

## GENERALIZED LAPLACIANS AND MULTIPLE TRIGONOMETRIC SERIES

BY

M. J. KOHN

ABSTRACT. V. L. Shapiro gave a  $k$ -variable analogue for Riemann's theorem on formal integration of trigonometric series. This paper derives Shapiro's results with weaker conditions on the coefficients of the series and extends the results to series which are Bochner-Riesz summable of larger order.

1. Introduction. Let  $f(x)$ ,  $x = (x_1, \dots, x_k)$ , be a complex valued function defined in a neighborhood of  $x_0 \in E^k$ ,  $k \geq 2$ . We will say  $f(x)$  has at  $x_0$  a generalized  $r$ th Laplacian equal to  $s$  if  $f(x)$  is integrable over each sphere  $|x - x_0| = t$ , for  $t$  small, and if

$$(2\pi)^{-k/2} \int_{|y|=1} f(x_0 + ty) ds(y) = a_0 + a_2 t^2 + \dots + a_{2r} t^{2r} + o(t^{2r})$$

as  $t \rightarrow 0$ , where  $a_{2r} = s/2^{k/2+2r-1} r! \Gamma(r + k/2)$ . This definition is due to Shapiro [4], who showed that if all partial derivatives of  $f$  of order  $2r$  exist and are continuous in a neighborhood of  $x_0$ , then the generalized  $r$ th Laplacian of  $f(x_0)$  exists and equals  $\Delta^r f(x_0)$ .

Let

$$(1.1) \quad T: \sum c_n e^{in \cdot x}$$

be a trigonometric series in  $k$  variables, with  $n = (n_1, \dots, n_k)$ ,  $x = (x_1, \dots, x_k)$ ,  $n \cdot x = n_1 x_1 + \dots + n_k x_k$ , and  $|n| = (n_1^2 + \dots + n_k^2)^{1/2}$ . For  $\alpha \geq 0$  we denote by

$$(1.2) \quad \sigma_R^\alpha(x) = \sum_{|n| < R} \left(1 - \left(\frac{|n|}{R}\right)^2\right)^\alpha c_n e^{in \cdot x}$$

the (spherical) Bochner-Riesz means of order  $\alpha$  of (1.1). We say  $T$  is summable  $(BR, \alpha)$  at  $x$  to sum  $s$  if  $\lim_{R \rightarrow \infty} \sigma_R^\alpha(x) = s$ .

Shapiro [4] proved the following result, which is a  $k$ -dimensional analogue of Riemann's theorem on twice integrated trigonometric series (see, for example, [5, Vol. I, p. 319]).

*Suppose  $T: \sum c_n e^{in \cdot x}$  is  $(BR, m)$  summable at  $x_0$  to the finite sum  $s$ ,*

Received by the editors December 20, 1971 and, in revised form, September 11, 1972.  
 AMS (MOS) subject classifications (1970). Primary 42A24, 42A48, 42A92.

Key words and phrases. Multiple trigonometric series, generalized Laplacian, Bochner-Riesz summable.

Copyright © 1973, American Mathematical Society

where  $m$  is an integer  $\geq 0$ , and where  $c_n = O(|n|^\alpha)$ , for some  $\alpha < m + 2 - k$ , as  $|n| \rightarrow \infty$ . Let  $r$  be an integer  $\geq m/2 + 1$ . Set

$$(1.3) \quad F(x) = \frac{c_0(x_1 + \cdots + x_k)^{2r}}{k^r[(2r)!]} + \sum_{|n| \neq 0} \frac{(-1)^r c_n}{|n|^{2r}} e^{in \cdot x}.$$

Then the generalized  $r$ th Laplacian of  $F(x)$  exists at  $x_0$  and is equal to  $s$ .

