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TWO-NORM SPACES AND DECOMPOSITIONS OF BANACH SPACES. II

BY

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ABSTRACT. Let X be a Banach space, Y a closed subspace of X^*. One says X is *Y-reflexive* if the canonical imbedding of X onto Y^* is an isometry and *Y-pseudo reflexive* if it is a linear isomorphism onto. If X has a basis and Y is the closed linear span of the corresponding biorthogonal functionals, necessary and sufficient conditions for X to be Y-pseudo reflexive are due to I. Singer. To every B-space X with a decomposition we associate a canonical two-norm space X_s and show that the properties of X_s, in particular its γ-completion, may be exploited to give different proofs of Singer's results and, in particular, to extend them to B-spaces with decompositions. This technique is then applied to a study of direct sum of B-spaces with respect to a BK space. Necessary and sufficient conditions for such a space to be reflexive are obtained.

1.0 **Introduction.** Given a Banach space (B-space for short) $(X, \|\ \|_1)$ with a Schauder decomposition $\{X_j\}$, a two-norm space X_s, called the *canonical two-norm space of* X, was introduced in [10] and the properties of X were then studied by investigating the properties of X_s. The aim of this paper is to continue these investigations and, in particular, study the so-called *pseudo reflexivity* of X with respect to $A(X_s)$, the γ-linear functionals on X_s. Some of our results generalize those of Singer [9] to the setting of Schauder decompositions providing, at the same time, a slightly different method of proof of these results.

The basic topic of our discussion is the notion of *γ-completion* of the canonical two-norm space. A canonical two-norm space may be defined whenever X is the direct sum of subspaces for a metric topology, not necessarily the norm topology. In §§5–7, we study spaces $X = B\text{-}\Sigma_n X_n$ of sequences $\{x_n\}$ where x_n belongs to a B-space X_n such that $\{\|x_n\|\}$ is an element of a given BK space B of real sequences. We extend the notions of boundedly complete and shrinking decompositions to this setting and obtain analogues of some of the results in Schauder decomposition theory (cf. [10]).

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2.0 **Notations and preliminaries.** A (*weak*) *decomposition* $\{X_j\}$ for a B-space $(X, |\ |)$ is a collection $\{X_j\}$ of nontrivial subspaces of X such that to each x in X, there corresponds a unique sequence $\{x_j\}$, x_j in X_j, and $x = \Sigma_j x_j$ in the norm (respectively weak) topology of X. It is called a *Schauder decomposition* of X, if for each j the projection map $P_j: X \to X_j$, defined by $P_j(x) = x_j$, is continuous. The decomposition $\{X_j\}$ is called *boundedly complete* if every sequence $\{x_j\}$, x_j in X_j, with the property

$$\sup_n \left\{ \left| \sum_{k=1}^{n} x_k \right| \right\} < \infty$$

defines an element of X, that is, $\Sigma_j x_j$ converges. The decomposition $\{X_j\}$ is called *shrinking for* f in X^* if $|f|_n^* \to 0$ as $n \to \infty$, where

$$|f|_n^* = \sup_n \left\{ |f(x)|: |x| \leq 1,\ x \in \sum_{j>n} X_j \right\}.$$

In this paper all decompositions are assumed to be *monotone*, that is, $|\Sigma_{k \leq n} x_k| \leq |\Sigma_k x_k|$ for all n. The reader will later observe that the proofs of analogues of our theorems to the nonmonotone case present no difficulties.

A *two-norm space* $X_s = (X, |\ |_1, |\ |_2)$ is a linear set X with two norms $|\ |_1$ and $|\ |_2$. In general, the second norm $|\ |_2$ is an F-norm although, in most cases of interest, it is the usual (homogeneous) norm. In this paper, all two-norm spaces considered are such that $(X, |\ |_1)$ is a Banach space and $(X, |\ |_2)$ is a normed linear space. An exception occurs in §4 where the second norm is a B_0^*-*norm*. We recall that a B_0^*-norm $|\ |_2$ on X is such that

$$|x|_2 = \sum_k 2^{-k}\{[x]_k/(1 + [x]_k)\},$$

$[\ ]_k$ being homogeneous pseudo norms such that $[x]_k = 0$ for all k implies $x = 0$. A sequence $\{x_n\} \subset X_s$ is said to be γ-convergent to x in X, written $x_n \xrightarrow{\gamma} x$, if $\{|x_n|_1\}$ is bounded and $x_n \to x$ in $|\ |_2$. A γ-*Cauchy sequence* is defined analogously and X_s is called γ-*complete* if it is sequentially complete for the convergence (γ). The space X_s is called *normal* if

$$x_n \xrightarrow{\gamma} x \Rightarrow |x|_1 \leq \liminf_n |x_n|_1.$$

A γ-*linear functional* on X_s is a linear functional which is continuous for the convergence (γ). The set of all γ-linear functionals on X_s will be denoted by $A(X_s)$. We shall say that the space X_s satisfies *property* (n_0) if $|\ |_2$ is coarser than $|\ |_1$, that is, every sequence that converges to 0 in $|\ |_1$ does so in $|\ |_2$ as well.

