\(\alpha_T\) IS FINITE FOR \(\aleph_1\)-CATEGORICAL \(T\)

BY

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ABSTRACT. Let \(T\) be a complete countable \(\aleph_n\)-categorical theory.

Definition. If \(\mathcal{A}\) is a model of \(T\) and \(A\) is a 1-ary formula in \(L(\mathcal{A})\) then \(A\)
has rank 0 if \(A(\mathcal{A})\) is finite. \(A(\mathcal{A})\) has rank \(n\) degree \(m\) iff for every set of 
m + 1 formulas \(B_1, \ldots, B_{m+1} \in S_1(L(\mathcal{A}))\) which partition \(A(\mathcal{A})\) some \(B_i(\mathcal{A})\) has 
rank \(\leq n - 1\). Theorem. If \(T\) is \(\aleph_1\)-categorical then for every \(\mathcal{A}\) a model of \(T\) 
and every \(A \in S_1(L(\mathcal{A}))\), \(A(\mathcal{A})\) has finite rank. Corollary. \(\alpha_T\) is finite. The 
methods derive from Lemmas 9 and 11 in "On strongly minimal sets" by Baldwin 
and Lachlan. \(\alpha_T\) is defined in "Categoricity in power" by Michael Morley.

In [4] Morley assigns an ordinal \(\alpha_T\) to each complete theory \(T\). He conjectures that if \(T\) is \(\aleph_1\)-categorical \(\alpha_T\) is finite. In this paper we prove this conjecture.

We assume familiarity with [1] and [4] but for convenience we list the principal results and definitions from those papers which are used here. Our notation is the same as in [1] with the following exceptions.

We deal with a countable first order language \(L\). We may extend the language \(L\) in several ways. If \(\mathcal{A}\) is an \(L\)-structure there is a natural extension \(L(\mathcal{A})\) of \(L\) obtained by adjoining to \(L\) a constant \(a\) for each \(a \in |\mathcal{A}|\) (the universe of \(\mathcal{A}\)). For each sentence \(A(a_1, \ldots, a_n) \in L(\mathcal{A})\) we say \(\mathcal{A}\) satisfies \(A(a_1, \ldots, a_n)\) and write \(\mathcal{A} \models A(a_1, \ldots, a_n)\) if in Shoenfield's notation \(\mathcal{A}(A(a_1, \ldots, a_n)) = T\) [7, p. 19]. If \(\mathcal{A}\) is an \(L\)-structure and \(X\) is a subset of \(|\mathcal{A}|\) then \(L(X)\) is the language obtained by adjoining to \(L\) a name \(x\) for each \(x \in X\). \((\mathcal{A}, X)\) is the natural expansion of \(\mathcal{A}\) to an \(L(X)\)-structure. A structure \(\mathcal{B}\) is an inessential expansion [7, p. 141] of an \(L\)-structure \(\mathcal{A}\) if \(\mathcal{B} = (\mathcal{A}, X)\) for some \(X \subseteq |\mathcal{A}|\).

\(S_n(L)\) denotes the set of formulas of \(L\) with free variables among \(v_0, \ldots, v_{n-1}\). If \(A\) is a formula such that \(u_1, \ldots, u_n\) in the natural order are the free variables in \(A\), then \(A(\mathcal{A})\) is the set of \(n\)-tuples \(b_1, \ldots, b_n\) such that

Presented to the Society, August 6, 1970; received by the editors February 24, 1971.

AMS (MOS) subject classifications (1970). Primary 02H05.

Key words and phrases. \(\aleph_1\)-categorical, strongly minimal, \(\alpha_T\).

(1) This material was contained in a Ph. D. thesis prepared under the direction of 
A. H. Lachlan and submitted to Simon Fraser University. The research was supported by a 
National Research Council Post Graduate Scholarship.
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$\mathfrak{A} \models A_{u_1, \ldots, u_n}(b_1, \ldots, b_n)$. If $p$ is a unary predicate symbol we abbreviate $pv_0(\mathfrak{A})$ by $p(\mathfrak{A})$.

A consistent set of $L$-sentences is a theory in $L$. If $T$ and $T'$ are theories in $L$ then $T'$ extends $T$ if $T \subseteq T'$. If $T$ is a theory in a language $L$ then $T'$ is an inessential extension of $T$ if there is a model $\mathfrak{A}$ of $T$ and a subset $X$ of $|\mathfrak{A}|$ such that $T' = Th(\mathfrak{A}, X)$ (i.e., the set of all sentences in $L(X)$ true of $(\mathfrak{A}, X)$). $T'$ is a principal extension of $T$ if $T'$ is an inessential extension of $T$ by a finite number of constants and a set of nonlogical axioms for $T'$ can be obtained by adjoining a finite set of sentences to a set of nonlogical axioms for $T$.

Let $\Gamma$ be a subset of $S_k(L)$. Then $\Gamma$ is a $k$-type in $T$ if there is some model $\mathfrak{A}$ of $T$ and elements $a_1, \ldots, a_k \in |\mathfrak{A}|$ such that $\mathfrak{A} \models A(a_1, \ldots, a_k)$ if and only if $A \in \Gamma$. If $\mathfrak{A}$ is a model of $T$ and $X \subseteq |\mathfrak{A}|$ then a $k$-type $\Gamma$ is realized in $X$ if there exists $x_1, \ldots, x_k \in X$ such that $\mathfrak{A} \models A(x_1, \ldots, x_k)$ for each $A \in \Gamma$. A $k$-type $\Gamma$ is a principal $k$-type in $T$ if there is a formula $A \in S_k(L(\mathfrak{A}))$ such that, for each formula $B$ in $\Gamma$, $\mathfrak{A} \models \forall v_0, \ldots, \forall v_{k-1}(A \rightarrow B)$. Since $T$ is complete there is one $0$-type truth.

Following Morley [4] we assume that each $T = \Sigma^*$ for some $\Sigma$ and thus that each $n$-ary formula $\Phi$ is equivalent in $T$ to an $n$-ary relation $A$. $\mathfrak{N}(T)$ is a set of all substructures of models of $T$. The following summarizes with slight changes in notation the second paragraph of §2 in [4]. If $\mathfrak{A}$ is an $L$-structure $\mathfrak{A}(\mathfrak{A})$ is the set of all open sentences in $L(\mathfrak{A})$ which are true in $(\mathfrak{A}, |A|)$. If $\mathfrak{A} \in \mathfrak{N}(T)$, $T(\mathfrak{A}) = \mathfrak{A}(\mathfrak{A}) \cup T$ is a complete theory in $L(\mathfrak{A})$. Let $S_k(\mathfrak{A})$ denote the Boolean algebra whose elements are the equivalence classes into which $S_k(L(\mathfrak{A}))$ is partitioned by the relation of equivalence in $T(\mathfrak{A})$, and whose operations of intersection, union, and complementation are those induced by conjunction, disjunction and negation respectively. The Stone space of $S_1(\mathfrak{A})$, the set of dual prime ideals of $S_1(\mathfrak{A})$, is a topological space denoted $S(\mathfrak{A})$. A dual prime ideal of $S_k(\mathfrak{A})$ is a $k$-type of $T(\mathfrak{A})$. This is a special case of the definition of $k$-type in the preceding paragraph. Note that, if $p \in S(\mathfrak{A})$ and $\mathfrak{A}'$ is an inessential expansion of $\mathfrak{A}$, $p$ is naturally a member of $S(\mathfrak{A}')$.

In [4] Morley makes the following definition. For each ordinal $\alpha$ and each $\mathfrak{A} \in \mathfrak{N}(T)$, subspaces $S^\alpha(\mathfrak{A})$ and $Tr^\alpha(\mathfrak{A})$ of $S(\mathfrak{A})$ are defined inductively by

1. $S^\alpha(\mathfrak{A}) = S(\mathfrak{A}) - \bigcup_{\beta < \alpha} Tr^\beta(\mathfrak{A})$,

2. $p \in Tr^\alpha(\mathfrak{A})$ if (i) $p \in S^\alpha(\mathfrak{A})$ and (ii) for every map $f^* : S(\mathfrak{B}) \rightarrow S(\mathfrak{A})$ where $\mathfrak{B} \in \mathfrak{N}(T)$ and $f$ is a monomorphism from $\mathfrak{A}$ into $\mathfrak{B}$, $f^* \circ p \cap S^\alpha(\mathfrak{B})$ is a set of isolated points in $S^\alpha(\mathfrak{B})$. (See [4, p. 519] for the definition of $f^*$).

If $i_{\mathfrak{B}}$ is an elementary embedding of $\mathfrak{A}$ into $\mathfrak{B}$ then $i_{\mathfrak{B}}^*$ maps $S(\mathfrak{B})$ onto $S(\mathfrak{A})$. Note that $q \in i_{\mathfrak{B}}^*^{-1}(p)$ is equivalent to $q \cap S_1(L(\mathfrak{A})) = p$. 