The purpose of this paper is to prove the conclusion of Shapiro's theorem under weaker hypotheses than those stated above. We are able to reduce the hypothesis on the decay of the coefficients of (1.1) and also on the order of summability required. Our result is

**Theorem.** Let the series (1.1) be summable  $(BR, \beta)$  at  $x_0$  to a finite sum  $s$ , where  $\beta$  is a real number  $\geq 0$ . Let  $r$  be an integer such that  $r > \frac{1}{2}(\beta + 1)$ , and suppose  $\sum |n|^\gamma |c_n|^2 < \infty$  for some  $\gamma > k - 1 - 4r$ . Then  $F(x)$  defined by

$$F(x) = \frac{c_0(x_1 + \cdots + x_k)^{2r}}{k^r[(2r)!]} + \lim_{R \rightarrow \infty} \sum_{0 < |n| < R} (-1)^r \frac{c_n}{|n|^{2r}} e^{in \cdot x}$$

has a generalized  $r$ th Laplacian at  $x_0$  equal to sum  $s$ .

We note, in particular, that our reduced condition on the growth of the coefficients  $c_n$  no longer implies that the series defining  $F(x)$  converges uniformly. Hence our improvement extends Shapiro's result to the case when  $F(x)$  is not necessarily continuous.

2. Before beginning the proof of the Theorem, we note that the Theorem is true if  $T$  is only a constant term; see, for example, [2, p. 289]. We may therefore assume  $c_0 = 0$ . We will also assume, as we may, that  $x_0 = 0$ ,  $s = 0$ , and that  $r$  is the smallest integer with  $2r > \beta + 1$ . Write  $\beta = m + \alpha$  where  $m$  is an integer and  $0 \leq \alpha < 1$ .

Write  $S_R = S_R(0) = \sum_{|n| < R} c_n$  and for  $\beta > 0$  put

$$(2.1) \quad S_R^\beta = \frac{1}{\Gamma(\beta)} \int_0^R (R-u)^{\beta-1} S_u \, du.$$

$S_R^\beta$ , as a function of  $R$ , is the fractional integral of order  $\beta$  of  $f(R) = S_R$ .

Hardy has shown that  $\sigma_R^\beta(x) \rightarrow s$  if and only if  $\bar{\sigma}_R^\beta(x) \rightarrow s$ , where

$$\bar{\sigma}_R^\beta(x) = \frac{\beta}{R^\beta} \int_0^R S_u(x) (R-u)^{\beta-1} \, du.$$

Thus the series (1.1) is  $(BR, \beta)$  summable to zero at  $x = 0$  if and only if

$$(2.2) \quad S_{R^\beta}^\beta = o(R^\beta),$$

as  $R \rightarrow \infty$ .

3. We begin with some lemmas. Lemmas 1 and 2 are modifications of lemmas from [4].

**Lemma 1.** Assume  $\sum |n|^\gamma |c_n|^2 < \infty$  for some  $\gamma > k - 4r - 1$  and suppose  $\sum c_n$  is  $(BR, m + 1)$  summable to zero. Then  $S_R^n = o(R^{2r+1/2})$  for  $n = 0, \dots, m$ , as  $R \rightarrow \infty$ .

**Proof.** Write  $\gamma = k - 4r - 1 + \epsilon$ , where  $\epsilon > 0$ . Then

$$(3.1) \quad \sum_{|n| < R} |c_n| = \sum_{|n| < R} |n|^{\gamma/2} |c_n| |n|^{-\gamma/2} \leq \left( \sum_{|n| < R} |n|^\gamma |c_n|^2 \right)^{1/2} \left( \sum_{|n| < R} |n|^{-\gamma} \right)^{1/2} \\ = C(R^{-\gamma+k})^{1/2} = CR^{1/2(-k+4r+1-\epsilon+k)} = o(R^{2r+1/2}).$$

Note that for  $j = 1, 2, \dots, m + 2$ ,

$$\sum_{|s| < R} c_s (R - |s| + j)^{m+1} = \sum_{|s| < R+j} - \sum_{R \leq |s| < R+j} = I + II.$$

$I = o(R^{m+1})$  since  $\sum c_s$  is  $(BR, m + 1)$  summable to 0.

$$|II| \leq C \sum_{|s| < R+j} |c_s| = o(R^{2r+1/2}), \text{ by (3.1).}$$

Since  $2r > \beta + 1 \geq m + 1$ , we have, combining I and II,  $\sum_{|s| < R} c_s (R - |s| + j)^{m+1} = o(R^{2r+1/2})$ .