Let $(X_j^*, |\ |_j^*) = (X, |\ |_j)^*$, $j = 1, 2$. For a normal two-norm space satisfying (n_0), it is known that

$$X_2^* \subseteq A(X_s) \subseteq X_1^*,$$

$A(X_s)$ is a closed subspace of X_1^* and that $A(X_s)$ is the closure of X_2^* in $(X_1^*, |\ |_1^*)$. Further, $A(X_s)$ is norm determining, that is,

$$|x|_1 = \sup\{|f(x)| : f \in A(X_s)\}.$$

The space X_s is called *γ-reflexive* if $(A(X_s), |\ |_1^*)^* = (A(X_s)^*, |\ |_1^{**})$ is isometrically isomorphic to $(X, |\ |_1)$ under the canonical map J, where for x in X, $J(x)f = f(x)$ for all f in $A(X_s)$. Whenever $A(X_s) = X_1^*$, X_s is called *saturated.*

Let $\{X_j\}$ be a decomposition for the B-space $(X, |\ |_1)$. The *canonical two-norm space* X_s of X is the two-norm space $(X, |\ |_1, |\ |_2)$, where for $x = \Sigma_j x_j$ in X,

$$|x|_2 = \sum_j |x_j|_1/2^j.$$

The norm $|\ |_2$ is called the *canonical second norm.* We shall continue to use the symbol X_1^* for $(X, |\ |_1)^*$ and X_2^* for $(X, |\ |_2)^*$. The reader should not confuse these symbols for the duals of the subspaces X_1 and X_2 respectively. This confusion can be easily avoided if he remembers that the duals of the subspaces themselves are nowhere considered in this paper.

The basic properties of the canonical two-norm space may be found in [10]. However, for the convenience of the reader, we recall the following notions and notations that are necessary for an understanding of the present paper.

Let $(X, |\ |_1)$ be a B-space with Schauder decomposition $\{X_j\}$, P_j the continuous projection of X onto X_j and let X_s be the canonical two-norm space. Then $A(X_s)$ has the Schauder decomposition $\{P_j^*(X_1^*)\}$. To simplify notation, we shall often write $A(X_s)_j$ for $P_j^*(X_1^*)$. The canonical two-norm space of $A(X_s)$ will be referred to as the *k-dual* of X_s and will be denoted by k-X_s. The Schauder decomposition of $A(k\text{-}X_s)$ will be denoted by $\{A(k\text{-}X_s)_j\}$ and the canonical two-norm space $A(k\text{-}X_s)_s$ of $A(k\text{-}X_s)$ by k^2-X_s. When $\{X_j\}$ is monotone, the canonical map J is an isometry (with respect to both norms) of X_s into k^2-X_s. When this map is also onto, one says X_s is *k-reflexive.* Finally, we use the symbol Σ^* to indicate convergence in the weak-* topology of a given dual space.

3.0 **The γ-completion of X_s.** Let $(X, |\ |_1)$ be a B-space with Schauder decomposition $\{X_j\}$ and canonical two-norm space X_s. As a consequence of [10, Theorem 3.6], it follows that X_s may not, in general, be γ-complete. We shall now show that X_s may be imbedded isometrically isomorphically into a γ-complete, normal two-norm space $C(X)_s$ such that X_s is γ-dense in $C(X)_s$.

Theorem 3.1. *The sets*

$$\left\{\{J(x_j)\} : x_j \in X_j,\ \sup_n \left[\left|\sum_{j\le n} J(x_j)\right|_1^{**}\right] < \infty\right\}$$

and

$$\left\{\{J(x_j)\}: \sum_j^* J(x_j) \text{ converges in } \sigma(A(X_s)^*, A(X_s))\right\}$$

are identical.

Proof. Consider $\{J(x_j)\} \subset A(X_s)^*$ such that $\sup_n |\Sigma_{j\le n} J(x_j)|_1^{**}$ is finite. Since $A(X_s)$ is the norm-closure of the set $\bigcup_k P_k^*(X_1^*)$ in X_1^* [2, p. 277], and $\{\Sigma_{k\le n} J(x_k)\}$ converges pointwise on $\bigcup_k P_k^*(X_1^*)$, it follows by the Banach-Steinhaus theorem that $\Sigma_k^* J(x_k)$ converges as desired.

Conversely the convergence of $\Sigma_k^* J(x_k)$ implies, by the Uniform Boundedness Principle, that $\sup_n |\Sigma_{j\le n} J(x_j)|_1^{**}$ is finite and the proof is complete.

Let $C(X) = \{\{J(x_j)\}: \Sigma_j^* J(x_j)$ converges in $\sigma(A(X_s)^*, A(X_s))\}$. As a consequence of Theorem 3.1, we may equip $C(X)$ with the norm $|\ |_1^{**}$ defined by

$$|\{J(x_j)\}|_1^{**} = \sup_n \left|\sum_{k\le n} J(x_k)\right|_1^{**}$$

and a canonical second norm $|\ |_2^{**}$ defined by

$$|\{J(x_j)\}|_2^{**} = \sum_j \frac{|J(x_j)|_1^{**}}{2^j} = \sum_j \frac{|x_j|_1}{2^j}.$$

Theorem 3.2. *Let $C(X)_s = (C(X), |\ |_1^{**}, |\ |_2^{**})$. Then $C(X)_s$ is γ-complete, normal and the map i defined for $x = \Sigma_j x_j$ in X by $i(x) = \{J(x_j)\}$ is a linear isomorphism of X into $C(X)$. Further, i is an isometry with respect to both norms and $i(X_s)$ is γ-dense in $C(X)_s$.*