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An element $p$ of $S(L(\bar{\mathfrak{A}}))$ is algebraic if $p \in \text{Tr}^{0}(\bar{\mathfrak{A}})$; $p$ is transcendental in rank $\alpha$ if $p \in \text{Tr}^{\alpha}(\bar{\mathfrak{A}})$. If $A \in S_{1}(L(\bar{\mathfrak{A}}))$, $\bigcup_{A} = \{p| p \in S(\bar{\mathfrak{A}}) \land A \in p\}$.

The following definitions are originally due to Marsh [3]. Let $\bar{\mathfrak{A}}$ be an $L$-structure and $X$ a subset of $|\bar{\mathfrak{A}}|$. The algebraic closure of $X$, denoted by $\text{cl}(X)$, is the union of all finite subsets of $|\bar{\mathfrak{A}}|$ definable in $((\bar{\mathfrak{A}}, X)$. $X$ spans $Y$ if $Y \subseteq \text{cl}(X)$. $X$ is independent if for each $x \in X$, $x \not\in \text{cl}(X - \{x\})$. $X$ is a basis for $Y$ if $X$ is an independent subset of $Y$ which spans $Y$. If every basis for $Y$ has the same cardinality $\mu$, we define the dimension of $Y$ to be $\mu$ and write $\text{dim}(Y) = \mu$.

Let $\bar{\mathfrak{A}}$ be an $L$-structure. A subset $X$ of $|\bar{\mathfrak{A}}|$ is minimal in $\bar{\mathfrak{A}}$ if $X$ is infinite, definable in $\bar{\mathfrak{A}}$, and for any subset $Y$ of $|\bar{\mathfrak{A}}|$ which is definable in $\bar{\mathfrak{A}}$ either $Y \cap X$ or $X - Y$ is finite.

If $D \in S_{1}(L(\bar{\mathfrak{A}}))$ and $X = D(\bar{\mathfrak{A}})$ then $X$ is strongly minimal in $\bar{\mathfrak{A}}$ if for any elementary extension $B$ of $\bar{\mathfrak{A}}$, $D(B)$ is minimal in $B$. Let $\bar{\mathfrak{A}}_{0}$ and $\bar{\mathfrak{A}}_{1}$ be models of a complete theory $T$. Since up to isomorphism any two models of $T$ have a common elementary extension, $D(\bar{\mathfrak{A}}_{0})$ is strongly minimal in $\bar{\mathfrak{A}}_{0}$ if and only if $D(\bar{\mathfrak{A}}_{1})$ is strongly minimal in $\bar{\mathfrak{A}}_{1}$. Thus, without ambiguity we define a formula $D \in S_{1}(L)$ to be strongly minimal in $T$ if there is a model $\bar{\mathfrak{A}}$ of $T$ such that $D(\bar{\mathfrak{A}})$ is strongly minimal in $\bar{\mathfrak{A}}$.

We will refer to the following theorem which is Theorem 5 in [1].

**Theorem 0.** If $\bar{\mathfrak{A}}$ is a model of an $\mathfrak{K}_{1}$-categorical theory $T$ then $\bar{\mathfrak{A}}$ is homogeneous.

Our first step in the proof of Morley's conjecture is to introduce a concept of the rank of a formula in a model of a theory. We will compare this notion with three other sorts of rank.

If $\bar{\mathfrak{A}}$ is an $L$-structure and $A$ is an element of $S_{1}(L(\bar{\mathfrak{A}}))$ then we defined $A$ to be minimal in $\bar{\mathfrak{A}}$ if $A(\bar{\mathfrak{A}})$ is infinite and, for each formula $B \in S_{1}(L(\bar{\mathfrak{A}}))$, $(B \land A)(\bar{\mathfrak{A}})$ or $(\neg B \land A)(\bar{\mathfrak{A}})$ is finite. We will define a notion of rank of a formula in a model such that minimal formulas have rank one.

Well order the class $X$ consisting of $\{-1\}$ and the direct product of the class of all ordinals with the positive integers by placing $-1$ first in the order and then following the natural lexicographic order. For each $L$-structure $\bar{\mathfrak{A}}$ define $f_{\bar{\mathfrak{A}}}: X \rightarrow 2^{S_{1}(L(\bar{\mathfrak{A}}))}$ by induction

$$f_{\bar{\mathfrak{A}}}(-1) = \{|A| S_{1}(L(\bar{\mathfrak{A}}))| A(\bar{\mathfrak{A}}) = \emptyset\}.$$

$A \in f_{\bar{\mathfrak{A}}}((\alpha, k))$ if and only if $A \not\in f(x)$ for any $x < (\alpha, k)$ and if for any set of $k + 1$ formulas $B_{1}, \cdots, B_{k+1}$ from $S_{1}(L(\bar{\mathfrak{A}}))$ such that the sets $B_{i}(\bar{\mathfrak{A}})$ partition $A(\bar{\mathfrak{A}})$ there exists an $x < (\alpha, 1)$ with one of the $B_{i} \in f(x)$. 
Let $T$ be totally transcendental, $\mathcal{A}$ a model of $T$, and $A \in S_1(L(\mathcal{A}))$. Call a formula $A$ rankless if $A$ is not in the range of $f_\mathcal{A}$. We claim there is no formula $A \in S_1(L(\mathcal{A}))$ such that $A$ is rankless. For, if so, we can construct for each finite binary sequence $\sigma$ a formula $A_\sigma$ such that (1) $A_\sigma$ is rankless and (2) if $\sigma' = \sigma \cup \langle dm \sigma, 0 \rangle$ and $\sigma'' = \sigma \cup \langle dm \sigma, 1 \rangle$ then $A_{\sigma'} = \neg A_{\sigma''}$. Let $X$ be the set of constants from $|\mathcal{A}|$ which occur in any $A_\sigma$. Then $X$ is countable but $S(X)$ is uncountable contrary to the hypothesis that $T$ is totally transcendental.

Thus if $\mathcal{A}$ is a model of a totally transcendental theory we may define for each $A \in S_1(L(\mathcal{A}))$ the rank of $A$ in $\mathcal{A}$ which we denote by $R_{\mathcal{A}}(A)$. $R_{\mathcal{A}}(A)$ is $-1$ if $A \in f_\mathcal{A}(-1)$. $R_{\mathcal{A}}(A)$ is $\langle \alpha, k \rangle$ if $A \in f_\mathcal{A}(\langle \alpha, k \rangle)$.

Notice that if $\mathcal{A} \subseteq \mathcal{B}$ and $A \in S_1(L(\mathcal{B}))$ then $R_{\mathcal{B}}(A) \leq R_{\mathcal{A}}(A)$. If $\mathcal{A}$ is a saturated model and $\mathcal{B} \supseteq \mathcal{A}$ then $R_{\mathcal{B}}(A) = R_{\mathcal{A}}(A)$. If $A(\mathcal{A}) \subseteq B(\mathcal{A})$ then $R_{\mathcal{B}}(A) \leq R_{\mathcal{B}}(B)$. Finally if $R_{\mathcal{A}}(A) = \langle \alpha, k \rangle$ and $(\beta, m) < (\alpha, k)$ then there is a formula $B \in S_1(L(\mathcal{A}))$ such that $B(\mathcal{A}) \subseteq A(\mathcal{A})$ and $R_{\mathcal{B}}(B) = (\beta, k)$. Let $\mathcal{A}$ be a structure with one binary relation $R$ such that $R$ is an equivalence relation and for each $n$ there is a unique equivalence class with exactly $n$ elements but there are no infinite equivalence classes in $\mathcal{A}$. Then $Th(\mathcal{A})$ is totally transcendental and $R_{\mathcal{A}}(v_0 = v_0) = (1, 1)$. But for each positive integer $k$ there is an elementary extension $\mathcal{B}_k$ of $\mathcal{A}$ with $R_{\mathcal{B}_k}(v_0 = v_0) = (1, k)$ and there is an elementary extension $\mathcal{B}$ with $R_{\mathcal{B}}(v_0 = v_0) = (2, 1)$. It is an immediate consequence of Theorem 2 that if $\mathcal{A}$ is a model of a $K_1$-categorical theory $T$, $A \in S_1(L(\mathcal{A}))$, and $\mathcal{B} \supseteq \mathcal{A}$ then $R_{\mathcal{B}}(A) = R_{\mathcal{A}}(A)$. In fact this remark appears to be equivalent to Theorem 2.

In [4], Morley introduced for a countable first order theory $T$, $X \in \Pi_1T$, and $p \in S(X)$ the concept of the transcendental rank of $p$. In [2] Lachlan interprets this notion in terms of the rank of a formula $A$ in $S_1(L(\mathcal{A}))$ as follows

$$r_\mathcal{A}(A) = \begin{cases} -1 & \text{if } A(\mathcal{A}) = \emptyset, \\ \sup \{ |A| \mid (3p) p \in U_A \land p \in Tr^\mathcal{A}(\mathcal{A}) \} & \text{otherwise.} \end{cases}$$

We relate $r_\mathcal{A}(A)$ to $R_{\mathcal{A}}(A)$ in the following theorem.

