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ADDENDUM TO "MODULAR REPRESENTATIONS OF METABELIAN GROUPS"

BY

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In this note the principal indecomposable modules of ΩG are determined where G is a finite metabelian group and Ω is an algebraically closed field with characteristic p dividing $|G|$. The notations are the same as of [1].

Let P be a p-Sylow subgroup of $K(H)$. Since $K(H)/K(H)'$ is abelian, there exist subgroups $V_1 \supseteq K(H)'$ and $V_2 \supseteq K(H)'$ such that $K(H)/K(H)' \cong V_1/K(H)' \times V_2/K(H)'$, $V_1/K(H)'$ is a p-group and $p \nmid |V_2/K(H)'|$. Let P_1 be a p-sylow subgroup of V_2, then $P_1 \subseteq K(H)'$ and thus P_1 is normal in V_2. Hence there exists a subgroup V of V_2 such that $V_2 = P_1 \circ V$, the semidirect product, and $p \nmid |V|$. Clearly $K(H) = \langle P, V\rangle$, $P \cap V = 1$, and $|V| = |K(H)|/|P|$.

For each $K(H)$, A/H cyclic and $p \nmid |A/H|$, fix a subgroup V with the above properties. Let τ' be a linear representation of $K(H)$ with $\ker \tau' \cap A = H$ such that τ'_K is conjugate to σ where $K = K(\Lambda)$. Then τ'^G is irreducible and $\tau'^G \in B(\sigma, H)$. Let $x \in G$ and define

$$e_x(\tau') = \frac{1}{|V|} \sum_{a \in V} \tau'(x^{-1}a^{-1}x)a$$

and $e_1(\tau') = e(\tau')$. We prove

Theorem 4. *All the principal indecomposable modules of ΩG are given by the collection of the ideals $\Omega G e(\tau')$ with $\tau' \in \bigcup M(H, K(H))$ where the union is over all subgroups H of A such that A/H is cyclic and $p \nmid |A/H|$.*

Proof. Let T' be an ordinary representation of $K(H)$ such that $\ker \tau' = \ker T' \supseteq P$ and for all $a \in K(H)$, $\overline{T'(a)} = \tau'(a)$, and T'_V be the restriction of T' to V. Define $T'^{(x)}(a) = T'(x^{-1}ax)$ where $x \in G$. Since $\ker T' \supseteq P$, it follows that $T'_V \neq T'^{(x)}_V$ if $x \notin K(H)$. Define

$$e_x(T') = \frac{1}{|V|} \sum_{a \in V} T'(x^{-1}a^{-1}x)a;$$

then $e_x(T')$ are minimal indempotents of $\overline{Q}V$ and $e_x(T') \cdot e_y(T') = 0$ if and only if $xK(H) \neq yK(H)$. Similarly, if τ'_1 is another linear representation of $K(H)$ not conjugate to τ' and $\ker \tau'_1 \cap A = H$, and if T'_1 is similarly defined then

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$e_x(T') \cdot e_z(T_1') = 0$ for any x and z in G.

Now $e(T') = e_1(T')$ is also an indempotent of $\bar{Q}G$. We have $\bar{Q}G \otimes_{\bar{Q}V} \bar{Q}Ve(T') \cong \bar{Q}Ge(T')$, since from the definitions of tensor products and balanced maps there is a $\bar{Q}G$-homomorphism of $\bar{Q}G \otimes_{\bar{Q}V} \bar{Q}Ve(T')$ onto $\bar{Q}Ge(T')$, and both modules are of $\bar{Q}$-dimension $|G|/|V| = p^{\alpha}|G/K(H)|$, $p^{\alpha} \| |K(H)|$, with $\{he(T')|\ h$ runs over a set of coset representatives of V in $G\}$ a $\bar{Q}$-basis for $\bar{Q}Ge(T')$. Hence $\bar{Q}Ge(T')$ affords $(T_V')^G$. Thus we have $e_x(\tau') \cdot e_y(\tau') = 0$ if and only if $xK(H) \neq yK(H)$, $e_x(\tau') \cdot e_z(\tau_1') = 0$ if $\ker \tau_1' \cap A = H$ and τ_1' is not conjugate to τ', and $\Omega Ge(\tau')$, a direct summand of ΩG, affording $(\tau_V')^G$ of degree $p^{\alpha}|G/K(H)|$. If χ is the character of T', then from the Frobenius reciprocity theorem, $1 = (\chi_V, (\chi^G)_V) = (\chi_V^G, \chi^G)$, or τ'^G is a composition factor of $(\tau_V')^G$. Assume $\Omega Ge(\tau') = U_1 \oplus \cdots \oplus U_t$, U_i some indecomposable components of ΩG, then τ'^G is afforded by a composition factor of some U_i or U_i belongs to $B(\sigma, H)$. But from Theorem 3 of [1], U_i is of degree $p^{\alpha}|G/K(H)|$ and hence $U_i = \Omega Ge(\tau')$ or $\Omega Ge(\tau')$, and $(\tau_V')^G$, are indecomposable.

Each $\tau'^G \in B(\sigma, H)$ is associated with $|G/K(H)|$ (= degree of τ'^G) distinct indecomposable components of ΩG, namely $\Omega Ge_x(\tau')$, $x \in G/K(H)$. Moreover, if $\Omega Ge(\tau_1')$ belongs to $B(\sigma_1, H_1)$, where $B(\sigma_1, H_1)$ is a block different from $B(\sigma, H)$, then $e(\tau') \cdot e(\tau_1') = 0$. Now the result follows by applying Theorems 1 and 2, which completes the proof.

Define

$$e(\sigma, H) = {\sum}' \sum_{x \in G/K(H)} e_x(\tau')$$

where the summation Σ' is over all distinct $\tau'^G \in B(\sigma, H)$. We have

Corollary. *All the indecomposable two-sided ideals (blocks) of ΩG are given by the collection of the ideals $\Omega Ge(\sigma, H)$ where H runs over all nonconjugate subgroups of A, A/H cyclic, $p \nmid |A/H|$, and σ runs over the elements of $C(H, K(\Lambda))$.*

BIBLIOGRAPHY

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