We observe that there exist numbers  $A_{jn}$  for  $j = 1, \dots, m + 2$  and  $n = 0, \dots, m$  such that  $\sum_{j=1}^{m+2} A_{jn} (z + j)^{m+1} = z^n$  for  $z \in \mathbb{C}$ . This is Lemma 3 of [4]. Hence

$$S_R^n = \frac{1}{n!} \sum_{|s| < R} c_s (R - |s|)^n = \frac{1}{n!} \sum_{|s| < R} c_s \sum_{j=1}^{m+2} A_{jn} (R - |s| + j)^{m+1} \\ = \sum_{j=1}^{m+2} \frac{1}{n!} A_{jn} \sum_{|s| < R} c_s (R - |s| + j)^{m+1} = \sum_{j=1}^{m+2} \frac{1}{n!} A_{jn} o(R^{2r+1/2}) = o(R^{2r+1/2}).$$

**Lemma 2.** Let  $J_s(t)$  denote the Bessel function of the first kind of order  $s$ , where  $s$  is an integer or half integer. Then for each  $n > 0$ ,

$$\frac{d^n}{dt^n} \frac{J_s(t)}{t^{s+2r}} = O(t^{-s-2r-1/2}),$$

as  $t$  tends to infinity.

**Proof.** The argument is the same as the proof of Lemma 2 of [4]. However we use here the fact that  $J_s(t) = O(t^{-1/2})$  as  $t \rightarrow \infty$ .

**Lemma 3.** If  $0 \leq \alpha < 1$ , then  $\int_0^u (u - z)^{-\alpha} z^\alpha dz = O(u)$  as  $u$  tends to infinity.

**Proof.**

$$\int_0^u (u - z)^{-\alpha} z^\alpha dz = \int_0^{u/2} + \int_{u/2}^u = A + B,$$

where

$$A = \int_0^{u/2} (u - z)^{-\alpha} z^\alpha dz = O(u^{-\alpha}) \int_0^{u/2} z^\alpha dz = O(u),$$

$$B = \int_{u/2}^u (u - z)^{-\alpha} z^\alpha dz = O(u^\alpha) \int_{u/2}^u (u - z)^{-\alpha} dz = O(u^\alpha)O(u^{1-\alpha}) = O(u).$$

**Lemma 4.** With  $S_u^\beta$  defined by (2.1), then for almost all  $u$

$$S_u^m = \frac{1}{\Gamma(1 - \alpha)} \frac{d}{du} \int_0^u (u - z)^{-\alpha} S_z^{\alpha+m} dz.$$

**Proof.** Put

$$I = \frac{1}{\Gamma(1 - \alpha)} \int_0^u (u - z)^{-\alpha} S_z^{\alpha+m} dz = \frac{1}{\Gamma(1 - \alpha)} \int_0^u (u - z)^{(1-\alpha)-1} S_z^{\alpha+m} dz.$$

Thus  $I$  is the fractional integral of order  $(1 - \alpha)$  of  $S_u^{\alpha+m}$ . But since  $S_u^{\alpha+m}$  is the fractional integral of order  $(\alpha + m)$  of  $S_u$ , therefore  $I$  is the fractional integral of  $S_u$  of order  $(\alpha + m) + (1 - \alpha) = m + 1$ . Hence for almost all  $u$ ,  $dI/du = S_u^m$ .

**4. Proof of the Theorem.** We must show

$$(2\pi)^{-k/2} \int_{z \in \Sigma} F(tz) ds(z) = a_0 + a_2 t^2 + \dots + a_{2(r-1)} t^{2(r-1)} + o(t^{2r})$$

as  $t$  tends to zero.

The hypothesis on the moduli of  $c_n$  implies  $\sum |n|^{k-1+\epsilon} |c_n| |n|^{-2r} < \infty$ , for some  $\epsilon > 0$ . Therefore by virtue of Theorem 1 of [3] the series defining  $F(x)$  in (1.3) converges spherically almost everywhere on each sphere  $|x| = t$ . By virtue of Theorem 2 of [3], with  $p = 1$ , we may integrate this series term by term over each sphere.

