof. Here $\{b_n\}$ is the basis for E_α mentioned earlier. Clearly T is linear, and the proof will be completed as soon as we show T is well defined, that is, $Ty \in E_\alpha$ for each $y \in E^*_\alpha$. Since T is symmetric, nonnegative-definite, it follows that

$$|(Ty)_i| \le \sum_{j=1}^\infty |t_{ij}||y(b_j)| \le \sum_{j=1}^\infty t_{ii}^{1/2} t_{jj}^{1/2} |y(b_j)| = t_{ii}^{1/2} \sum_{j=1}^\infty t_{jj}^{1/2} |y(b_j)|.$$

By assumption $\sum_{j=1}^{\infty} \alpha[(t_{jj}^{1/2})^2] = \sum_{j=1}^{\infty} \alpha(t_{jj})$ is finite, hence $\{t_{jj}^{1/2}\} \in E_\alpha$ But $y \in E_\alpha^*$, $\{t_{jj}^{1/2}\} \in E_\alpha$ imply [21, Theorem 3, p. 82], that $\sum_{j=1}^{\infty} t_{jj}^{1/2} |y(b_j)| = A < \infty$. Hence $|(Ty)_i| \leq A\, t_{ii}^{1/2}$. Since $\alpha(\cdot)$ is increasing we get $\alpha(|(Ty)_i|^2) \leq \alpha(A^2 t_{ii})$. Now using the fact that $\alpha(\cdot)$ satisfies (2.1.1), for some k finite, $\alpha(A^2 t_{ii}) \leq M^k \alpha(t_{ii})$. Therefore

$$\sum_{i=1}^{\infty} \alpha(|(Ty)_i|^2) \leq M^k \sum_{i=1}^{\infty} \alpha(t_{ii}) < \infty,$$

which implies that $Ty \in E_\alpha$, and hence the proof is completed.

We know (Lemma 2.1.2) that E_α is a Borel subset of H_λ and the $\|\cdot\|_\Gamma$-topology is stronger than $\|\cdot\|_\lambda$-topology on E_α. Hence it follows that H_λ^* is a subset of E_α^*. We now identify H_λ^* with H_λ and prove the following lemma.

2.1.5. **Lemma.** *Every α-operator T on E_α^* is a trace class operator on H_λ.*

Proof. Since T is an α-operator on E_α^* we get $\sum_{i=1}^{\infty} \alpha(t_{ii}) < \infty$ which implies $\{t_{ii}\} \in S_\alpha$, and since $\lambda = \{\lambda_i\} \in S_{\alpha_c}$ it follows [21, Theorem 3, p. 82] that $\sum_{i=1}^{\infty} \lambda_i t_{ii}$ is finite. Observe that $\operatorname{trace}_{H_\lambda} T = \sum_{i=1}^{\infty} \lambda_i t_{ii}$, which is finite, hence T is a trace class operator on H_λ.

2.1.6. **Remark.** Let T be an α-operator on E_α^* corresponding to a Gaussian measure ν on E_α [13, Theorem 5.2]. Then T is a nonnegative, symmetric trace-class operator on H_λ (Lemma 2.1.5). Hence $T^{1/2}$ is a Hilbert-Schmidt operator on H_λ and this implies (Remark 1.1.7(b)), $\nu = n \circ T^{-1/2}$ is a probability measure on H_λ where n is the canonical normal distribution on H_λ. Furthermore the form of the c.f.'s of $n \circ T^{-1/2}$ and ν [13, Theorem 5.2] gives that the probability measure $n \circ T^{-1/2}$ is countably additive on $(E_\alpha, \mathcal{B}(E_\alpha))$.

2. In this section we prove a theorem which gives a class of inversion formulae for a probability measure on the space E_α extending some work of L. Gross in [7].

Let $P_\lambda(\cdot)$ be the product probability measure on l, the space of all real sequences with topology of coordinate convergence, such that the ith coordinate is Gaussian with mean zero and variance λ_i where $\lambda = \{\lambda_i\} \in S_{\alpha_c}^+$. Each $\lambda \in S_{\alpha_c}^+$ adequate for E_α and for $y \in l$ we define the "stochastic linear functional" on E_α in the following manner:

$$(y, x)^{\approx} = \lim_N \sum_{i=1}^{N} x_i y_i. \tag{2.2.1}$$

The stochastic linear functional $(y, x)^{\approx} = \lim_N \sum_{i=1}^{N} x_i y_i$ is Borel measurable on $l \times E_\alpha$ and $P_\lambda \times \mu\{(y, x)\colon (y, x)^{\approx}$ exists and is finite$\} = 1$.