**Theorem 1.** Let $\mathcal{A}$ be a model of a totally transcendental theory $T$ and $A \in S_1(L(\mathcal{A}))$.

(i) $r_\mathcal{A}(A) \geq \sup \{ |A| \mid (3k) \exists \mathcal{B} \geq \mathcal{A} \land R_{\mathcal{B}}(A) = (\alpha, k) \}.$

(ii) For some $\mathcal{B}$ an elementary extension of $\mathcal{A}$ and some integer $k$, $R_{\mathcal{B}}(A) = (r_{\mathcal{A}}(A), k)$.

(iii) $r_\mathcal{A}(A) = \sup \{ |A| \mid (3k) \exists \mathcal{B} \geq \mathcal{A} \land R_{\mathcal{B}}(A) = (\alpha, k) \}.$

(iv) For some elementary extension $\mathcal{B}$ of $\mathcal{A}$ and some positive integer $k$, $R_{\mathcal{B}}(A) = \sup \{ R_{\mathcal{B}}(A) \mid (\mathcal{B}) = (r_{\mathcal{A}}(A), k) \}.$
(v) If \( R(\varphi) = (\alpha, k) \) there is an elementary extension \( \mathcal{B} \) of \( \mathfrak{A} \) and a formula \( B \in S_1(L(\mathcal{B})) \) such that \( B(\mathcal{B}) \subseteq A(\mathfrak{A}) \) and \( R(\mathcal{B}) = (\alpha, 1) = \sup \{ R_C(\mathcal{B}) \mid C \supseteq \mathcal{B} \} \).

To prove this theorem we need the following extension of a lemma in [2].

**Lemma 1.** Let \( T \) be a first order theory, \( \mathfrak{A} \) a model of \( T \), \( A \in S_1(L(\mathfrak{A})) \) and suppose \( R(\varphi) = \alpha \) then for each \( \beta < \alpha \) there exists an elementary extension \( \mathcal{B} \) of \( \mathfrak{A} \) such that \( i^{-1} \{ U_A \cap Tr^\beta(\mathcal{B}) \} \) is infinite.

**Proof.** If the lemma is false there exists a model of \( T \) and a formula \( A \in S_1(L(\mathfrak{A})) \) with \( R(\varphi) = \alpha \) and some \( \beta < \alpha \) such that, for each \( \mathcal{B} \supseteq \mathfrak{A} \), \( i^{-1} \{ U_A \cap Tr^\beta(\mathcal{B}) \} \) is finite. Suppose \( q \in Tr^\beta(\mathcal{B}) \). Then for each \( C \supseteq \mathcal{B} \), \( i^{-1} (q) \cap S(\mathcal{C}) \) is a set of isolated points in \( S(\mathcal{C}) \). But then if \( A \in q \), \( i^{-1} (q) \cap S(\mathcal{C}) \) is a set of isolated points in \( S(\mathcal{C}) \) since \( i^{-1} (U_A) = i^{-1} (U_A) \) and \( i^{-1} (U_A) \cap Tr^\beta(\mathcal{B}) \) is finite. Thus \( q \in Tr^\beta(\mathcal{B}) \) but \( q \) was chosen in \( Tr^\beta(\mathcal{B}) \) so this is impossible. Hence \( i^{-1} \{ U_A \cap Tr^\beta(\mathcal{B}) \} \) is empty and by induction for each \( \gamma \geq \beta + 1 \), for each \( C \supseteq \mathfrak{A} \), \( Tr^\gamma(\mathcal{C}) \cap i^{-1} \{ U_A \} \) is empty. So \( R(\varphi) \neq \alpha \).

**Proof of Theorem 1.** (i) The proof proceeds by induction on \( R(\varphi) \). If \( R(\varphi) = -1 \) then \( F_\varphi \sim \exists \varphi \exists \varphi \) and so the theorem holds. Suppose, as the induction hypothesis, the theorem holds for a formula \( A \) if \( R(\varphi) = \gamma \) is less than \( \alpha \). We first prove that, for each \( \mathcal{B} \supseteq \mathfrak{A} \), \( R(\varphi) = \beta + 1 \). Then suppose \( q \in Tr^\beta(\mathcal{B}) \). Then there exists a sequence of formulas \( (A_i) \) each \( A_i \in S(\mathcal{L}(\mathfrak{J})) \) such that \( A_i \subseteq A_{i+1} \) and \( A_i \wedge A_j \) is \( \varnothing \) if \( i \neq j \). Then by induction, for each \( \gamma \geq \beta + 1 \), \( R(\varphi) = \gamma \) so there exists \( p_i \in U_A \cap S(\mathcal{B}) \). Then for each \( i \), since \( S(\mathcal{B}) \) is compact and \( U_A \) is closed, there exists \( p_i \), an accumulation point of the \( p_i \), such that \( p_i \in U_A \cap S(\mathcal{B}) \).

**Case 1.** \( \alpha \) is a successor ordinal, say \( \alpha = \lambda + 1 \). Since \( R(\varphi) = (\lambda + 1, 1) \) there exists a sequence of formulas \( (A_i) \) each \( A_i \subseteq A_{i+1} \), such that \( A_i \subseteq A_{i+1} \), \( A_i \wedge A_{i+1} \) is \( \varnothing \) if \( i \neq j \) and \( R(\varphi) = (\lambda, 1) \). Then by induction, for each \( \gamma \geq \beta + 1 \), \( R(\varphi) = \gamma \) so there exists \( p_i \in U_A \cap S(\mathcal{B}) \). Then for each \( i \), since \( S(\mathcal{B}) \) is compact and \( U_A \) is closed, there exists \( p_i \), an accumulation point of the \( p_i \), such that \( p_i \in U_A \cap S(\mathcal{B}) \).

**Case 2.** \( \alpha \) is a limit ordinal. \( \alpha \) has cofinality \( \omega \) since \( \alpha < \omega \) [2]. Then there exists a sequence of ordinals \( (\alpha_i) \) each \( \alpha_i \subseteq \omega \) and a sequence of formulas \( (A_i) \) each \( A_i \subseteq A_{i+1} \), such that \( A_i \subseteq A_{i+1} \), \( A_i \wedge A_{i+1} \) is \( \varnothing \) if \( i \neq j \) and \( R(\varphi) = (\alpha, 1) \) for each \( i \), and the \( \alpha_i \) increase monotonically to \( \alpha \). Then by induction, \( R(\varphi) \geq \alpha \) so there exists a type \( p_i \in U_A \cap Tr^\alpha(\mathcal{B}) \). Since \( U_A \) is closed and \( S(\mathcal{B}) \) is compact there exists \( p_i \), an accumulation point of the \( p_i \) for each \( i \). But \( p \notin Tr^\gamma(\mathcal{B}) \) for any \( \gamma < \alpha \) so \( p \in U_A \cap S(\mathcal{B}) \).

Since \( U_A \) is closed there exists \( p \), an accumulation point of the \( p_i \) and
\( p \in U_A \cap S^{a+1}(\mathfrak{B}_1) \) since each \( p_i \in U_A \cap \text{Tr}^a(\mathfrak{B}_1) \). Hence \( i^{*\mathfrak{B}_1}(p) \in U_A \cap S^{a+1}(\mathfrak{B}_1) \). But then \( r_{\mathfrak{B}_1}(A) \geq a + 1 \) so (i) is proved.

(ii) Now we show that there exists \( \mathfrak{B} \geq \mathfrak{A} \) such that for some \( k \), \( R_{\mathfrak{A}}(A) = (\alpha, k) \). By Lemma 1 since \( r_{\mathfrak{A}}(A) = \alpha \), for each \( \gamma < \alpha \) there exists an elementary extension \( \mathfrak{A}_\gamma \) of \( \mathfrak{A} \) such that \( i^{*\mathfrak{A}_\gamma}(U_A) \cap \text{Tr}^\gamma(\mathfrak{A}_\gamma) \) is infinite. Hence there exists a sequence of formulas \( (A_i^\gamma)_{i<\omega} \) such that \( A_i^\gamma \in S_1(L(\mathfrak{A}_\gamma)) \), \( A_i^\gamma \subseteq A(\mathfrak{A}_\gamma) \), and \( r_{\mathfrak{A}_\gamma}(A_i^\gamma) = \gamma \). By induction there exists \( \mathfrak{A}_\alpha \) such that for each \( \gamma \) and some \( k \), \( iR_{\mathfrak{A}_\gamma,i}(A_i^\gamma) = (\gamma, k) \). Without loss of generality we may assume \( (|\mathfrak{A}_\gamma, i| - |\mathfrak{A}_\delta, i|) \cap (|\mathfrak{A}_\delta, j| - |\mathfrak{A}_\delta|) = \emptyset \) if \((\gamma, i) \neq (\delta, j)\). There exists a model \( \mathcal{C} \) such that for each \((\gamma, i), \mathcal{C} \geq \mathfrak{A}_\gamma \), by the compactness theorem. Then for each \( \gamma < \omega \) there exists a \( \delta \) such that \( R_{\mathfrak{A}}(A^\gamma_\delta) \geq (\gamma, k) \). So \( R_{\mathfrak{A}}(A) \geq (\alpha, 1) \). Since each \( \mathfrak{B} \geq \mathfrak{A} \), \( R_{\mathfrak{B}}(A) \leq (\alpha + 1, 1) \) by (i), for some \( k \), \( R_{\mathfrak{C}}(A) = (\alpha, k) \) and \( \mathcal{C} \) is the required model.