Bochner [1] observed  $(2\pi)^{-k/2} \int_{z \in \Sigma} e^{int \cdot z} ds(z) = J_s(|n|t)/(|n|t)^s$ , where  $s = \frac{1}{2}(k - 2)$ . Hence,

$$\begin{aligned} (2\pi)^{-k/2} \int_{\Sigma} F(tz) ds(z) &= \lim_{R \rightarrow \infty} \sum_{0 < |n| < R} (-1)^r \frac{c_n}{|n|^{2r}} (2\pi)^{-k/2} \int_{\Sigma} e^{in \cdot tz} ds(z) \\ (4.1) \quad &= \lim_{R \rightarrow \infty} (-1)^r \sum_{0 < |n| < R} \frac{c_n}{|n|^{2r}} \frac{J_s(|n|t)}{(|n|t)^s} = (-1)^r t^{2r} \lim_{R \rightarrow \infty} \sum_{0 < |n| < R} c_n \gamma(|n|t), \end{aligned}$$

where  $\gamma(u) = u^{-s-2r} J_s(u)$ .

We apply summation by parts to  $\sum_{|n| < R} c_n \gamma(|n|t)$   $m$  times.

$$\begin{aligned}
 \sum_{|n| < R} c_n \gamma(|n|t) &= S_R \gamma(Rt) - \int_0^R S_u \frac{d}{du} \gamma(ut) du \\
 &= S_R \gamma(Rt) - S_R^1 \frac{d}{dR} \gamma(Rt) + \int_0^R S_u^1 \frac{d^2}{du^2} \gamma(ut) du \\
 &\quad \vdots \\
 &= S_R \gamma(Rt) - S_R^1 \frac{d}{dR} \gamma(Rt) + \dots + (-1)^m S_R^m \frac{d^m}{dR^m} \gamma(Rt) \\
 &\quad + (-1)^{m+1} \int_0^R S_u^m \frac{d^{m+1}}{du^{m+1}} \gamma(ut) du.
 \end{aligned}$$

Using Lemmas 1 and 2, with  $j = 1, \dots, m$ ,  $S_R^j (d^j/dR^j) \gamma(Rt) = o(R^{2r+1/2})$ .  $O(R^{-(k-2)/2-2r-1/2}) = o(1)$ , as  $R$  tends to infinity. Thus,

$$\lim_{R \rightarrow \infty} \sum_{0 < |n| < R} c_n \gamma(|n|t) = (-1)^{m+1} \int_0^\infty S_u^m \frac{d^{m+1}}{du^{m+1}} \gamma(ut) du,$$

and returning to (4.1) we get

$$\begin{aligned}
 (2\pi)^{-k/2} \int_{\Sigma} F(tz) ds(z) &= (-1)^r t^{2r} \sum_{0 < |n| < R} c_n \gamma(|n|t) \\
 (4.2) \qquad \qquad \qquad &= (-1)^{r+m+1} t^{2r} \int_1^\infty S_u^m \frac{d^{m+1}}{du^{m+1}} \gamma(ut) du.
 \end{aligned}$$

We use the formula  $J_s(z) = \sum_{n=0}^\infty ((-1)^n (1/2 z)^{s+2n} / n! \Gamma(s+n+1))$ . Hence

$$\begin{aligned}
 z^{-s} J_s(z) &= \sum_{n=0}^\infty \frac{(-1)^n (2)^{-s-2n} z^{2n}}{n! \Gamma(s+n+1)} \\
 &= c_0 + c_2 z^2 + \dots + c_{2n} z^{2n} + \dots.
 \end{aligned}$$