Now let T be an α-operator on E_α^*, then by Lemma 2.1.5, T is a trace class operator on H_λ. Denote by C_t the positive square root of $I + t^2 f(t)^2 T$ on H_λ

where $f(t)$ is a positive admissible function on $(0, \infty)$ [17, Theorem, p. 265]. Let μ be a probability measure on $(E_\alpha, \mathfrak{B}(E_\alpha))$, and define

$$\mu_t(A) = \frac{1}{a_t}\int_A h(t)\exp[-t^2\|C_t^{-1}s\|_\lambda^2/2]\,d\mu(s), \qquad A \in \mathfrak{B}(E_\alpha),$$

where $h(t)$ is a positive function on $(0, \infty)$, and

$$a_t = \int_{E_\alpha} h(t)\exp[-t^2\|C_t^{-1}s\|_\lambda^2/2]\,d\mu(s).$$

From Main Theorem I, we obtain

2.2.2. **Main Theorem II.** *Let*

(i) E_α *be as in* §2.2 *and* $\lambda \in S^+_{\alpha_c}$;

(ii) μ *be a probability measure on* $(E_\alpha, \mathfrak{B}(E_\alpha))$ *with c.f.* ϕ;

(iii) $f(t)$ *be a positive admissible function on* $(0, \infty)$, *and* $h(t)$ *be a positive function on* $(0, \infty)$;

(iv) T *be a positive-definite* α*-operator on* E^*_α;

(v) X *be an* E_α*-valued random variable distributed as* $\nu = n \circ T^{-1/2}$ *where* n *is the canonical normal distribution on* H_λ;

(vi) Y_t *and* Y *be* E_α*-valued random variables distributed as* μ_t *and* μ *respectively*;

(vii) E_{P_λ} *denote the integral with respect to* P_λ *on* l.

Then for all real-valued, bounded, $\|\cdot\|_\Gamma$*-continuous functions* G *on* E_α *the following are equivalent*:

$$(2.2.3)\quad \begin{cases} \text{(a)} \quad \displaystyle\int_{E_\alpha} G(s)\,d\mu(s) = \lim_{t\to\infty} h(t)(\det C_t)_{H_\lambda} E_{P_\lambda} \\ \qquad\qquad \cdot \left\{\phi^{\approx}(ty)\left(\displaystyle\int_{E_\alpha} G(f(t)x)\exp[-itf(t)(y, x)^{\approx}]\,d\nu(x)\right)\right\} \\ \text{(b)} \quad \{\mu_t : t \in (0, \infty)\} \text{ is a } \lambda\text{-family as } t \to \infty, \end{cases}$$

$$(2.2.4)\quad \begin{cases} \text{(a)} \quad f(t)C_t^{-1}X \xrightarrow{\mathfrak{D}} 0 \quad \text{as } t \to \infty \\ \text{(b)} \quad Y_t \xrightarrow{\mathfrak{D}} Y \qquad\qquad \text{as } t \to \infty. \end{cases}$$

We now put another condition on α, and prove the following lemma to reduce (2.2.4)(a) in the form similar to that of L. Gross [7, (10), p. 36].

2.2.5. **Lemma.** *Let*

(i) E_α *be as in* §2.2 *and* $\lambda \in S^+_{\alpha_c}$;

(ii) T *be a positive* α*-operator on* E^*_α;

(iii) X *be distributed as* $\nu = n \circ T^{-1/2}$ *where* n *is the canonical normal distribution on* H_λ.

(iv) *Assume further that there exists a constant* C *such that*

$$\int_{-\infty}^{+\infty} \alpha(u^2)\,d\mu(u) \le C\alpha\left[\int_{-\infty}^{+\infty} u^2\,d\mu(u)\right] \tag{2.2.6}$$

for all Gaussian measures μ *on* $(-\infty, +\infty)$ *with mean zero.*

Then the following are equivalent:

$$f(t)C_t^{-1}X \xrightarrow{\mathfrak{D}} 0 \quad as\ t \to \infty \tag{2.2.7}$$

$$\sum_{i=1}^{\infty} \alpha(f(t)^2(C_t^{-2}T)_{ii}) \to 0 \quad as\ t \to \infty \tag{2.2.8}$$

where $(C_t^{-2}T)_{ii} = (C_t^{-2}T\,b_i, b_i)_{H_\lambda}$.