Proof. Let $\{p_k\}$ denote the sequence of projections associated with the Schauder decomposition of $A(X_s)$. By [10, Theorem 6.2] this decomposition is monotone so that the decomposition constant K_2 of $A(X_s)$ is 1. Hence for any $F = \Sigma_k^* p_k^*(F)$ in $A(X_s)^*$,

$$\text{(A)} \qquad |F|_1^{**} \ge \sup_n \left|\sum_{k\le n} p_k^*(F)\right|_1^{**}.$$

The reverse inequality being trivial, we obtain equality in (A). Let now $\{F_n\}$ be a γ-Cauchy sequence in $C(X)_s$ where $F_n = \{J(x_{n,k})\}$, $x_{n,k} \in X_k$. Then $\sup_n |F_n|_1^{**} = K < \infty$ and $|J(x_{n,k}) - J(x_{m,k})|_1^{**} \to 0$ as $n, m \to \infty$, $k = 1, 2, \cdots$. Since $J(X_k)$ is complete, there exists y_k in X_k such that $J(x_{n,k}) \to J(y_k)$, $k = 1, 2, \cdots$. Let $f \in A(X_s)$ with $|f|_1^* = 1$. For any arbitrary but fixed m,

$$\left|\sum_{k\le n} J(y_k)f\right| = \lim_n \left|\sum_{k\le n} J(x_{n,k})f\right|$$
$$\le \underline{\lim}_n \left|\sum_{k\le m} J(x_{n,k})\right|_1^{**} \le \liminf_n |F_n|_1^{**} \le K$$

so that, by Theorem 3.1, $\{J(y_k)\} \in C(X)$. Hence $C(X)_s$ is γ-complete and this proof may also be used to show that it is normal.

The map i is obviously well defined. It is 1-1 since $\{X_j\}$ is a Schauder decomposition of X. That it is an isometry with respect to $|\,|_1$ and $|\,|_2$ is a consequence of the normality of X_s and the fact that $\Sigma_k x_k = x$ in X whenever $\Sigma_k J(x_k) = J(x)$ in $A(X_s)^*$ and conversely.

Finally for $\{J(x_k)\} \in C(X)$, $\Sigma_{k\le n} J(x_k) \in i(X)$ and $\{\Sigma_{k\le n} J(x_k)\} \xrightarrow{\gamma} \{J(x_k)\}$. Hence $i(X_s)$ is γ-dense in $C(X)_s$ and the proof is complete.

We note that if $\{X_j\}$ is nonmonotone, i reduces to a linear isomorphism with respect to both norms.

As an illustration of this last theorem, observe that the unit vectors $\{e_n\}$, $e_n = \{\delta_{n,j}\}$, form a nonboundedly complete basis for c_0 so that c_{0s} is not γ-complete. Its completion is $m_s = (m, \sup_n |a_n|, \Sigma\, |a_n|/2^n)$. Here, however, $m = l^* = A(c_{0s})^*$. We shall soon see that this is a characteristic property of bases in B-spaces.

We also note that $C(X)$ is isometrically isomorphic to the set of all sequences $\{x_n\}$, $x_n \in X_n$, such that $\sup_n |\Sigma_{k\le n} x_k|_1 < \infty$ with the norm (cf. [9])

$$|\{x_n\}| = \sup_n \left| \sum_{k\le n} x_k \right|_1 .$$

Definition 3.3. *Let X be a B-space with a (monotone) Schauder decomposition. The two-norm space $C(X)_s$ of Theorem 3.2 is called the canonical γ-completion of X_s. The space $C(X)$ is called the bounded completion of X.*

Theorem 3.4. *Let $\{X_j\}$ be a (monotone) Schauder decomposition of a B-space X. Then $C(X)$ is identical with $A(X_s)^*$ if and only if X_s is k-reflexive.*

Proof. Let I be the identity map on $A(X_s)^* = \bigoplus \Sigma_j^* A(k\text{-}X_s)_j$ where $\{A(k\text{-}X_s)_j\}$ denotes the Schauder decomposition of $A(k\text{-}X_s)$. By definition, $C(X) = \Sigma_j^* J(X_j)$, where $J(X_j) \subseteq A(k\text{-}X_s)_j$ [10, Theorem 4.4]. Hence $I\colon C(X) \to A(X_s)^*$. We thus have

$$\begin{aligned} I \text{ is onto} &\Leftrightarrow \bigoplus \sum_j{}^* J(X_j) = \bigoplus \sum_j{}^* A(k\text{-}X_s)_j \\ &\Leftrightarrow J(X_j) = A(k\text{-}X_s)_j, \qquad j = 1, 2, \cdots, \\ &\Leftrightarrow J : X_s \xrightarrow{\text{onto}} k^2\text{-}X_s \\ &\Leftrightarrow X_s \text{ is } k\text{-reflexive} \end{aligned}$$

by [10, Theorem 4.7] and this concludes the proof.

Corollary 3.5. *For a B-space X with a basis, X_s is γ-complete if and only if it is γ-reflexive.*

Proof. Every γ-reflexive space is obviously γ-complete. On the other hand, the γ-completeness of X_s implies the boundedly completeness of the basis for X. Hence $X \cong C(X) = A(X_s)^*$ by Theorem 3.4, that is X_s is γ-reflexive.