Define  $P(z) = c_0 + c_2 z^2 + \dots + c_{2(r-1)} z^{2(r-1)}$  and let  $\lambda(z) = (z^{-s} J_s(z) - P(z)) / z^{2r}$ . Then  $\lambda(z)$  is an entire function in the plane and

$$(4.3) \qquad \qquad \qquad \gamma(z) = \lambda(z) + z^{-2r} P(z).$$

Substituting into (4.2),

$$\begin{aligned}
 (2\pi)^{-k/2} \int_{\Sigma} F(tz) ds(z) &= (-1)^{r+m+1} t^{2r} \int_1^\infty S_u^m \frac{d^{m+1}}{du^{m+1}} \left[ \frac{P(ut)}{(ut)^{2r}} + \lambda(ut) \right] du \\
 (4.4) \qquad \qquad \qquad &= (-1)^{r+m+1} \int_1^\infty S_u^m \frac{d^{m+1}}{du^{m+1}} \frac{P(ut)}{u^{2r}} du + (-1)^{r+m+1} t^{2r} \int_1^\infty S_u^m \frac{d^{m+1}}{du^{m+1}} \lambda(ut) du \\
 &= A(t) + t^{2r} B(t).
 \end{aligned}$$

$$\begin{aligned}
 A(t) &= (-1)^{r+m+1} \int_1^\infty S_u^m \frac{d^{m+1}}{du^{m+1}} \frac{P(ut)}{u^{2r}} du \\
 (4.5) \quad &= (-1)^{r+m+1} \int_1^\infty S_u^m \frac{d^{m+1}}{du^{m+1}} \left\{ \sum_{j=0}^{r-1} c_j \frac{(ut)^{2j}}{u^{2r}} \right\} du \\
 &= (-1)^{r+m+1} \sum_{j=0}^{r-1} t^{2j} \int_1^\infty S_u^m \frac{d^{m+1}}{du^{m+1}} c_j u^{2(j-r)} du = \sum_{j=0}^{r-1} A_{2j} t^{2j}.
 \end{aligned}$$

We will show  $A_{2j}$  is finite for each  $j$ .

$$A_{2j} = c_j' \int_0^\infty S_u^m u^{2(j-r)-m-1} du = c_j' \int_1^\infty S_u^m u^{-n_j} du,$$

with  $m+3 \leq n_j \leq 2r+m+1$ . By Lemma 4,

$$S_u^m = (1/\Gamma(1-\alpha))(d/du) \int_0^u (u-z)^{-\alpha} S_z^{\alpha+m} dz,$$

for almost all  $u$ . Hence

$$A_{2j} = c_j'' \int_1^\infty u^{-n_j} \frac{d}{du} \int_0^u (u-z)^{-\alpha} S_z^{\alpha+m} dz du.$$

Integrating by parts,

$$A_{2j} = c_j'' u^{-n_j} \int_0^u (u-z)^{-\alpha} S_z^{\alpha+m} dz \Big|_1^\infty - c_j'' \int_1^\infty u^{-n_j-1} \int_0^u (u-z)^{-\alpha} S_z^{\alpha+m} dz du = \alpha_j + \beta_j.$$

By (2.2), we have  $S_R^{\alpha+m} = o(R^{\alpha+m})$ , so

$$\begin{aligned}
 \alpha_j &= c_j'' u^{-n_j} \int_0^u (u-z)^{-\alpha} o(z^{\alpha+m}) dz \Big|_1^\infty \\
 &= o(u^{-n_j+m}) \int_0^u (u-z)^{-\alpha} z^\alpha dz \Big|_1^\infty = o(u^{-n_j+m}) O(u) \Big|_1^\infty
 \end{aligned}$$

by Lemma 3. Since  $n_j \geq m+3$ ,  $\alpha_j = O(1)$ .

To estimate  $\beta_j$  we again use Lemma 3.

$$\begin{aligned}
 \beta_j &= c_j'' \int_1^\infty u^{-n_j-1} \int_0^u (u-z)^{-\alpha} o(z^{\alpha+m}) dz du \\
 &= c_j'' \int_1^\infty u^{-n_j-1} o(u^m) \int_0^u (u-z)^{-\alpha} o(z^\alpha) dz du \\
 &= c_j'' \int_1^\infty o(u^{-n_j-1+m}) O(u) du = O(1).
 \end{aligned}$$

This shows that each  $A_{2j} = \alpha_j + \beta_j$  is finite. Combining this last statement with (4.4) and (4.5), we obtain

$$(2\pi)^{-k/2} \int_{\Sigma} F(tz) ds(z) = \sum_{j=0}^{r-1} A_{2j} t^{2j} + t^{2r} B(t).$$