Proof. We shall first prove that (2.2.7) implies (2.2.8). Let $\{t_n\}$ be a sequence in $(0, \infty)$ such that $t_n \to \infty$ as $n \to \infty$. Let $f(t_n)C_{t_n}^{-1}X$ be distributed as ν_{t_n}, then ν_{t_n} is defined on $(E_\alpha, \mathfrak{B}(E_\alpha))$ and by (2.2.7), $\{\nu_{t_n}: n = 1, 2, \cdots\}$ converges weakly on E_α to δ_0. Since E_α is a Borel subset of H_λ, $\{\nu_{t_n}: n = 1, 2, \cdots\}$ can be regarded as probability measures on H_λ, and since topology of H_λ is weaker than that of E_α, ν_{t_n} converges weakly on H_λ to δ_0. Hence $\nu_{t_n} \overset{W}{\Rightarrow} \delta_0$ on H_λ as $n \to \infty$ implies [7, Theorem 2, p. 8] that

$$\sum_{i=1}^{\infty} \lambda_i(f(t_n)^2(C_{t_n}^{-2}T)_{ii}) = \mathrm{trace}_{H_\lambda}(f(t_n)^2C_{t_n}^{-2}T) \to 0 \quad \text{as } n \to \infty. \tag{2.2.9}$$

Also, from condition (2.2.6) and the fact that $\{\nu_{t_n}\}$ is compact on E_α we get [13, Theorem 9.1] that for each $\epsilon > 0$ there exists an N_0 such that

$$\sum_{i=N_0}^{\infty} \alpha(f(t_n)^2(C_{t_n}^{-2}T)_{ii}) < \epsilon \quad \text{for all } t_n. \tag{2.2.10}$$

Therefore (2.2.9) and (2.2.10) imply that, in view of the fact that $\lambda_i > 0$,

$$\sum_{i=1}^{\infty} \alpha(f(t_n)^2(C_{t_n}^{-2}T)_{ii}) \to 0 \quad \text{as } n \to \infty.$$

For the converse, let $\{t_n\}$ be a sequence in $(0, \infty)$ such that $t_n \to \infty$ as $n \to \infty$. Then for sufficiently large n_0

$$\sum_{i=1}^{\infty} \alpha(f(t_n)^2(C_{t_n}^{-2}T)_{ii}) < \infty, \qquad n \ge n_0. \tag{1.2.11}$$

For $n \ge n_0$, $f(t_n)C_{t_n}^{-2}X$ has characteristic functional $\exp\{-\tfrac{1}{2}f^2(t_n)(C_{t_n}^{-2}Ty, y)_{H_\lambda}\}$ for $y \in H_\lambda$ regarded as a Gaussian measure on H_λ and hence, by continuity of the characteristic functional and (2.2.11) and Lemma 5.4 of [13], $\exp\{-\tfrac{1}{2}f^2(t_n)(C_{t_n}^{-2}Ty, y)\}$, $y \in E_\alpha$, is the characteristic functional of the Gaussian measure $f(t_n)C_{t_n}^{-2}X$ on E_α. Now Lemma 5.3 of [13] implies (2.2.7).

2.2.12. **Corollary.** *If, in addition to* (i)–(vii) *of Theorem* 2.2.2, *the function* $\alpha(\cdot)$ *satisfies* (2.2.6), *then the following are equivalent*:

$$(2.2.13)\quad \begin{cases} \text{(a)} \quad \int_{E_\alpha} G(s)\,d\mu(s) = \lim_{t\to\infty} h(t)(\det C_t)_{H_\lambda} E_{P_\lambda} \\ \qquad\qquad \left\{\phi^{\approx}(ty)\left(\int_{E_\alpha} G(f(t)x)\exp[-itf(t)(y, x)^{\approx}]\,d\nu(x)\right)\right\} \\ \text{(b)} \quad \{\mu_t : t \in (0, \infty)\} \text{ is a } \lambda\text{-family as } t \to \infty, \end{cases}$$

$$(2.2.14)\quad \begin{cases} \text{(a)} \quad \sum_{i=1}^{\infty} \alpha(f(t)^2(C_t^{-2}T)_{ii}) \to 0 \quad \text{as } t \to \infty \\ \text{(b)} \quad Y_t \xrightarrow{\mathfrak{D}} Y \qquad\qquad \text{as } t \to \infty. \end{cases}$$

Proof. By Lemma 2.2.5, condition (2.2.14)(a) is equivalent to (2.2.4)(a), hence the proof follows from Theorem 2.2.2.