Corollary 3.6 [9, p. 143]. *Let X be a B-space which has a basis $\{x_n\}$ and let $Y = [y_n]$ be the closed linear subspace of X_1^* spanned by the y_n, $y_n(x_j) = \delta_{n,j}$. Then Y^* is isomorphic to the B-space of scalars*

$$E = \left\{ \{b_n\} : \sup_n \left| \sum_{k \le n} b_k x_k \right|_1 < \infty \right\}$$

where the norm is defined by $\|\{b_n\}\| = \sup_n |\Sigma_{k \le n} b_k x_k|_1$.

Proof. This follows from our remarks made earlier and the fact that the γ-completion of X_s in this case is $Y^* = A(X_s)^*$.

Corollary 3.7. *Let X be a B-space with a (monotone) shrinking Schauder decomposition $\{X_j\}$. Then X_1^{**} is isomorphic (respectively isometrically isomorphic to) to $C(X)$ if and only if each X_j is reflexive.*

Proof. By [10, Corollary 3.5], $A(X_s) = X_1^*$ so that $A(X_s)^* = X_1^{**}$ and the conclusion now follows from Theorem 3.4.

We may remark that if each X_j is of dimension 1, Corollary 3.7 reduces to a result of R. C. James (see [9, p. 143]).

Theorem 3.8. *Let $(X, |\ |_1)$ be a B-space with a Schauder decomposition $\{X_j\}$. If i is defined as in Theorem 3.2, then $i(X) = \{\{J(x_j)\}: \Sigma_j J(x_j)$ converges in $A(X_s)^*\}$. Further $\{X_j\}$ is boundedly complete if and only if i is bijective, that is, $i(X) = C(X)$.*

Proof. For $x = \Sigma_j x_j$ in X, $i(x) = \{J(x_j)\}$ and $\Sigma_j J(x_j)$ converges in $A(k\text{-}X_s)$, hence in $A(X_s)^*$. Conversely, if $\{J(y_j)\} \in C(X)$ such that $\Sigma_j J(y_j)$ converges, then $\Sigma_j J(y_j) \in [\bigcup_j J(X_j)] = J(X)$, where [ ] denotes closed linear span. Thus for some y in X, $J(y) = \Sigma_j J(y_j)$ and $i(y) = \{J(y_j)\}$.

If $\{X_j\}$ is boundedly complete for X, then so is $\{J(X_j)\}$ for $J(X)$. Hence

$$\{J(x_j)\} \in C(X) \Leftrightarrow \sup_n \left| \sum_{j \le n} J(x_j) \right|_1^{**} < \infty \Leftrightarrow \sum_j J(x_j) \text{ converges}$$

$$\Leftrightarrow \{J(x_j)\} \in i(X) \Leftrightarrow C(X) = i(X).$$

The following result was observed in [10].

Theorem 3.9. *Let $\{X_j\}$ be a monotone Schauder decomposition for a B-space X. The canonical two-norm space X_s is γ-reflexive if and only if X_s is k-reflexive and γ-complete.*

Proof. X_s is γ-reflexive $\Leftrightarrow$ $J: X \to A(X_s)^*$ is 1-1 and onto $\Leftrightarrow$ i is bijective and $A(X_s)^* = C(X) \Leftrightarrow X_s$ is k-reflexive and $\{X_j\}$ is boundedly complete by Theorems 3.4 and 3.8.

4.0 **Pseudo-reflexivity.** Let $(X, \|\ \|_1)$ be a B-space and Y a closed linear subspace of X_1^*. Following Singer [9], we shall say that X is *Y-pseudo reflexive* if the canonical map J from X into Y^* is a linear isomorphism onto Y^*. If J is an isometry as well, X will be called *Y-reflexive*. Singer has shown [9, p. 140] that a necessary and sufficient condition for X to be Y-reflexive is that the closed unit ball S of X be complete for the topology $\sigma(X, Y)$ and that Y be an X-total subspace of X_1^*.

Let us observe that given a total, norming and closed linear subspace Y of X_1^*, if there exists a norm $\|\ \|_2$ on X such that $A(X_s) = Y$, then X is Y-reflexive if and only if X_s is γ-reflexive. In general, however, such a norm need not exist. If Y is separable, a suitable second norm may be defined on X such that for the resulting two-norm space X_s we have $A(X_s) = Y$ [3. p. 124]. In this case we can give a different proof of the following theorem of Singer [9, p. 142]:

Theorem. *Let $(X, \|\ \|_1)$ be a B-space and Y a separable linear subspace of X_1^*. Then X is Y-reflexive if and only if S, the closed unit ball of X, is sequentially complete for the topology $\sigma(X, Y)$ and Y is an X-total subspace of X_1^*.*

Proof. The necessity is clear and it is also obvious that Y must be norming. On the other hand, to show that these conditions are sufficient, let $|x|_2 = \Sigma_n |f_n(x)|/2^n$ where $\{f_n\}$ is dense in $\{f \in Y: |f|_1^* = 1\}$. Then [3, p. 124], X_s is normal, γ-precompact and $A(X_s) = Y$. Convergence in $\|\ \|_2$ being equivalent to convergence of each of the sequences $\{f_n(\cdot)\}$, $n = 1, 2, \cdots$, the sequential completeness of S for the topology $\sigma(X, Y)$ implies that X_s is γ-complete. Hence X_s is γ-compact whence it is γ-reflexive [2, §5.2]. Hence X is Y-reflexive.