5. The proof of the Theorem will be complete when we show  $B(t)$  tends to zero with  $t$ . Using Lemma 4,

$$\begin{aligned}
 B(t) &= (-1)^{r+m+1} \int_1^\infty S_u^m \frac{d^{m+1}}{du^{m+1}} \lambda(ut) du \\
 &= (-1)^{r+m+1} \int_0^\infty \frac{1}{\Gamma(\alpha+m)} \frac{d}{du} \int_0^u (u-z)^{-\alpha} S_z^{\alpha+m} dz \frac{d^{m+1}}{du^{m+1}} \lambda(ut) du.
 \end{aligned}$$

(We are justified in replacing the limits of integration in the outer integral by  $0 \leq u < \infty$  because  $S_u^m = 0$  for  $0 \leq u < 1$ .) We integrate the expression for  $B(t)$  by parts.

$$\begin{aligned}
 B(t) &= \frac{(-1)^{r+m+1}}{\Gamma(\alpha+m)} \int_0^u (u-z)^{-\alpha} S_z^{\alpha+m} dz \cdot \frac{d^{m+1}}{du^{m+1}} \lambda(ut) \Big|_0^\infty \\
 &\quad + \frac{(-1)^{r+m}}{\Gamma(\alpha+m)} \int_0^\infty \int_0^u (u-z)^{-\alpha} S_z^{\alpha+m} dz \cdot \frac{d^{m+2}}{du^{m+2}} \lambda(ut) du \\
 &= B_1(t) + B_2(t).
 \end{aligned}$$

We assert that, for  $t \neq 0$ ,  $B_1(t)$  is zero. Since  $\lambda(z)$  is entire in the plane,  $d^{m+1}\lambda(ut)/du^{m+1}$  remains bounded as  $u$  tends to zero. Hence at zero the expression defining  $B_1(t)$  is zero. To evaluate the expression defining  $B_1(t)$  at  $u = \infty$  we use (4.3) and (2.2).

$$\int_0^u (u-z)^{-\alpha} S_z^{\alpha+m} dz \cdot \frac{d^{m+1}}{du^{m+1}} \lambda(ut) = o(u^m) \int_0^u (u-z)^{-\alpha} z^\alpha dz \cdot \frac{d^{m+1}}{du^{m+1}} \left[ \gamma(ut) - \frac{P(ut)}{(ut)^{2r}} \right].$$

The integral in this last expression is  $O(u)$  by Lemma 3. By Lemma 2,  $d^{m+1}\gamma(ut)/du^{m+1} = O(u^{-m-1})$ , and since  $P(ut)/(ut)^{2r}$  is a quotient of polynomials with the degree of the denominator greater by two than the degree of the numerator,  $(d^{m+1}/du^{m+1}) P(ut)/(ut)^{2r} = O(u^{-2-m-1})$  as  $u \rightarrow \infty$ . Hence, near  $u = \infty$ , the expression defining  $B_1(t)$  is  $o(u^m)O(u)[O(u^{-m-1}) + O(u^{-m-1})] = o(1)$ .

It remains to be shown that  $B_2(t)$  tends to zero with  $t$ .

$$\begin{aligned}
 B_2(t) &= \frac{(-1)^{r+m}}{\Gamma(\alpha+m)} \int_0^\infty \int_0^u (u-z)^{-\alpha} S_z^{\alpha+m} dz \frac{d^{m+2}}{du^{m+2}} \lambda(ut) du \\
 &= \frac{(-1)^{r+m}}{\Gamma(\alpha+m)} \int_0^\infty S_z^{\alpha+m} \left\{ \int_z^\infty (u-z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \lambda(ut) du \right\} dz \\
 &= \frac{(-1)^{r+m}}{\Gamma(\alpha+m)} \int_0^\infty S_z^{\alpha+m} H(z, t) dz = \frac{(-1)^{r+m}}{\Gamma(\alpha+m)} \left\{ \int_0^{1/t} + \int_{1/t}^\infty \right\} = \frac{(-1)^{r+m}}{\Gamma(\alpha+m)} \{M + N\}.
 \end{aligned}$$