(iii) This follows immediately from (i) and (ii).

(iv) We must find \( \mathfrak{B} \geq \mathfrak{A} \) and a positive integer \( k \), such that \( R_{\mathfrak{B}}(A) = (r_{\mathfrak{A}}(A), k) \). By (ii) choose \( \mathfrak{B}_0 \geq \mathfrak{A} \) such that, for some \( k \), \( R_{\mathfrak{B}_0}(A) = (r_{\mathfrak{B}}(A), k) \). Then applying (i) for each \( \mathfrak{C} \geq \mathfrak{B}_0 \) there is an integer \( k \) such that \( R_{\mathfrak{C}}(A) = (r_{\mathfrak{C}}(A), k) \). It suffices to show that the set of such \( k \) is bounded. If not, there exists an increasing sequence of positive natural numbers \( m_n \) and a sequence of models \( \mathfrak{B}_n \) such that \( \mathfrak{B}_n \geq \mathfrak{B}_0 \) and \( R_{\mathfrak{B}_n}(A) = (r_{\mathfrak{B}_n}(A), n) \). We may assume that, if \( m \neq n \), \( (|\mathfrak{B}_n| - |\mathfrak{B}_m|) \cap (|\mathfrak{B}_n| - |\mathfrak{B}_0|) = \emptyset \). By the compactness theorem there exists a model \( \mathcal{D} \) which elementarily extends each \( \mathfrak{B}_n \). But then \( R_{\mathcal{D}}(A) \geq (r_{\mathcal{D}}(A) + 1, 1) \) contrary to (i). Hence there exists a maximum \( k \) and an elementary extension \( \mathfrak{B} \) of \( \mathfrak{B}_0 \) such that \( R_{\mathfrak{B}}(A) = (r_{\mathfrak{B}}(A), k) \).

(v) We will construct a sequence of models \( \mathfrak{B}_i \) and formulas \( B_i \in S_1(L(\mathfrak{B}_{i-1})) \) such that \( B_{i+1}(\mathfrak{B}_i) \subseteq B_i(\mathfrak{B}_i) \), \( R_{\mathfrak{B}_i}(B_{i+1}) = (\alpha, 1) \), \( R_{\mathfrak{B}_i}(B_{i+1}) = \sup \{ R_{\mathcal{C}}(B_{i+1}) \mid \mathfrak{C} \geq \mathfrak{B}_i \} \) and if \( R_{\mathfrak{B}_i}(B_i) > (\alpha, 1) \) then \( R_{\mathfrak{B}_{i+1}}(B_{i+1}) < R_{\mathfrak{B}_i}(B_i) \). Since there is no infinite descending sequence in a well ordered set, for some \( i \), \( R_{\mathfrak{B}_i}(B_i) = (\alpha, 1) \) and letting \( \mathfrak{B} = \mathfrak{B}_i \) and \( B = B_i \) proves (v). Let \( \mathfrak{B}_0 \geq \mathfrak{A} \) and \( B_0 = A \). Suppose \( \mathfrak{B}_i \) and \( B_i \) have been chosen for \( i < n \). Let \( B_n \in S_1(L(\mathfrak{B}_{n-1})) \) such that \( B_n(\mathfrak{B}_{n-1}) \subseteq B_{n-1}(\mathfrak{B}_{n-1}) \) and \( R_{\mathfrak{B}_{n-1}}(B_{n-1}) = (\alpha, 1) \). Then by (iv) choose \( \mathfrak{B}_n \geq \mathfrak{B}_{n-1} \) such that \( R_{\mathfrak{B}_n}(B_n) = \sup \{ R_{\mathfrak{C}}(B_n) \mid \mathfrak{C} \geq \mathfrak{B}_n \} \). If \( R_{\mathfrak{B}_n}(B_n) > (\alpha, 1) \) then both \( R_{\mathfrak{B}_n}(B_n \wedge B_{n+1}) \) and \( R_{\mathfrak{B}_n}(B_n \wedge B_{n+1}) \) are greater than or equal to \( (\alpha, 1) \). Hence, if \( R_{\mathfrak{B}_{n+1}}(B_{n+1}) = R_{\mathfrak{B}_n}(B_n), R_{\mathfrak{B}_{n+1}}(B_n) > R_{\mathfrak{B}_n}(B_n) \) contrary to the choice of \( \mathfrak{B}_n \).
At the suggestion of the referee we include the following comparison of the rank defined here with that defined by Shelah in his paper on the uniqueness of prime models [6].

Shelah chooses a sufficiently saturated model $\mathcal{M}$ of $T$ (for $T$ totally transcendental a countable saturated model suffices) and defines for $A \in S_1(L(\mathcal{M}))$,

(A) $\rho(A) = -1$ iff $\mathcal{M} \not\models \exists \nu_0 A$.

(B) $\rho(A) = \alpha$ iff

(1) $\mathcal{M} \models \exists \nu_0 A$,

(2) for no $\beta < \alpha$, $\rho(A) = \beta$,

(3) for no $B \in S_1(L(\mathcal{M}))$ do both $A \land B$ and $A \land \lnot B$ satisfy (1) and (2).

(C) $\rho(A) = \infty$ if $\rho(A)$ is not defined by (A) and (B). $\infty$ is assumed greater than each ordinal.

Shelah proves that if $T$ is totally transcendental then $\rho(A) < \infty$. The following theorem indicates the relation between $R_{\mathfrak{d}}(A)$ and $\rho(A)$ if $Th(\mathfrak{f})$ is totally transcendental.

Theorem 1'. Let $T$ be a totally transcendental theory and $\mathfrak{f}$ a saturated model of $T$ then, for $A \in S_1(L(\mathfrak{f}))$, $R_{\mathfrak{d}}(A) = (\alpha, k)$ if and only if $\rho(A) = \omega \cdot \alpha + m$ where $2^m \leq k < 2^{m+1}$. $R_{\mathfrak{d}}(A) = 1$ if and only if $\rho(A) = -1$.

Proof. Since $\mathfrak{f}$ is a saturated model of $T$ we may take $\mathfrak{f}$ for $\mathcal{M}$ in the definition of $\rho(A)$. The result is evident if $R_{\mathfrak{d}}(A) = -1$. For the rest we induct on $R_{\mathfrak{d}}(A)$. It is easy to verify that $R_{\mathfrak{d}}(A) = (0, 1)$ if and only if $\rho(A) = 0$.

Now suppose the conclusion holds for each $A \in S_1(L(\mathfrak{f}))$ with $R_{\mathfrak{d}}(A) < (\alpha, k)$ and choose an $A \in S_1(L(\mathfrak{f}))$ with $R_{\mathfrak{d}}(A) = (\alpha, k)$.

Case 1. Let $k = 1$. To show $\rho(A) \geq \omega \cdot \alpha$ it suffices by [6, Theorem 1.1 A, B], as $T$ is totally transcendental, to show there is an increasing sequence of ordinals $\langle \gamma_i \rangle_{i < \omega}$ tending to $\omega \cdot \alpha$ and a collection of formulas $B_i \in S_1(L(\mathfrak{f}))$ such $\rho(A \land B_i) \geq \gamma_i$ and $\rho(A \land \lnot B_i) \geq \gamma_i$. Let $\langle \delta_i, k_i \rangle$ be an increasing sequence tending to $\langle \omega, 1 \rangle$. For each $i$, choose $B_i, B'_i \in S_1(L(\mathfrak{f}))$ such that $B_i(\mathfrak{f}) \subseteq A(\mathfrak{f}), B'_i(\mathfrak{f}) \subseteq A(\mathfrak{f}), B_i(\mathfrak{f}) \cap B'_i(\mathfrak{f}) = \emptyset$ and $R_{\mathfrak{d}}(B_i) = R_{\mathfrak{d}}(B'_i) = (\delta_i, k_i)$. Then by induction $\rho(A_i \land B_i) = \omega \cdot \delta_i + m_i$ and $\rho(A \land \lnot B_i) \geq \omega \cdot \delta_i + m_i$ where $2^{m_i} \leq k_i < 2^{m_i+1}$. Let $\gamma_i = \omega \cdot \delta_i + m_i$; we have an appropriate sequence.