We now investigate the Y-reflexivity of a B-space $(X, \|\ \|_1)$ with a Schauder decomposition $\{X_j\}$, where $Y = A(X_s)$ and X_s is the canonical two-norm space of X. The following theorem characterizes the $A(X_s)$-reflexivity of X.

Theorem 4.1. *Let X be a B-space with a (monotone) Schauder decomposition $\{X_j\}$ and let X_s be the canonical two-norm space of X. The following are equivalent:*

(1) *X is $A(X_s)$-pseudo reflexive (respectively $A(X_s)$-reflexive).*

(2) X_s *is γ-semi reflexive (respectively γ-reflexive).*

(3) $\{X_j\}$ *is boundedly complete and* X_j *is reflexive for each* j.

(4) $A(X_s)^* = \Sigma_j J(X_j) = \Sigma_j A(k\text{-}X_s)_j$.

(5) $\{A(k\text{-}X_s)_j\}$ *is shrinking for* $A(X_s)^*$ *and* X_j *is reflexive for each* j.

(6) $X_1^{**} = J(X) \oplus A(X_s)^{\perp}$, *where* $A(X_s)^{\perp}$ *is the set* $\{F \in X_1^{**}: F(A(X_s)) = 0\}$.

Proof. (1) $\Leftrightarrow$ (2) by definition, (2) $\Leftrightarrow$ (3) by Theorems 3.9, [10, Theorems 3.6, 4.7]. On the other hand X_s is γ-reflexive $\Leftrightarrow A(X_s)^* = J(X) = \Sigma_j J(X_j) \subseteq \Sigma_j A(k\text{-}X_s)_j = A(k\text{-}X_s) \subseteq A(X_s)^*$ whence (2) $\Leftrightarrow$ (4). Furthermore, (4) $\Leftrightarrow A(X_s)^* = \Sigma_j A(k\text{-}X_s)_j$ and $J(X_j) = A(k\text{-}X_s)_j$ for every j and this is equivalent to (5). Finally, the equivalence of (1) and (6) is a special case of the following theorem.

Theorem 4.2. *Let* $X_s = (X, \|\ \|_1, \|\ \|_2)$ *be a normal two-norm space satisfying* (n_0) *where* $\|\ \|_2$ *is a* B_0^**-norm. Then* X_s *is γ-reflexive if and only if* $X_1^{**} = J(X) \oplus A(X_s)^{\perp}$, *where* $A(X_s)^{\perp}$ *is the annihilator of* $A(X_s)$ *in* X_1^{**}.

Proof. If X_s is γ-reflexive, then $(A(X_s), \|\ \|_1^*)^* = J(X) \cong X$. Hence X_1^{**} is isometrically isomorphic to the third conjugate of $A(X_s)$. Hence by a theorem of Diximier (see for instance [4, p. 426]), the projection

$$\pi: X_1^{**} \longrightarrow J(X)$$

defined by $\pi(F) = F|_{J(X)}$ for F in X_1^{**} is continuous with norm 1 and kernel $A(X_s)^{\perp}$. Hence $X_1^{**} = J(X) \oplus A(X_s)^{\perp}$.

Conversely, since X_s satisfies (n_0), $A(X_s)$ is a closed subspace of X_1^*. Therefore, given $f \in A(X_s)^*$, by the Hahn-Banach theorem there exists $F \in X_1^{**}$ with the same norm as that of f and such that $F|_{A(X_s)} = f$. Hence there exist $x \in X$ and $g \in A(X_s)^{\perp}$ such that $F = J(x) + g$. Now for any ξ in $A(X_s)$ we have $F(\xi) = J(x)(\xi)$, that is, $f(\xi) = J(x)(\xi)$. Thus $f \equiv J(x)$ on $A(X_s)$ so that we may conclude $A(X_s)^* \subseteq J(X)$. Since the other inclusion is obvious, we have $A(X_s)^* = J(X)$, that is X_s is γ-reflexive.

5.0 **Direct sum of B-spaces.** We now consider B-spaces X which arise as a countable direct sum of other B-spaces (X_n). In our case the structure of X and its norm will be determined by an underlying space B, a BK space. Many sequence spaces may be considered as examples of such spaces. Several such spaces have also been used as counterexample spaces by many authors.

The spaces X_n may be considered as imbedded in X, hence treated as subspaces of X. In general, they do not form a decomposition of X for norm convergence. Nevertheless, each x in X shall correspond to a unique sequence $\{x_n\}$, $x_n \in X_n$. Also $\{X_n\}$ will form a 'generalized decomposition' of X in the sense $\bigcup_k P_k^*(X^*)$ shall be X-total in X^*, where $P_k: X \longrightarrow X_k$. We shall see that a natural two-norm space X_s may be defined as before and the usual arguments of two-norm

spaces in Schauder decompositions adapted to spaces of this kind.

As an example, consider the space (m). Here each $X_n = \mathbf{R}$ and the space B is (m) itself. The vectors (e_n) form a generalized basis of (m). A natural extension of the definition of boundedly complete bases shows that (e_n) is boundedly complete for (m). Note also that the associated two-norm space (m_s) is γ-complete. We shall observe later that this is a special case of the analogue of [10, Theorem 3.6].