2.2.15. **Remark.** In case the matrix $\{t_{ij}\}$ of the α-operator T is diagonal, condition (2.2.14)(a) would be replaced by

$$(2.2.16)\qquad \sum_{i=1}^{\infty} \alpha\left(\frac{f(t)^2 t_{ii}}{1 + t^2 f(t)^2 t_{ii}}\right) \to 0 \quad \text{as } t \to \infty.$$

2.2.17. **Definition.** If $2 \le p < \infty$ then a linear operator T from l_q $(1/p + 1/q = 1)$ into l_p is an S_p-operator if T can be represented as an infinite symmetric, nonnegative definite matrix $\{t_{ij}\}$ such that $\sum_{i=1}^{\infty} (t_{ii})^{p/2}$ is finite. Here, by nonnegative definite, we mean that $\sum_{i,j=1}^{n} t_{ij}x_i x_j \ge 0$ for all $(x_1, x_2, \cdots, x_n) \in R^n$ and all integers n. Thus for $p = 2$, an S_2-operator is an S-operator.

For $p > 2$, l_p is an E_α space (Orlicz space) with $\alpha(s) = s^{p/2}$ [21, p. 78]. Furthermore, for this $\alpha(\cdot)$, condition (2.2.6) is satisfied. The α-operators on $E_\alpha^* = l_p^*$ are the same as S_p-operators, and $S_{\alpha_c}^+ = (l_{p/2}^*)^+$. Now Corollary 2.2.12 gives us the following corollary.

2.2.18. **Corollary.** *Let* $p \ge 2$, *and let*

(i) *T be a positive-definite S_p-operator on l_p^**;

(ii) *μ be a probability measure on $(l_p, \mathfrak{B}(l_p))$ with c.f. ϕ*;

(iii) *$f(t)$ be a positive admissible function on $(0, \infty)$, and $h(t)$ be a positive function on $(0, \infty)$*;

(iv) *$\nu = n \circ T^{-1/2}$ where n is the canonical normal distribution on H_λ*;

(v) *E_{P_λ} denote the integral with respect to P_λ on l.*

Then for all real-valued, bounded, l_p-continuous (that is, continuous in l_p norm) functions G on l_p the following are equivalent:

$$
(2.2.19)\quad \begin{cases} \text{(a)} & \int_{l_p} G(s)\,d\mu(s) = \lim_{t\to\infty} h(t)(\det C_t)_{H_\lambda} E_P \cdot \left\{ \phi^{\approx}(ty)\left(\int_{l_p} G(f(t)x)\exp[-itf(t)(y,x)^{\approx}]\,d\nu(x) \right) \right\} \\ \text{(b)} & \{\mu_t : t \in (0,\infty)\} \textit{ is a } \lambda\textit{-family as } t \longrightarrow \infty, \end{cases}
$$

$$
(2.2.20)\quad \begin{cases} \text{(a)} & \sum_{i=1}^{\infty} [f(t)]^p [(C_t^{-2}T)_{ii}]^{p/2} \longrightarrow 0 \quad \textit{as } t \longrightarrow \infty \\ \text{(b)} & \textit{The measures } \{\mu_t : t \in (0,\infty)\} \textit{ converge weakly to } \mu \textit{ on } l_p \textit{ as } t \longrightarrow \infty. \end{cases}
$$

2.2.21. **Remark.** (a) If the matrix $\{t_{ij}\}$ of the S_p-operator T is diagonal, then (2.2.20)(a) would be replaced by

$$
(2.2.22)\qquad \sum_{i=1}^{\infty} [f(t)]^p \left(\frac{t_{ii}}{1 + t^2 f(t)^2 t_{ii}} \right)^{p/2} \longrightarrow 0 \quad \text{as } t \longrightarrow \infty.
$$

(b) It can easily be shown that in Theorem 4 of L. Gross [7] one can, without loss of generality, assume the Hilbert-Schmidt operators in (10) and (11) of [7, p. 36] to be diagonal.