But for each formula $B \in S_1(L(\mathfrak{f}))$ either $R_{\mathfrak{d}}(A \land B) < (\alpha, 1)$ or $R_{\mathfrak{d}}(A \land \lnot B) < (\alpha, 1)$. Say $R_{\mathfrak{d}}(A \land B) = (\beta, k) < (\alpha, 1)$. Then by induction $\rho(A \land B) = \omega \cdot \beta + m < \omega \cdot \alpha$ where $2^m \leq k < 2^{m+1}$. Hence $\rho(A) \leq \omega \cdot \alpha$ so $\rho(A) = \omega \cdot \alpha$.

Case 2. Suppose $k > 1$, and $2^m \leq k < 2^{m+1}$. Let $B \in S_1(L(\mathfrak{f}))$, then either $R_{\mathfrak{d}}(A \land B) < (\alpha, 2^m)$ or $R_{\mathfrak{d}}(A \land \lnot B) < (\alpha, 2^m)$ since $R_{\mathfrak{d}}(A \land B) = (\alpha, 2^m)$ and $R_{\mathfrak{d}}(A \land \lnot B) = (\alpha, 2^m)$ implies $R_{\mathfrak{d}}(A) > (\alpha, 2^{m+1}) > (\alpha, k)$. Hence by induction $\rho(A \land B) < \omega \cdot \alpha + m$ or $\rho(A \land \lnot B) < \omega \cdot \alpha + m$. Thus $\rho(A) < \omega \cdot \alpha + m$. 

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There exist formulas $B_1, \ldots, B_k \in S_1(L(\bar{q}))$ such that the $B_i(L)$ partition $A(L)$ and each $R_\bar{q}(B_i) = (\alpha, 1)$. Let $B = \bigvee_{i=1}^{2m-1} B_i$. Then by induction $\rho(A \land B) = \omega \cdot \alpha + (m - 1)$ and $\rho(A \land \sim B) \geq \omega \cdot \alpha + (m - 1)$ so by [6, Theorem 1.1B] $\rho(A) \geq \omega \cdot \alpha + m$. Thus $\rho(A) = \omega \cdot \alpha + m$.

**Corollary to Main Theorem.** If $T$ is $\aleph_1$-categorical, $\bar{q} \models T$ and $A \in S_1(L(\bar{q}))$, $\rho(A) < \omega \cdot \omega$.

**Proof.** This is immediate from Theorem 1' and Theorem 3.

We now restrict our attention to $\aleph_1$-categorical theories. In particular, we will deal with an $\aleph_1$-categorical theory $T$ with a specified strongly minimal formula $D$ such that, for each model $\bar{q}$ of $T$, $D(\bar{q}) \cap \text{cl}(\emptyset)$ is infinite.

We want to assign to each formula $B \in S_1(L(\bar{q}))$ a formula $B^*$ which "witnesses" the rank of $B$. In order to do this we consider formulas $A \in S_1(L)$ for each $l$. To each $A$ and for each $n$ we assign a class $\Gamma^{(n)}_A$ of possible witnesses. Each $\Gamma_A^{(n)}$ is a set of $l$-ary formulas such that there is a positive integer $k$ with $R_\bar{q}(A(a_1, \ldots, a_l)) = (n, k)$ if and only if, for some $A^* \in \Gamma^{(n)}_A$, $(\bar{q} \models A(a_1, \ldots, a_l))$.

The simplest cases are as follows. If $A(L)$ is finite, $A^*$ tells how many elements are in $A(L)$. If $A$ is strongly minimal $A^*$ expresses $A$ as a "uniform union of finite sets" over the fixed strongly minimal set $D$. In the following definition $A^*$ will be in $\Phi^{(n)}_A$ just when $R_\bar{q}(A) = (n, 1)$. The definition of $\Theta^{(n)}_A$ arises from the intuition that $R_\bar{q}(A) = (n, k)$ when $A(L)$ is a union of finitely many definable sets with rank $(n, 1)$.

For each natural number $l$, for each $A \in S_{l+1}(L)$ and to $-1$ and each natural number $n$ assign a set of formulas as follows

$$\Gamma^{(-1)}_A = \{ \exists v_0 A \},$$

$$\Phi^{(0)}_A = \{ \exists v_0 A \land \exists^k v_0 A \mid 0 < k < \omega \},$$

$$\Phi^{(n)}_A = \{ \exists v_{l+1}, \ldots, \exists v_k (\forall v_0 (A \leftrightarrow \exists v_{k+1} (C \land D(v_{k+1})) \land C^*))$$

$$\land (\forall v_0 (A \rightarrow \exists^p v_{k+1} (C \land D(v_{k+1}))))$$

$$\land (\exists^p v_{k+1} \exists^q v_0 (D(v_{k+1}) \land C \land (\sim A \lor C^*)))) \mid 0 < p < \omega, l \leq k < \omega, C \in S_{k+2}(L), \text{ and } C^* \in \Gamma^{(n-1)}_C,$$

$$\Theta^{(n)}_A = \{ \exists v_{l+1}, \ldots, \exists v_k (\forall v_0 (A \leftrightarrow (A_1 \lor \cdots \lor A_s)) \land A^*_1 \land \cdots \land A^*_s) \mid l \leq k < \omega, A_i \in S_{k+2}(L), s < \omega \text{ each } A^*_i \in \bigcup_{r < n} \Gamma^{(r)}_A \cup \Phi^{(n)}_A$$

$$\land \text{ and some } A^*_i \in \Phi^{(n)}_A,\},$$

$$\Gamma^{(n)}_A = \Phi^{(n)}_A \cup \Theta^{(n)}_A.$$
Note that if $A \in S_{I+1}(L)$ and $A^* \in \Gamma_A^{(n)}$ for some $n$, then $A^*$ has free variables $v_1, \ldots, v_l$. Thus when we write $A^*(a_1, \ldots, a_l)$ we mean the result of substituting $a_i$ for $v_i$ for $i = 1, 2, \ldots, l$. We abbreviate $A_{v_1, \ldots, v_l}(a_1, \ldots, a_l)$ by $A(a_1, \ldots, a_l)$. Thus $A(a_1, \ldots, a_l) \in S_1(L|\{a_1, \ldots, a_l\})$.

**Theorem 2.** Let $T$ be an $\kappa_1$-categorical theory and $D$ a strongly minimal formula in $T$ such that, in each model $B$ of $T$, $D(B) \cap cl(\emptyset)$ is infinite. Let $\bar{\mathfrak{A}}$ be a model of $T$, $m \in \{-1\} \cup \omega$, $A \in S_{I+1}(L)$, and $a_1, \ldots, a_l \in |\bar{\mathfrak{A}}|$. The following two propositions are equivalent.

(i) There exists a formula $A^* \in \Gamma_A^{(n)}$ such that $\bar{\mathfrak{A}} \models A^*(a_1, \ldots, a_l)$.

(ii) For some $k R_{\mathfrak{A}}(A(v_0, a_1, \ldots, a_l)) = (m, k)$ if $m \geq 0$. If $m = -1$, $R_{\mathfrak{A}}(A(v_0, a_1, \ldots, a_l)) = -1$.

Notice that there is no loss of generality in this theorem because of our assumption that $T$ has a strongly minimal formula $D$ and that, for each model $\mathfrak{B}$ of $T$, $D(\mathfrak{B}) \cap cl(\emptyset)$ is infinite. For, let $T$ be an arbitrary $\kappa_1$-categorical theory in a first order language $L$. Then there is a principal extension $T'$ of $T$ with a strongly minimal formula $D'$. Let $\bar{\mathfrak{A}}$ be a prime model of $T'$. Let $X$ be an infinite subset of $D'(\bar{\mathfrak{A}})$. Then $Th(\bar{\mathfrak{A}}, X) = T''$ is a theory of the specified kind. Suppose $\mathfrak{B}$ is a model of $T''$, $A \in S_{I+1}(L)$, $A^* \in \Gamma_A^{(m)}$ for some $m$, and $a_1, \ldots, a_l \in |\mathfrak{B}|$. Then $\mathfrak{B} \models A^*(a_1, \ldots, a_l)$ if and only if $\mathfrak{B} \models A^*(a_1, \ldots, a_l)$. Moreover, $R_{\mathfrak{B}}(A(v_0, a_1, \ldots, a_l)) = R_{\mathfrak{A}}(A(v_0, a_1, \ldots, a_l))$. Thus it suffices to prove the theorem for $T''$.

**Proof of theorem.** The proof proceeds by induction on $m$. If $m = -1$, $\bar{\mathfrak{A}} \models A^*(a_1, \ldots, a_l)$ for some $A^* \in \Gamma_A^{(-1)}$ if and only if $A(v_0, a_1, \ldots, a_l)(\bar{\mathfrak{A}}) = \emptyset$ which is equivalent to $R_{\mathfrak{A}}(A(v_0, a_1, \ldots, a_l)) = -1$. We assume the theorem is true for $m < n$ and prove (i) implies (ii) for $m = n$. Then we prove a lemma. Finally we assume the theorem holds for $m < n$ and prove (ii) implies (i) for $m = n$.