We begin with

Definition 5.1. A *BK space* $(B, |\ |_B)$ is a Banach space of real sequences such that $|\{a_n\}|_B \geq \Sigma_n |a_n|/(2^n(1 + |a_n|))$ for elements $\{a_n\}$ in B. The space B is called *normal* if for any two sequences $\{a_n\}$ and $\{b_n\}$, $|a_n| \leq |b_n|$ and $\{b_n\} \in B$ $\Rightarrow \{a_n\} \in B$. We say B is *ordered* if for any two elements $\{a_n\}$, $\{b_n\}$ in B with $|a_n| \leq |b_n|$, we have $|\{a_n\}|_B \leq |\{b_n\}|_B$.

The above definition is equivalent to the usual definition of a BK space since it implies (and is implied by) the fact that the coordinate functionals are continuous.

All BK spaces B considered in this paper are assumed to have the following properties:

(1) B is ordered and normal,

(2) B contains the unit vectors $\{e_n = \{\delta_{n,j}\}\}$ and $|e_n|_B = 1$ for all n, and

(3) if $a = \{a_n\} \in B$, then $|a|_B = \sup_n \{|\,|a_j|\,\epsilon_{jn}|_B\}$ where $\epsilon_{jn} = 1$ for $j \leq n$, $= 0$ for $j > n$.

This last postulate has the advantage of making $|\ |_B$ "monotone". In most cases of interest, the natural norm on the BK space is either equal or equivalent to a norm introduced in the above manner.

Definition 5.2. *Let $(X_n, |\ |_n)$, $n = 1, 2, \cdots$, be a sequence of B-spaces and $(B, |\ |_B)$ a BK space. By the direct sum with respect to B of (X_n), we shall mean the set of all those sequences $x = \{x_n\}$ with $x_n \in X_n$ and such that $\{|x_n|_n\} \in B$. This set will be denoted by $B\text{-}\Sigma_n X_n$, and when there is no cause for confusion about the underlying space B, merely by $\Sigma_n X_n$.*

Theorem 5.3. *Let $X = B\text{-}\Sigma_n X_n$ and for $x = \{x_n\} \in X$, define $\|x\| = |\{|x_n|_n\}|_B$. Then $(X, \|\ \|)$ is a B-space.*

Proof. We need only show that $(X, \|\ \|)$ is complete. Let $\{x_n\}$ be a Cauchy sequence in X where $x_n = \{x_n^{(j)}\}_{j=1}^{\infty}$. Then given $\epsilon > 0$, $\exists N(\epsilon)$ such that $n, m > N$ implies

$$\|x_n - x_m\| = \sup_p |\{|x_n^{(j)} - x_m^{(j)}|_j \epsilon_{jp}\}|_B < \epsilon.$$

Since B is ordered, for $n, m > N$ and any j,

$$|x_n^{(j)} - x_m^{(j)}|_j = |\{|x_n^{(k)} - x_m^{(k)}|_k \delta_{j,k}\}|_B$$

$$\leq |\{|x_n^{(k)} - x_m^{(k)}|_k \epsilon_{kj}\}|_B < \epsilon.$$

Hence $\{x_n^{(j)}\}$ is Cauchy in X_j for each j, so that $x_n^{(j)} \to x_0^{(j)} \in X_j$ uniformly for all j. Since $\{|x_n^{(j)}|_j\} \in B$ for each n and B is ordered, it follows that $\{|x_n^{(j)}|_j\}_{n=1}^{\infty}$ is Cauchy in B and converges to $\{|x_0^{(j)}|_j\}$. Since B is complete, $\{|x_0^{(j)}|_j\} \in B$ whence $\{x_0^{(j)}\} \in X$. It is routine to show that x_n converges to $\{x_0^{(j)}\}$ and this completes the proof.

Consider the subspace $Y_n \subseteq X$ consisting of all elements of the form $\{x_n \delta_{n,j}\}$, $\delta_{n,j}$ being the Kronecker δ. The map t_n from X_n onto Y_n defined by $t_n(z) = \{z\delta_{n,j}\}$, $z \in X_n$, is clearly an isometry. We shall, therefore, occasionally identify X_n and Y_n in the future and consider X_n as a subspace of X.

Definition 5.4. *Let $X = B\text{-}\Sigma_n X_n$. The canonical two-norm space of X is the two-norm space $(X, \| \|, \| \|_2)$ where, for $x = \{x_n\} \in X$, $\|x\|_2 = \Sigma_n |x_n|_n/2^n$.*

Theorem 5.5. *Let X_s be the canonical two-norm space of $X = B\text{-}\Sigma_n X_n$. Then X_s is normal and satisfies (n_0).*

Proof. Let $x_n \xrightarrow{\gamma} \xi$ in X_s where $x_n = \{x_{n,j}\}_{j=1}^{\infty}$, $\xi = \{\xi_j\}$, $x_{n,j}, \xi_j \in X_j$, $j = 1, 2, \cdots$. For each p,

$$|\{|\xi_j|_j \epsilon_{jp}\}|_B = \left|\sum_{j=1}^{p} |\xi_j|_j e_j\right|_B$$

$$= \left|\lim_n \left(\sum_{j=1}^{p} |x_{n,j}|_j e_j\right)\right|_B$$

$$= \lim_n \left|\sum_{j=1}^{p} |x_{n,j}|_j e_j\right|_B \leq \liminf_n \{\|x_n\|\}.$$

It follows that $\|\xi\| \leq \liminf_n \|x_n\|$ showing that X_s is normal. That X_s satisfies (n_0) is clear since, for any x in X, $|x_j|_j \leq \|x\|$ (B is ordered) whence $\|x\|_2 \leq \|x\|$.