(c) In the special case $E_\alpha = l_2$ we have $S_{\alpha_c} = l_\infty$ and for simplicity we take $\lambda = (1, 1, \cdots)$, then $H_\lambda = l_2$. Now using Lemma 4.3 of [14], and the fact that $T^{1/2}$ is a Hilbert-Schmidt operator on l_2 whenever T is an S_2-operator on l_2, we get Theorem 4 of [7].

(d) From (b), it follows that, in case of Hilbert space, condition (2.2.22) is restatement of condition (10) of [7, p. 36].

III. INVERSION FORMULAE FOR BANACH SPACES WITH A SHRINKING SCHAUDER BASIS

1. In this part of the paper we prove inversion formulae for Banach spaces with a Schauder basis which is shrinking. We obtain the inversion formulae of the form of the Main Theorem II in terms of the characteristic functional instead of its extension and without the use of the Lévy Continuity Theorem, and hope that one might be able to use this theorem to give a proof of the Lévy Continuity Theorem.

We start with the preliminaries. Some of the concepts given below are due to L. Gross and taken by us from his work in [6].

2. A tame (cylinder) set in a real separable Hilbert space H can be described as a set of the form $C = P^{-1}(E)$ where P is a finite-dimensional orthogonal projection on H with range $\mathfrak{R}$, say, and E is a Borel set in $\mathfrak{R}$. The cylinder set measure ν (see [6, p. 32]) associated with the canonical normal distribution is called Gauss measure on H, and for the above tame set C we have

$$\nu(C) = (2\pi)^{-k/2} \int_E e^{-\|x\|^2/2} dx$$

where k is the dimension of $\mathcal{R}$ and dx is Lebesgue measure on R_k.

3.2.1. **Definition.** A seminorm $\|x\|_1$ on H is called a measurable seminorm if for every real number $\epsilon > 0$ there exists a finite-dimensional projection P_0 such that for every finite-dimensional projection P orthogonal to P_0 we have

(3.2.2) $$\operatorname{Prob}(\|Px\|_1^\sim > \epsilon) < \epsilon$$

where $\|Px\|_1^\sim$ denotes the random variable on the probability space (Ω, m) corresponding to the tame function $\|Px\|_1$ and Prob refers to the probability of the indicated event with respect to the probability measure m associated with the canonical normal distribution.

Observe that the condition (3.2.2) can also be written as

$$\nu(\{x\colon \|Px\|_1 > \epsilon\}) < \epsilon$$

where ν is Gauss measure on H (see [6, p. 33]). We note that a measurable norm is a measurable seminorm which is a norm.

3.2.3. **Definition** [3, p. 69]. A Schauder basis $\{b_i\}$ in a Banach space B is called *shrinking* basis for B if, for each β in B^*, $\lim_{n\to\infty} p_n(\beta) = 0$ where $p_m(\beta) =$ norm of β restricted to the range of $x - \sum_{i=1}^m \beta_i(x) b_i$; that is, $p_m(\beta) = \operatorname{Sup}\{\beta(x)\colon \sum_{i=1}^m \beta_i(x) b_i = 0$ and $\|x\|_B \leq 1\}$.

Throughout this part B will denote a real Banach space with a shrinking Schauder basis $\{b_n\}$ such that $\|b_n\|_B = 1$. As before we will write the expansion of x as $\sum_{n=1}^\infty \beta_n(x) b_n$, and $\|x\|_B = \lim_{k\to\infty} \|\sum_{n=1}^k \beta_n(x) b_n\|_B$. Each $\lambda \in l_1^+$ is adequate for B. Let H_λ denote the completion of B under the inner product (1.2.5) for $\lambda \in l_1^+$.

Let n be the canonical normal distribution on H_λ^* into the set of all random variables defined on B^*, that is, for each $x \in H_\lambda^*$, $n(x)$ is a random variable on B^* which is distributed normally with mean zero and variance $\|x\|_{H_\lambda^*}^2$. We identify H_λ^* with H_λ, hence for each $x \in H_\lambda$, $n(x)$ is a random variable on B^* distributed normally with mean zero and variance $\|x\|_\lambda^2$.

The basis elements b_i's can be considered as coordinate functionals on B^* [3, Lemma 1, p. 70]. Then $n(b_i) = (b_i, \cdot)$ is a random variable defined on B^* which is distributed normally with mean zero and variance $\|b_i\|_\lambda^2 = (b_i, b_i) = \lambda_i$.