To prove (i) implies (ii) consider a formula $A \in S_{I+1}(L)$ and a formula $A^* \in \Gamma_A^{(n)}$ such that $\bar{\mathfrak{A}} \models A^*(a_1, \ldots, a_l)$ with $a_1, \ldots, a_l \in |\bar{\mathfrak{A}}|$. Notice first that it suffices to prove the case in which $A^* \in \Phi_A^{(n)}$. For, suppose that (i) implies (ii) has been shown for each integer $l$, each $A \in S_j(L)$ and each $A^* \in \Phi_A^{(n)}$ and that $A^* \in \Theta_A^{(n)}$. Then since $\bar{\mathfrak{A}} \models A^*(a_1, \ldots, a_l), A(v_0, a_1, \ldots, a_l)(\bar{\mathfrak{A}}) = \bigcup_{i=1}^k (A_i(v_0, a_1, \ldots, a_k)(\bar{\mathfrak{A}}))$ for some $a_{l+1}, \ldots, a_k$ in $|\bar{\mathfrak{A}}|$ and some $A_1, \ldots, A_k$. Moreover, for each $i$, $\bar{\mathfrak{A}}$ satisfies $A^*(a_1, \ldots, a_k)$ and each $A^* \in \bigcup_{i=1}^{n-1} \Gamma_A^{(n-1)} \cup \Phi_A^{(n)}$. So for each $i$ there exists $n_i \leq n$ and a $k_i$ such that $R_{\mathfrak{A}}(A_i(v_0, a_1, \ldots, a_l)) = (n_i, k_i)$ and for some $i$ there exists $k$ such that $R_{\mathfrak{A}}(A_i(a_1, \ldots, a_l)) = (n, k)$, by induction and the assumption that the theorem holds for each $B^* \in \Phi_B^{(n)}$. But then $R_{\mathfrak{A}}(A(a_1, \ldots, a_l)) = (n, m)$ for some integer $m$. License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Thus to prove (i) implies (ii) when \( m = n \), let \( A \in S_{l+1}(L) \) and suppose
\[ \bar{\theta} \models A^*(a_1, \ldots, a_l) \text{ where } A^* \in \Phi^*_A(n). \]
Letting \( A' = A(v_0, a_1, \ldots, a_l) \) we wish to prove that, for some \( q \), \( R_{\theta}(A') = (n, q) \). From the definition of \( \Phi^*_A(n) \) we see \( A^* \) has the form
\[
3v_{l+1}, \ldots, 3v_k \forall v_0(A \leftrightarrow 3v_{k+1}(C \land D(v_{k+1}) \land C^*)) \]
\[ \land \left( \forall v_0(A \rightarrow 3^p v_{k+1}(C \land D(v_{k+1}))) \right) \]
\[ \land 3^{p^p} v_{k+1}3v_C(D(v_{k+1}) \land C \land (\sim A \lor C^*)) \]
where \( p \) is a positive integer, \( l \leq k < \omega \), \( C \) is in \( S_{k+2}(L) \) and \( C^* \) is in \( \Gamma^+_C^{(n-1)} \).
Since \( \bar{\theta} \models A^*(a_1, \ldots, a_l) \) there exist \( a_{l+1}, \ldots, a_k \in |\bar{\theta}| \) such that, for all but \( p \) elements \( b \) of \( D(\bar{\theta}) \), \( \bar{\theta} \models C^*(a_1, \ldots, a_k, b) \). Thus, for any \( \bar{\theta}' \supseteq \bar{\theta} \) and \( d \in D(\bar{\theta}' \setminus D(\bar{\theta})) \), \( \bar{\theta}' \models C^*(a_1, \ldots, a_k, d) \).

By induction, for some \( s \), \( R_{\bar{\theta}'}(C^'_v v_{k+1}(d)) = (n-1, s) \) where \( C' = C(v_0, a_1, \ldots, a_k, v_{k+1}) \). Then \( R_{\bar{\theta}'}(A') \subseteq (n, s) \). For, if not there exist \( L \)-formulas \( B_1, \ldots, B_s \) where each \( B_i \) has free variables \( v_0, v_{k+2}, \ldots, v_m \) with the following properties. There exist constants \( a^i_{k+2}, \ldots, a^i_m \in |\bar{\theta}| \) such that if \( B_i = B_i(v_0, v_{k+2}, \ldots, v_m), \) \( B_i(\bar{\theta}) \subseteq A'(\bar{\theta}), \) \( B_i(\bar{\theta}) \cap B_j(\bar{\theta}) = \emptyset \) if \( i \neq j \), and \( R_{\bar{\theta}'}(B_i) > (n, 1) \). We will show that this condition implies for each elementary extension \( \bar{\theta}' \) of \( \bar{\theta} \), each \( d \in D(\bar{\theta}' \setminus D(\bar{\theta})) \), and each \( i \) that \( R_{\bar{\theta}'}(B_i \land C^v_{k+1}(d)) \geq (n-1, 1) \). This in turn implies \( R_{\bar{\theta}'}(C^'_v v_{k+1}(d)) > (n-1, s) \) which is a contradiction allowing us to conclude that \( R_{\bar{\theta}'}(A') \subseteq (n, s) \).

Suppose \( R_{\bar{\theta}'}(B_i) > (n, 1) \) and for some \( \bar{\theta}' \supseteq \bar{\theta} \) and some \( d \in D(\bar{\theta}' \setminus D(\bar{\theta})) \), \( R_{\bar{\theta}'}(B_i \land C^v_{k+1}(d)) < (n-1, 1) \). By induction there exists a formula \( (B'_i \land C^*) \in \Gamma_{B_i \land C}^{(r)} \) for some \( r < n-1 \) such that \( \bar{\theta}' \models (B'_i \land C^*)(a_1, \ldots, a_{k'}, d, a^i_{k+2}, \ldots, a^i_m) \).
Since \( D \) is strongly minimal, there exists \( p_1 \in \omega \) which may be assumed larger than \( p \) such that, for all but \( p \) members of \( D(\bar{\theta}') \), \( \bar{\theta}' \models (B'_i \land C^*)(a_1, \ldots, a_{k'}, b, a^i_{k+2}, \ldots, a^i_m) \). Consider the formulas
\[
F = 3v_{k+1}(D(v_{k+1}) \land (B_i \land C) \land (B'_i \land C^*)) ,
\]
\[
G = (\forall v_0(F \leftrightarrow F)) \land (\forall v_0(F \rightarrow 3^{p^p} v_{k+1}(D(v_{k+1}) \land (B_i \land C)))) \land (3^{p^p} v_{k+1}3v_C(D(v_{k+1}) \land (B_i \land C) \land (\sim F \lor \sim (B_i \land C)^*))) ,
\]
\[
H = 3v_0F .
\]
If \( r = -1 \) let \( F^* = H \); otherwise let \( F^* = G \). Then \( F^* \in \Gamma^+_F \cup \Gamma^+_F^{+1} \) and
\( \bar{\theta} \models F^*(a_1, \ldots, a_{k'}, a^i_{k+2}, \ldots, a^i_m) \) so if \( F' \) is the formula \( (v_0, a_1, \ldots, a_{k'}, a^i_{k+2}, \ldots, a^i_m) \) by induction there is an integer \( q \) such that \( R_{\bar{\theta}'}(F') = (r+1, q) < (n, 1) \). For each element \( c \in B_i'(\bar{\theta}') \) there exists an element \( b \) in \( D(\bar{\theta}') \) such that \( \bar{\theta}' \models A^*(c_1, \ldots, a_k, b) \) since \( B_i'(\bar{\theta}') \subseteq A'(\bar{\theta}') \) and \( \bar{\theta}' \models A^*(a_1, \ldots, a_k) \). Let

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Let \( b_1, \ldots, b_q \) be an enumeration of the elements \( b \in D(\mathfrak{g}) \) such that

\[ \mathfrak{g} \models C^*(a_1, \ldots, a_k, b) \land \sim (B_i \land C^*)(a_1, \ldots, a_k, b, a_{k+2}, \ldots, a_i). \]

We know there are only finitely many such \( b \) from above. Then

\[ R^*_\mathfrak{g}(B_i \land C^*_{v+1}(b)) \leq R^*_\mathfrak{g}(C^*_{v+1}(b)) = (n-1, u) \]

for some \( u < \omega \) by induction. But

\[ \mathfrak{g} \models \forall \nu_0 \left( B_i' \leftrightarrow F' \lor \bigvee_{j=1}^q (B_i \land C^*_{v+1}(b)) \right). \]

So \( B_i'(\mathfrak{g}) \) is the union of a finite number of definable sets each with rank less than \( (n, 1) \) and thus \( R^*_\mathfrak{g}(B_i') \leq (n, 1) \) contrary to assumption. Thus we conclude as outlined above \( R^*_\mathfrak{g}(A') \leq (n, s) \). Since \( \mathfrak{g} \models \forall \nu_0 \exists \nu_1^0 \nu_1^0(C', R^*_\mathfrak{g}(A') \geq (n, 1)) \)

Therefore there exists an \( l, 1 \leq l \leq s \), such that \( R^*_\mathfrak{g}(A') = (n, l) \). We have shown (i) implies (ii) when \( m = n \).