We first consider  $M$ . In this integral  $z$  ranges between 0 and  $1/t$ , and

$$H(z, t) = \int_z^\infty (u-z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \lambda(ut) du = \int_z^{1/t} + \int_{1/t}^\infty = P + Q.$$

$\lambda(z)$  is entire, so for  $|ut| < 1$ ,  $d^{m+2}\lambda(ut)/du^{m+2} = O(t^{m+2})$ . Hence

$$(5.2) \quad \begin{aligned} P &= \int_z^{1/t} (u-z)^{-\alpha} O(t^{m+2}) du = O(t^{m+2})(1/t-z)^{1-\alpha} \\ &= O(t^{m+1})(1/t-z)^{1-\alpha}(t(1/t-z)) = O(t^{m+1})(1/t-z)^{1-\alpha}, \end{aligned}$$

since  $t(1/t-z) = 1-tz \leq 1$ .

$$\begin{aligned} Q &= \int_{1/t}^\infty (u-z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \lambda(ut) du \\ &= \int_{1/t}^\infty (u-z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \gamma(ut) du + \int_{1/t}^\infty (u-z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \frac{P(ut)}{(ut)^{2r}} du = Q_1 + Q_2. \end{aligned}$$

By Lemma 2,

$$\frac{d^{m+2}}{du^{m+2}} \gamma(ut) = t^{m+2} O((ut)^{-s-2r-1/2}) = t^{m+3/2-2r-s} O(u^{-s-2r-1/2}).$$

Hence,

$$\begin{aligned} Q_1 &= \int_{1/t}^\infty (u-z)^{-\alpha} t^{m+3/2-2r-s} O(u^{-s-2r-1/2}) du \\ &= (1/t-z)^{-\alpha} t^{m+3/2-2r-s} \int_{1/t}^\infty O(u^{-s-2r-1/2}) du = (1/t-z)^{-\alpha} O(t^{m+1}). \end{aligned}$$

Since  $(d^{m+2}/du^{m+2})[P(ut)(ut)^{-2r}] = t^{-2}O(u^{-m-4})$ , therefore

$$Q_2 = \int_{1/t}^\infty (u-z)^{-\alpha} t^{-2} O(u^{-m-4}) du = (1/t-z)^{-\alpha} O(t^{-2+m+3}).$$

Combining the estimates for  $Q_1$  and  $Q_2$  we get  $Q = (1/t-z)^{-\alpha} O(t^{m+1})$ , and with (5.1) and (5.2) we conclude, for  $0 \leq z \leq 1/t$ ,  $H(z, t) = (1/t-z)^{-\alpha} O(t^{m+1})$ . Therefore

$$\begin{aligned} M &= \int_0^{1/t} S_z^{\alpha+m} H(z, t) dz = \int_0^{1/t} o(z^{\alpha+m}) O(t^{m+1})(1/t-z)^{-\alpha} dz \\ &= o(1/t)^{\alpha+m} t^{m+1} \int_0^{1/t} (1/t-z)^{-\alpha} dz = o(1). \end{aligned}$$

We now consider  $N = \int_{1/t}^\infty S_z^{\alpha+m} H(z, t) dz$ . In this integral  $z$  is larger than  $1/t$ .

$$\begin{aligned} H(z, t) &= \int_z^\infty (u-z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \lambda(ut) du \\ &= \int_z^\infty (u-z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \gamma(ut) du + \int_z^\infty (u-z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \frac{P(ut)}{(ut)^{2r}} du = R + S. \end{aligned}$$

To compute  $R$  we apply Lemma 2.