Let P_λ be the cylinder set measure on the field $\mathcal{C}$ generated by tame sets of B^* induced by the above canonical normal distribution on H_λ.

3.2.4. **Lemma.** P_λ *is countably additive on* $\mathcal{C}$.

Proof. Without loss of generality we assume $\|\beta_i\|_{B^*} = 1$. By Lemma 1 of [3, p. 70], $\{\beta_i\}$ is a basis for B^*, hence we have $\|y\|_{B^*} = \lim_{n\to\infty} \|\Sigma_{i=1}^n b_i(y)\beta_i\|_{B^*}$. Let $\|y\|_n = \|\Sigma_{i=1}^n b_i(y)\beta_i\|_{B^*}$ and observe that

$$P_\lambda\{y: \|y\|_{B^*} < \epsilon\} = P_\lambda\{y: \lim_{n\to\infty} \|y\|_n < \epsilon\} > P_\lambda\left\{y: \lim_{n\to\infty} \sum_{i=1}^n |b_i(y)| < \epsilon\right\}.$$

Let $X(y) = \lim_{n\to\infty} \Sigma_{i=1}^n |b_i(y)|$, then X is a random variable on B^* since $E\{\Sigma_{i=1}^n [b_i(y)]^2\} = \Sigma_{i=1}^n \lambda_i < \Sigma_{i=1}^\infty \lambda_i < \infty$, and the series is a series of independent random variables ([15, p. 234] and [2, Theorem 9.5.5]). In view of the property of Laplace transform, Theorem 6.6.2 of [2], we observe that the distribution of X is absolute normal with mean $\Sigma_{i=1}^\infty E[|b_i(y)|]$ and variance $\Sigma_{i=1}^\infty \lambda_i$. Hence the distribution of X puts mass around zero, and therefore

$$P_\lambda\{y: \|y\|_{B^*} < \epsilon\} > P_\lambda\{X < \epsilon\} > 0. \tag{3.2.5}$$

From the definition of H_λ we see that $H_\lambda^* = H_\lambda$ is a dense subset of B^*, and B^* is the completion of H_λ in the Banach norm $\|\cdot\|_{B^*}$ on B^*. Furthermore, $\|y\|_n$ is a tame function on H_λ and hence it is a measurable norm.

From (3.2.5) and the fact that $\|y\|_n$ is a nondecreasing sequence of measurable norms on H_λ, it follows that $\|y\|_{B^*}$ is a measurable norm on H_λ [9, Corollary 4.4].

So far we have shown that $\|y\|_{B^*}$ is a measurable norm on H_λ and B^* is the completion of H_λ in this norm. Hence by Theorem 1 of [6], P_λ is countably additive on $\mathcal{C}$.

Let $x \in B$, $y \in B^*$; then

$$(y, x) = y(x) = y\left(\sum_{i=1}^\infty \beta_i(x) b_i\right) = \sum_{i=1}^\infty \beta_i(x) y(b_i) = \sum_{i=1}^\infty \beta_i(x) b_i(y).$$

It follows that $(\cdot, x) = \Sigma_{i=1}^\infty \beta_i(x) b_i(\cdot)$ is a random variable defined on B^* which is distributed normally with mean zero and variance $\Sigma_{i=1}^\infty \lambda_i \beta_i^2(x)$ under P_λ.

Now let $x \in H_\lambda$, $y \in B^*$ and define the "stochastic linear functional" $(y, x)^{\approx}$ as follows:

$$(y, x)^{\approx} = \lim_{N\to\infty} \sum_{i=1}^N \hat{\beta}_i(x) b_i(y) \tag{3.2.6}$$

where as before $\hat{\beta}_i$ is extension of β_i to H_λ. From (3.2.6) we have $(y, x)^{\hat{\approx}} = y(x)$ for $x \in B$, $y \in B^*$.

3.2.7. **Lemma.** *The stochastic linear functional*

$$(y, x)^{\hat{\approx}} = \lim_{N\to\infty} \sum_{i=1}^N \hat{\beta}_i(x) b_i(y)$$

is Borel measurable on $B^* \times H_\lambda$ *and if*

$$F = \{(y, x) \colon (y, x)^{\hat{\approx}} \text{ exists and is finite}\},$$

then $P_\lambda \times \mu(F) = 1$ *where* μ *is a probability measure on* H_λ.