An immediate corollary of the above theorem is that the canonical map $J: X \to A(X_s)^*$ is an isometry. In what follows we write X_1^* for $(X, \| \|)^*$. We shall have no occasion to consider the space $(X_1, | |_1)^*$ and once again the reader is asked not to confuse the latter space for X_1^*.

Definition 5.6. *Let $X = B\text{-}\Sigma_n X_n$. The sequence (X_n) is said to be boundedly complete if for any sequence $\{x_n\}$, $x_n \in X_n$, $\sup_n |\{|x_j|_j \epsilon_{jn}\}|_B < \infty$ implies $\{x_n\} \in X$.*

Definition 5.7. *Let $X = B\text{-}\Sigma_n X_n$ and let Z_n be the subspace of X consisting of sequences whose first n coordinates are zeros. For $f \in X_1^*$ let $|f|_{1,n}^* = \sup\{|f(x)|: x \in Z_n, \|x\| = 1\}$. Then (X_n) is said to be shrinking for $f \in X_1^*$ if $\lim_n |f|_{1,n}^* = 0$.*

In the special case when (Y_n) forms a Schauder decomposition of X, the above definitions reduce to the usual definitions of boundedly completeness and the shrinking property. We also note that under this definition the unit vectors form a generalized boundedly complete basis for (m) and are shrinking for every element in l. This is a special case of the following

Theorem 5.8. *Let $X = B\text{-}\Sigma_n X_n$. Then the sequence (X_n) is boundedly complete if and only if X_s is γ-complete. Further (X_n) is shrinking for $f \in X_1^*$ if and only if $f \in A(X_s)$.*

The proof is similar to those of [10, Theorems 3.4, 3.6]. We indicate the lines of the proof of the second assertion leaving that of the first to the reader.

Proof of second assertion. Given $\epsilon > 0$ and $f \in A(X_s)$ we can find $\delta > 0$ such that $\|x\| \leq 1$ and $\|x\|_2 < \delta$ imply $|f(x)| < \epsilon$. Choose $N > 0$ such that $n > N$ implies $\Sigma_{j>n} 2^{-j} < \delta$. Then for $\xi = \{\xi_j\}$ in Z_n with $\|\xi\| = 1$ we have $\|\xi\|_2 < \delta$ so that $|f(\xi)| < \epsilon$. This shows $|f|_{1,n}^* < \epsilon$ for $n > N$.

For the other part, we identify $x_{n,j} \in X_j$ with $\{x_{n,j}\delta_{j,p}\}_{p=1}^{\infty}$ in Y_j, $j = 1, 2, \ldots$. Let $f \in X_1^*$, $x_n \to 0\,(\gamma)$ in X_s and suppose $\|x_n\| \leq 1$. Find $N > 0$ such that $|f|_{1,N}^* < \epsilon/2$, $\epsilon > 0$ being arbitrary. There exists M such that $|f(x_{n,j})| < \epsilon/2N$ whenever $n > M$ and $1 \leq j \leq N$. Clearly for $n > M$ we have

$$|f(x_n)| \leq \left|\sum_{j=1}^{N} f(x_{n,j})\right| + \left|f\left(x_n - \sum_{j=1}^{N} x_{n,j}\right)\right| < N \cdot \frac{\epsilon}{2N} + \frac{\epsilon}{2} = \epsilon$$

showing that f is γ-linear. This completes the proof.

For $x = \{x_n\} \in X$, the map $S_n: X \to X$ defined by $S_n(x) = \{x_j \epsilon_{jn}\}$ is clearly continuous. Hence the projection operator $P_n: X \to Y_n$, $P_n(x) = \{x_j \delta_{j,n}\}$ is also continuous. Since $P_n^*(X_1^*) \subseteq A(X_s)$ trivially and $A(X_s)$ is closed, $\bigoplus \Sigma_{k=1}^{\infty} P_k^*(X_1^*) \subseteq A(X_s)$. On the other hand for any f in $A(X_s)$, $|f|_{1,n}^* \to 0$ by the last theorem so that $|f - \Sigma_{k \leq n} P_k^*(f)|_1^* \to 0$ as $n \to \infty$. From these considerations we get

Theorem 5.9. *Let $X = B\text{-}\Sigma_n X_n$, X_s its canonical two-norm space. Then $\{P_k^*(X_1^*)\}$ is a Schauder decomposition of $A(X_s)$.*

6.0 The γ-completion of X_s, $X = B\text{-}\Sigma_n X_n$. As in the case of B-spaces with Schauder decompositions, the γ-completion of the canonical two-norm space X_s where $X = B\text{-}\Sigma_n X_n$ may be constructed. One easily verifies that in this case

the space $C(X)$ may be identified with the space $B\text{-}\Sigma_n' X_n$ of all sequences $\{x_n\}$, $x_n \in X_n$, such that

$$N(\{x_n\}) = \sup_n \{\|\{|x_j|_j \epsilon_{jn}\}\|_B\}$$

is finite. The norm in the B-space $C(X)$ is $N(\ )$. It is also possible to describe the space $C(X)$ in a different way.