Lemma 2. Let \( \mathfrak{g} \models T, A \in S_{l+1}(L), a_1, \ldots, a_l \in |\mathfrak{g}|, A' = A(v_0, a_1, \ldots, a_l) \) and \( \alpha \leq \omega \). Suppose the theorem holds for each \( m < \alpha \) and that for each \( \mathfrak{g} \models \exists \nu_0 \nu_1^0 \nu_1^0(C', R^*_\mathfrak{g}(A') \leq (n, s) \)

there is some \( k \) such that \( R^*_\mathfrak{g}(A') = (\alpha, k) \), then there exists \( \alpha < \alpha \) and \( A* \in \Gamma_{\mathfrak{A}}(r+1) \) such that \( \mathfrak{g} \models A*(a_1, \ldots, a_l) \).

Proof. Adjoin a new unary predicate symbol \( q \) to \( L \) to form \( L' \) and a new constant symbol \( f \) to \( L' \) to form \( L'' \). Let \( \Delta \) be the set of \( L' \) sentences which are true in an \( L' \) structure \( \mathfrak{C} \) just if there is an elementary substructure \( \mathfrak{C}^* \) of the reduct of \( \mathfrak{C}' \) to \( L \) such that \( |\mathfrak{C}^*| = q(C') \). Let \( D^n \) be the \( L' \) sentence \( \exists \nu_0 \nu_1^0 \nu_1^0(D \land \sim q) \). Let \( \Gamma_1 \) be the set of sentences

\[ \{ \text{elementary diagram of } \mathfrak{g} \cup \Delta \cup D^n \cup \{ \nu_0 \nu_1^0 \nu_1^0(q(a)) \ a \in |\mathfrak{g}| \}. \]

If \( k < \omega \) and \( F \in S_{k+2}(L) \) consider the following formulas.

Let \( m = l + k \).

Let \( F_1 \in S_{m+2}(L) \) be the formula

\[ F(v_0, v_{l+1}, \ldots, v_{m}, v_{m+1}) \land A. \]

Let \( F_1^* \) be in \( S_{m+1}(L) \).

Let \( G(F, F_1^*) = \exists \nu_{m+1}^0(D(v_{m+1}) \land F_1 \land F_1^*) \).

Let \( G^*(F, F_1^*, p) \) be

\[ (\forall \nu_0 (G(F, F_1^*) \leftrightarrow G(F, F_1^*))) \land (\forall \nu_0 (G(F, F_1^*) \rightarrow \exists \nu_0^1 \nu_0^1(D(v_{m+1}) \land F_1))) \land 3^p v_{m+1} \exists \nu_0 (D(v_{m+1}) \land (\sim G(F, F_1^*) \lor \sim F_1^*)) \]

Then if \( F_1^* \) is in \( \Gamma_{F_1^*}^{(s)} \), \( G^*(F, F_1^*, p) \) is in \( \Gamma_{G(F, F_1^*)}^{(s+1)} \). Let \( \Gamma_2 \) be the set of sentences
\[ \Gamma_1 \cup \{ \forall \alpha(f) \land \sim q(f) \} \cup \left\{ \forall (G(F, F^*)_{\alpha}, a_1, \ldots, a_\nu, b_{l+1}, \ldots, b_m) \right. \\
\left. \land G^*(F, F^*, p)(a_1, \ldots, a_\nu, b_{l+1}, \ldots, b_m) \right\} \\
\text{for } k \in \omega \text{ let } F \in S_{k+2}(L), \\
F^*_1 \in \bigcup_{u < \alpha} \Gamma_{F_1}^{(u)} b_{l+1}, \ldots, b_m, \in |\bar{a}| \right\}. \\
\]

Now we show that \( \Gamma_2 \) is inconsistent by finding for each \( L^\alpha \) structure \( C^\alpha \) such that \( C^\alpha \models \Gamma_1 \), for each element \( f \in (A^\alpha \land \sim q)(C^\alpha) \) formulas \( F \) and \( F^*_1 \), an integer \( p \), and constants \( c_{l+1}, \ldots, c_m \) such that

\[ C^\alpha \models G(F, F^*)_{\alpha}(f, q_1, \ldots, a_\nu, c_{l+1}, \ldots, c_m) \]
\[ \land G^*(F, F^*, p)(a_1, \ldots, a_\nu, c_{l+1}, \ldots, c_m). \]

Let \( C^\alpha \models \Gamma^*_1 \) and \( |\bar{B}| = q(C^\alpha). \) Let \( C = C^\alpha | L. \) \( \bar{B} \) is an \( L \)-structure. Let \( C_1 \) be an \( L \)-structure prime over \( |\bar{a}| \cup \{ f \} \). Then \( D(C_1) = D(\bar{B}) \neq \varnothing. \) For, suppose \( D(C_1) \subseteq D(\bar{B}) \) and let \( \bar{B}_1 \) be prime over \( D(C_1). \) \( (\bar{B}_1, C_1) \) exist by 4.3 of [7]. Then \( C_1 = \bar{B}_1 \) for if not \( \bar{B}_1 \subseteq C_1 \) while \( D(\bar{B}_1) = D(\bar{C}_1). \) But then \( \bar{C}_1 \) and \( C_1 \) are models of \( T \) which satisfy the hypothesis of the two cardinal theorem so \( T \) is not \( \aleph_1 \)-categorical. For, by the two cardinal theorem [5] there is a model \( \bar{\alpha} \) of \( T \) with \( \kappa(\bar{\alpha}) = \aleph_1 \) and \( \kappa(D(\bar{\alpha})) = \aleph_0. \) But there is certainly a model \( \bar{B} \) of \( T \) with \( \kappa(\bar{B}) = \aleph_1 \) and \( \kappa(D(\bar{B})) = \aleph_0. \) Thus there exists \( d \in D(\bar{C}_1) \land \sim D(\bar{B}). \)

Let \( C \in S_{k+2}(L) \) and \( c_1, \ldots, c_k \in |\bar{\alpha}| \) such that \( C(f, c_1, \ldots, c_k, v_{k+1}) \) generates the principal \( 1 \)-type in \( Th(C, |\bar{a}| \cup \{ f \}) \) realized by \( d. \) Then \( C(f, c_1, \ldots, c_k, v_{k+1})(C) \) is finite. For if not, since \( D \) is strongly minimal and contains infinitely many algebraic points there exists an algebraic point \( b \in |\bar{\alpha}| \) such that \( C \models C(f, c_1, \ldots, c_k, b). \) Since \( b \) is algebraic there exists a formula \( B \in S_1(L) \) and an integer \( t \) such that \( C \models B(b) \land \exists v_0 B. \) But since \( C \models C(f, c_1, \ldots, c_k, b), \) \( C(f, c_1, \ldots, c_k, v_{k+1}) \) generates a principal type and \( C(f, c_1, \ldots, c_k, v_{k+1})(C) \) is infinite, \( B(C) \) is infinite. So for some \( q < \omega, \)

\[ C \models C(f, c_1, \ldots, c_k, v_{k+1}) \land \exists v_0 B. \]

Let \( C_1 \) be the following member of \( S_{m+2}(L). \)

\[ C_{v_1, \ldots, v_{k+1}}(v_{l+1}, \ldots, v_{m+1}) \land A \land \exists v_{m+1} C_{v_1, \ldots, v_{k+1}}(v_{l+1}, \ldots, v_{m+1}). \]

Let \( C_1' \) be obtained from \( C_1 \) by substituting \( a_1, \ldots, a_\nu \) for \( v_1, \ldots, v_\nu \) and \( c_1, \ldots, c_k \) for \( v_{l+1}, \ldots, v_m. \) For any \( b \in D(C) = D(\bar{B}), \) \( R_C(C_1'_{v_{m+1}}(b)) = R_C(C_1'_{v_{m+1}}(d)) \) since any such \( b \) realizes the same \( 1 \)-type in
**A. T** IS FINITE FOR \( \mathfrak{K} \)-CATEGORICAL \( T \)