Proof. Similar to Lemma 4.1 of [13].

Let μ be a probability measure on B, then μ can be regarded as a probability measure on H_λ. Let ψ be the c.f. of μ when μ is regarded as a probability measure on H_λ; then

$$\psi(y) = \int_{H_\lambda} e^{i(y,x)} d\mu(x) \quad \text{for all } y \in H_\lambda^*.$$

Now ψ is a function defined on $H_\lambda^* \subseteq B^*$, and we would like to extend ψ to be defined on B^*.

For each $x \in H_\lambda$, $(y, x)^{\hat{\approx}}$ is defined on B^* a.e. P_λ, and is equal to $y(x)$ with μ measure one for each $y \in H_\lambda^*$. We call

$$\int_{H_\lambda} e^{i(y,x)^{\hat{\approx}}} d\mu(x) \qquad (y \in B^*)$$

the extension of ψ to B^*. Clearly on H_λ^* we have $\int_{H_\lambda} e^{i(y,x)^{\hat{\approx}}} d\mu(x) = \int_{H_\lambda} e^{i(y,x)} d\mu(x) = \psi(y)$. Since μ is actually defined on B we have

$$\int_{H_\lambda} e^{i(y,x)^{\hat{\approx}}} d\mu(x) = \int_B e^{i(y,x)^{\hat{\approx}}} d\mu(x) \quad \text{a.e. } P_\lambda.$$

But $(y, x)^{\hat{\approx}} = (y, x)$ for all $x \in B$, $y \in B^*$, hence

$$\int_{H_\lambda} e^{i(y,x)^{\hat{\approx}}} d\mu(x) = \int_B e^{i(y,x)} d\mu(x) = \phi(y) \quad \text{a.e. } P_\lambda \tag{3.2.8}$$

where ϕ is the c.f. of μ when μ is regarded as a probability measure on B. Thus c.f. of a probability measure μ on B when considered as a random variable on B^* is equal almost everywhere P_λ to the extension of the c.f. of μ when μ is regarded as a probability measure on H_λ.

3.2.9. **Remarks.** (a) Let l be as before. Since $\{b_n\}$ is a shrinking basis for B, the coordinate functional β_n's form a basis for B^* [3, Lemma 1, p. 70]. Hence there exists an isomorphism U^* from B^* to a Borel measurable subset of l, say Ω^* [11, §2, pp. 123–127]. Therefore B^* can be identified with a Borel measurable subset of l, and hence P_λ can be regarded as a countably additive cylinder set measure on l through this identification.

(b) By (a), Lemma 4.3 of [14], and the fact that P_λ sits actually on B^* we get $\psi(y)^\sim = \phi(y)$ a.e. P_λ where $\psi(\cdot)^\sim$ is Gross extension of the uniformly τ-continuous function $\psi(\cdot)$ with respect to the canonical normal distribution n on H_λ.

In view of the above remarks and $\lambda \in l_1^+$ being adequate for B, the following theorem giving inversion formulae for a probability measure on a Banach space B with a *shrinking Schauder* basis follows from Main Theorem I. It differs from the Main Theorem I in the sense that (3.2.11)(a) is stronger than (1.4.3)(a). This can easily be seen since $P_\lambda(B^*) = 1$ for $\lambda \in l_1^+$. Furthermore, we do not use the Lévy Continuity Theorem in the proof, since in this case the analogue of Lemma 1.2.11 can be obtained from Theorem 2.1 of [4]. We hope that one might be able to use this theorem to get a proof for the Lévy Continuity Theorem in this case. Let ν, T_ν, C_t, μ_t and a_t be as in Part I.

3.2.10. Main Theorem III. *Let*

(i) B *be a Banach space with shrinking Schauder basis* $\{b_i\}$ *and* $\lambda \in l_1^+$;

(ii) – (v) *be as in Main Theorem I*;

(vi) E_{P_λ} *denote the integral with respect to* P_λ *on* B^* (*see Lemma* 3.2.4).