$$(d^{m+2}/du^{m+2}) \gamma(ut) = t^{m+3/2-s-2r} O(u^{-s-2r-1/2}).$$

Hence,



$$\begin{aligned}
 R &= t^{m+3/2-s-2r} \int_z^\infty (u-z)^{-\alpha} O(u^{-s-2r-1/2}) du \\
 &= t^{m+3/2-s-2r} \left[ \int_z^{2z} + \int_{2z}^\infty \right]. \\
 \int_z^{2z} (u-z)^{-\alpha} O(u^{-s-2r-1/2}) du &= O(z^{-s-2r-1/2}) \int_z^{2z} (u-z)^{-\alpha} du \\
 &= O(z^{-s-2r-1/2}) O(z^{1-\alpha}) = O(z^{-s-2r+1/2-\alpha}), \\
 \int_{2z}^\infty (u-z)^{-\alpha} O(u^{-s-2r-1/2}) du &= O(z^{-\alpha}) \int_{2z}^\infty u^{-s-2r-1/2} du \\
 &= O(z^{-\alpha}) O(z^{-s-2r+1/2}) = O(z^{-s-2r+1/2-\alpha}).
 \end{aligned}$$

Thus,  $R = t^{m+3/2-s-2r} O(z^{-s-2r+1/2-\alpha}) = t^{-1/2} O(z^{-\alpha-m-3/2}) (tz)^{m+2-s-2r}$ . Since  $|tz| \geq 1$  in the interval of integration and since  $2r \geq m+2$  and  $s \geq 0$ , the last factor on the right of this last equation is  $\leq 1$ . Hence,

$$(5.3) \quad R = t^{-1/2} O(z^{-\alpha-m-3/2}).$$

To compute  $S$ , we again use the fact that  $(d^{m+2}/du^{m+2})P(ut)/(ut)^{2r} = t^{-2}O(u^{-m-4})$ ; and therefore

$$\begin{aligned}
 S &= \int_z^\infty (u-z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \frac{P(ut)}{(ut)^{2r}} du \\
 &= t^{-2} \int_z^\infty (u-z)^{-\alpha} O(u^{-m-4}) du = t^{-2} \left\{ \int_z^{2z} + \int_{2z}^\infty \right\}.
 \end{aligned}$$

$$\begin{aligned}
 \int_z^{2z} (u-z)^{-\alpha} O(u^{-m-4}) du &= O(z^{-m-4}) \int_z^{2z} (u-z)^{-\alpha} du \\
 &= O(z^{-m-4}) O(z^{1-\alpha}) = O(z^{-m-\alpha-3}).
 \end{aligned}$$

$$\int_{2z}^\infty (u-z)^{-\alpha} O(u^{-m-4}) du = O(z^{-\alpha}) \int_{2z}^\infty u^{-m-4} du = O(z^{-\alpha}) z^{-m-3}.$$

Thus  $S = t^{-2} O(z^{-m-\alpha-3}) = t^{-1/2} O(z^{-m-\alpha-3/2})$ . Combining with (5.3), we see that for  $1/t < z$ ,  $H(z, t) = R + S = t^{-1/2} O(z^{-m-\alpha-3/2})$ .

Substituting into the integral defining  $N$ ,

$$\begin{aligned}
 N &= \int_{1/t}^\infty S_z^{\alpha+m} H(z, t) dz = \int_{1/t}^\infty o(z^{\alpha+m}) t^{-1/2} O(z^{-m-\alpha-3/2}) dz \\
 &= t^{-1/2} \int_{1/t}^\infty o(z^{-3/2}) dz = t^{-1/2} o(1/t)^{-1/2} = o(1).
 \end{aligned}$$

We have shown that  $B_2(t)$  tends to zero with  $t$ . This completes the proof of the Theorem.

## BIBLIOGRAPHY

1. S. Bochner, *Summation of multiple Fourier series by spherical means*, *Trans. Amer. Math. Soc.* **40** (1936), 175–207.
2. R. Courant, *Methods of mathematical physics. Vol. II: Partial differential equations*, Interscience, New York, 1962. MR 25 #4216.
3. M. Kohn, *Spherical convergence and integrability of multiple trigonometric series on hypersurfaces*, *Studia Math.* **44** (1972), 345–354.
4. V. L. Shapiro, *Circular summability  $C$  of double trigonometric series*, *Trans. Amer. Math. Soc.* **76** (1954), 223–233. MR 15, 866.
5. A. Zygmund, *Trigonometric series. Vols. 1, 2*, 2nd ed., Cambridge Univ. Press, New York, 1968. MR 38 #4882.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MINNESOTA  
55455