Since \( D(C) - D(\mathfrak{B}) \) is infinite and \( \bar{a}_1, \ldots, a_n, c_1, \ldots, c_k \) as \( d \) and \( C \) is homogeneous by Theorem 0. So for some \( u < a \) and some \( k, R_{\mathfrak{B}}(C_1^{u+1}(d)) = (u, k) \). Thus by hypothesis, there exists a formula \( C^* \in \Gamma(u) \) such that \( C \models C^*(a_1, \ldots, a_n, c_1, \ldots, c_k, d) \). Let \( p \) be the maximum of \( q \) and the cardinality of \( \sim C^*(a_1, \ldots, a_n, c_1, \ldots, c_k) \) which is a finite subset of \( D(C^*) \). Then

\[
C^* \models A'(\bar{a}) \land q(\bar{a}) \land G(C, C^*(\bar{a})) \land G^*(C, C^*, p)(a_1, \ldots, a_n, c_1, \ldots, c_k)
\]

so \( C^* \) does not model \( \Gamma_2 \) but \( C^* \) was an arbitrary model of \( \Gamma_1 \), so \( \Gamma_2 \) is inconsistent. By the compactness theorem, there exists \( k \in \omega, F_1, \ldots, F_s \) in \( S_{k+2}(L) \) and \( F_1^* \in V \) for some \( t < a \) such that

\[
\Gamma_1 \models \left( \forall v_0 \left( A'(v_0) \land q(v_0) \rightarrow \bigvee_{i=1}^{s} G(F_i, F_1^*)(a_1, \ldots, a_n, c_1, \ldots, c_k) \right) \right)
\]

\[
\land \left( \bigwedge_{i=1}^{s} G^*(F_i, F_1^*, p_i)(a_1, \ldots, a_n, c_1, \ldots, c_k) \right).
\]

\( c_1, \ldots, c_k \) list the constants occurring in some \( F_i \) and are assumed to occur in each \( F_i \) for notational convenience.

Let \( B' = \bigvee_{i=1}^{s} G(F_i, F_1^*)(v_0, a_1, \ldots, a_n, c_1, \ldots, c_k) \). If \( (A' \land \sim B')(\bar{a}) \) is infinite then there are models of \( T \) of arbitrarily large cardinality with \( (A' \land \sim B')(\bar{a}) \neq \emptyset \). Thus there is a model \( \mathfrak{C} \) of \( \Gamma_1 \) with \( (A' \land \sim B')(\mathfrak{C}) \neq \emptyset \). But this is impossible. Let \( H \) be

\[
\forall v_0 \left( A' \leftrightarrow \left( \bigvee_{i=1}^{s} G(F_i, F_1^*)(a_1, \ldots, a_n, c_1, \ldots, c_k) \right) \lor (A' \land \sim B) \right)
\]

\[
\land \left( \bigwedge_{i=1}^{s} G^*(F_i, F_1^*, p_i) \right) \land \left( \exists v_0 \left( A \land \left( \bigvee_{i=1}^{s} G(F_i, F_1^*) \right) \right) \right).
\]

Then \( \bar{a}, H \) so

\[
\bar{a} \models 3v_l, \ldots, 3 v_{l+k} c_1, \ldots, c_k (v_l, \ldots, v_{l+k})
\]

and

\[
\exists v_l \ldots 3v_{l+k} H c_1, \ldots, c_k (v_l, \ldots, v_{l+k}) \in \Gamma(u+1)
\]

where \( u = \max (u_i) < a \).

We return to the proof of Theorem 2. The induction hypothesis asserts that (i) is equivalent to (ii) if \( m < n \). We have already proved (i) implies (ii) if \( m = n \).
and now we wish to show (ii) implies (i) if \( m = n \). Suppose \( A \in S_{\lambda+1}(L), a_1, \ldots, a_\lambda \in |A| \), \( A' = A(a_1, \ldots, a_\lambda) \) and, for some \( k \), \( R_\lambda(A') = (n, k) \). The definition of \( \mathcal{G}_\lambda^{(n)} \) allows us to assume that \( k = 1 \). We will find a formula \( A* \in \Gamma_\lambda^{(n)} \) such that \( \bar{\alpha} \ni A^*(a_1, \ldots, a_\lambda) \).

By Theorem 1 (v) there is an elementary extension of \( \bar{B} \) of \( \bar{\alpha} \) and a formula \( B' \in S_1(L(\bar{B})) \) such that \( B'(\bar{B}) \subseteq A'(\bar{B}) \) and \( R_\lambda(B') = (n, 1) \). Now \( B' \) and \( \bar{B} \) satisfy the hypothesis of Lemma 2 so there exists \( B* \in \Gamma_\lambda^{(k+1)} \) for some \( k < n \) such that \( \bar{\alpha} \ni B^*(b_1, \ldots, b_s) \). If \( k < n - 1 \) by the induction hypothesis \( R_\lambda(B') < (n, 1) \) so \( k = n - 1 \). \( \bar{\alpha} \ni B^*(b_1, \ldots, b_s) \land \forall \nu_0(B(b_1, \ldots, b_s) \rightarrow A') \) and \( \bar{B} \) is an elementary extension of \( \bar{\alpha} \) so for some \( c_1, \ldots, c_s \in |\bar{\alpha}|, \bar{\alpha} \ni B^*(c_1, \ldots, c_s) \land \forall \nu_0(B(c_1, \ldots, c_s) \rightarrow A') \). Since \( B* \in \Gamma_\lambda^{(n)} \), we have proved (ii) implies (i) for \( m = n \), for some \( l \), \( R_\lambda(B(c_1, \ldots, c_s)) = (n, l) \). \( l \) must equal 1 since \( B(c_1, \ldots, c_s)(\bar{\alpha}) \subseteq A'(\bar{\alpha}) \) and \( R_\lambda(A') = (n, 1) \). If \( C' = C(\nu_0, a_1, \ldots, a_\mu, c_1, \ldots, c_s) = A' \land B(\nu_0, c_1, \ldots, c_s) \) then \( R_\lambda(C') < (n, 1) \). So by induction there exists \( C* \in \bigcup_{j=1}^{n-1} \Gamma_\lambda^{(j)} \) such that \( \bar{\alpha} \ni C^*(c_1, \ldots, a_\mu, c_1, \ldots, c_s) \). Hence letting

\[
A^* = \exists \nu_{i+1}, \ldots, \exists \nu_{i+s}((\forall \nu_0(A \rightarrow B(\nu_0, \nu_{i+1}, \ldots, \nu_{i+s}) \lor C)) \land B^* \land C*).
\]

\( A^* \) is in \( \Gamma_\lambda^{(n)} \) and \( \bar{\alpha} \ni A^*(a_1, \ldots, a_\lambda) \) proving the theorem.

Recall that \( \alpha_T \) is defined to be the least ordinal such that, for all \( \bar{\alpha} \in \mathfrak{N}(T) \) and \( \beta > \alpha_T \), \( S_\beta^{(\alpha_\lambda)}(\bar{\alpha}) = S_\beta(\bar{\alpha}) \). In [4] Morley proved \( \alpha_T \) exists and is less than \( (2^{\aleph_0})^+ \) for every complete theory. In [2] Lachlan shows that \( \alpha_T \leq \omega_1 \) for each complete theory. We apply Theorem 2 to prove the following conjecture of Morley.

**Theorem 3.** If \( T \) is \( \kappa_1 \)-categorical then \( \alpha_T \) is finite.

**Proof.** If for some \( \bar{\alpha} \) and some \( \beta > \omega \) there exists \( p \in S_\beta^{(\bar{\alpha})} \), then since \( T \) is totally transcendental for some \( \gamma \geq \beta \), \( p \in T_{\gamma}^{(\bar{\alpha})} \) and by Lemma 1 there exists \( \bar{B} \geq (\bar{\alpha}, q \in T_{\gamma}^{(\bar{\alpha})} \cap r_\lambda^{-1}(p) \) and there is a formula \( A' = A(\nu_0, a_1, \ldots, a_\lambda) \) in \( S_1(L(\bar{B})) \) with \( R_\lambda(A') = \omega \). By Theorem 1, there exists \( \bar{C} \geq \bar{B} \) and an integer \( k \) such that, for every elementary extension \( \bar{C}_1 \) of \( \bar{C} \), \( R_{\bar{C}_1}(A) = (\omega, k) \). Now by Lemma 2 with \( \alpha = \omega \), there exists an \( n < \omega \) and a formula \( A* \in \Gamma_\lambda^{(n+1)} \) such that \( \bar{C} \ni A^*(a_1, \ldots, a_\lambda) \). By Theorem 2, for some \( k \), \( R_\lambda(A') = (n + 1, k) \). This is a contradiction so there is no \( \bar{C} \) and no \( \beta > \omega \) and no \( p \) with \( p \in S_\beta^{(\bar{\alpha})} \). Hence \( \alpha_T < \omega \).

This proof relied on Theorem 0 which is shown in [1] to be equivalent to Vaught's conjecture that \( \kappa_1 \)-categorical theory has either 1 or \( \aleph_0 \)-countable models. According to Morley this conjecture had already been verified under the assumption that \( \alpha_T \) was finite. In fact, it is easy to deduce Lemma 13 of [1] which is crucial to the proof of Vaught's conjecture from our Theorem 3.
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