Then for all real-valued, bounded, $\|\cdot\|_B$*-continuous functions* G *on* B *the following are equivalent*:

$$(3.2.11)\quad \begin{cases} \text{(a)} & \displaystyle\int_B G(s)\,d\mu(s) \\ & \displaystyle = \lim_{t\to\infty} h(t)(\det C_t)E_{P_\lambda}\left\{\phi(ty)\left(\int_B G(f(t)x)\exp[-itf(t)(y,x)]\,d\nu(x)\right)\right\} \\ \text{(b)} & \{\mu_t : t \in (0,\infty)\} \text{ is } \lambda\text{-family as } t \to \infty. \end{cases}$$

$$(3.2.12)\quad \begin{cases} \text{(a)} & f(t)C_t^{-1}X \xrightarrow{\mathfrak{D}} 0 \quad \text{as } t \to \infty \\ \text{(b)} & Y_t \xrightarrow{\mathfrak{D}} Y \quad \text{as } t \to \infty. \end{cases}$$

REFERENCES

1. P. Billingsley, *Convergence of probability measures*, Wiley, New York, 1968. MR 38 #1718.

2. Chung Kai Lai, *A course in probability theory*, Harcourt, Brace & World, New York, 1968. MR 37 #4842.

3. M. M. Day, *Normed linear spaces*, 2nd rev. ed., Academic Press, New York; Springer-Verlag, Berlin, 1962. MR 26 #2847.

4. A. D. DeAcosta, *Existence and convergence of probability measures in Banach spaces*, Trans. Amer. Math. Soc. 152 (1970), 273–298. MR 42 #2516.

5. I. M. Gel'fand and N. Ja. Vilenkin, *Generalized functions*. Vol. 4: *Some applications of harmonic analysis*, Fizmatgiz, Moscow, 1961; English transl., Academic Press, New York, 1964. MR 26 #4173; MR 30 #4152.

6. L. Gross, *Abstract Wiener spaces*, Proc. Fifth Berkeley Sympos. Math. Statist. and Probability (Berkeley, Calif., 1965/66), vol. II: Contributions to Probability Theory, part 1, Univ. of California Press, Berkelely, Calif., 1967, pp. 31–42. MR 35 #3027.

7. ———, *Harmonic analysis on Hilbert space*, Mem. Amer. Math. Soc. No. 46 (1963). MR 28 #4304.

8. ———, *Integration and nonlinear transformations in Hilbert space*, Trans. Amer. Math. Soc. 94 (1960), 404–440. MR 22 #2883.

9. ———, *Measurable functions on Hilbert space*, Trans. Amer. Math. Soc. 105 (1962), 372–390. MR 26 #5121.

10. P. R. Halmos, *Measure theory*, Van Nostrand, Princeton, N. J., 1968.

11. J. Kampé de Fériet, *Mesures de probabilité sur les espaces de Banach possédant une base dénombrable*, J. Math. Pures Appl. (9) 39 (1960), 119–163. MR 23 #A1223.

12. J. Kuelbs, *Gaussian measures on a Banach space*, J. Functional Analysis 5 (1970), 354–367. MR 41 #4639.

13. J. Kuelbs and V. Mandrekar, *Harmonic analysis on F-spaces with a basis*, Trans. Amer. Math. Soc. 169 (1972), 113–152.

14. ———, *Harmonic analysis on certain vector spaces*, Trans. Amer. Math. Soc. 149 (1970) 213–231.

15. M. Loève, *Probability theory*, 3rd ed., Van Nostrand, Princeton, N. J., 1963. MR 34 #3596.

16. K. R. Parthasarathy, *Probability measures on metric spaces*, Probability and Math. Statist., no. 3, Academic Press, New York, 1967. MR 37 #2271.

17. F. Riesz and B. Sz.-Nagy, *Leçons d'analyse fonctionnelle*, Akad. Kiadó, Budapest, 1953; English transl., Ungar, New York, 1955. MR 15, 132; MR 17, 175.

18. H. L. Royden, *Real analysis*, Macmillan, New York, 1966.

19. W. Rudin, *Real and complex analysis*, McGraw-Hill, New York, 1966. MR 35 #1420.

20. V. Sazonov, *Remarks on characteristic functionals*, Teor. Verojatnost. i Primenen. 3 (1958), 201–205 = Theor. Probability Appl. 3 (1958), 188–192. MR 20 #4882.

21. A. C. Zaanen, *Linear analysis. Measure and integral, Banach and Hilbert space, linear integral equations*, Interscience, New York; North-Holland, Amsterdam; Noordhoff, Groningen, 1953. MR 15, 878.

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