The F-space $(s, |\ |_s)$ of all scalar sequences with the Fréchet metric $|\ |_s$ has the unit vectors $\{e_k\}$ for a basis. Since B is dense in s, $(B, |\ |_s)^*$ is the linear span $[e_n]$. It is clear that $B_s = (B, |\ |_B, |\ |_s)$ is normal ($|\ |_B$ being 'monotone') so that $A(B_s)$ is the closure of $[e_n]$ in $(B, |\ |_B)^*$ [2, p. 277]. Hence $\{e_n\}$ is a basis for $A(B_s)$ and a weak-* basis for $A(B_s)^*$. It is easily verified that $A(B_s)^*$ is an ordered, normal BK space consisting of scalar sequences $\{a_k\}$ with $|\{a_k\}|_0 = \sup_n \{|a_k \epsilon_{kn}|_B\}$ finite. But this is precisely the space $C(B)$ so that $C(B) = A(B_s)^*$, $|\ |_0$ being its norm. We now have

Theorem 6.1. $B\text{-}\Sigma_n' X_n = C(B)\text{-}\Sigma_n X_n$.

Proof. We need only show that the set on the right is contained in the one on the left. The normality of B_s shows that the canonical map $B \longrightarrow A(B_s)^* = C(B)$ is an isometry. Let $|||\ |||$ denote the norm in $C(B)\text{-}\Sigma_n X_n$ and suppose $\{x_k\}$ is an element of it. Then,

$$\begin{aligned} |||\{x_k \epsilon_{kn}\}||| &= \|\{|x_k|_k \epsilon_{kn}\}\|_{C(B)} \\ &= \|\{|x_k|_k \epsilon_{kn}\}\|_B = N(\{x_k \epsilon_{kn}\}), \end{aligned}$$

showing that $\{x_k\} \in B\text{-}\Sigma_n' X_n$ and completing the proof.

7.0 **The α-duals.** Given the space $X = B\text{-}\Sigma_n X_n$, we shall call the canonical two-norm space of the space $A(X_s)$ as the *α-dual* of X_s and denote it by $\alpha\text{-}X_s$. Higher α-duals are defined by induction. We note that $\alpha^2\text{-}X_s$ coincides with the k-dual (see §1) of $\alpha\text{-}X_s$. In the special case when (Y_n) forms a Schauder decomposition of X, the α-duals of X_s are identical with its k-duals.

It was shown in §6 that for any BK space B, $A(B_s)$ has the unit vectors $\{e_k\}$ for a basis. Thus one may define the α-duals of B_s (called $T(B_s)$, $T^2(B_s)$ etc., in [6]). It was shown in [6] that in general $\alpha^2\text{-}B_s$ is isometrically imbedded in B_s and that this isometry is onto B_s if and only if B has $\{e_k\}$ for a basis.

We shall show here that these results may be extended to our present setting. For $X = B\text{-}\Sigma_n X_n$, we call X_s "*α-reflexive*" if $\alpha^2\text{-}X_s$ is isometrically isomorphic to X_s under the canonical map. We begin by showing

Theorem 7.1. *Let $X = B\text{-}\Sigma_n X_n$. Then, if X_j is reflexive for each j, J^{-1}: $\alpha^2\text{-}X_s \xrightarrow{\text{onto}} X_s$, where J denotes the canonical map from X_s into $A(X_s)^*$.*

Proof. *Let* $\{A(X_s)_k, p_k\}$ be the Schauder decomposition of $A(X_s)$, $\{A(X_s)^*_k, p^*_k\}$ the Schauder decomposition of $A(\alpha\text{-}X_s)$ in the norm topology of $A(X_s)^*$. By the reflexivity of X_j for each j, it follows that

$$A(\alpha\text{-}X_s) = \left[\bigcup_j A(X_s)^*_j\right] = \left[\bigcup_j J(Y_j)\right].$$

Since J is an isometry from X into $A(X_s)^*$, $J^{-1}A(\alpha\text{-}X_s) = [\bigcup_j Y_j] \subseteq X$, the closure in [ ] being with respect to the norm in X. This completes the proof.

Theorem 7.2. *Let* $X = B\text{-}\Sigma_n X_n$. *If* B_s *is* α*-reflexive then* (X_n) *is a Schauder decomposition of* X *in the sense that* $X = \bigoplus \Sigma_n Y_n$. *Further if* B_s *is* α*-reflexive,* X_j *is reflexive for each* j *if and only if* X_s *is* α*-reflexive.*

Proof. Let $x = \{x_n\} \in X$, $x_n \in X_n$. Let $y_n = \{x_j \delta_{j,n}\}$. We shall show that $\|x - \Sigma_{j\le n} y_j\| \to 0$. Indeed,

$$\left\|x - \sum_{j\le n} y_j\right\| = \sup_m \left\{\left|\sum_{j=n+1}^{m} |x_j|_j e_j\right|_B\right\}$$
$$\le \left|\sum_{j>n} |x_j|_j e_j\right|_B \to 0$$

since by the α-reflexivity of B_s, $\{e_j\}$ is a basis for B and $\{x_n\} \in X \Rightarrow \{|x_n|_n\} \in B \Rightarrow \Sigma_n |x_n|_n e_n$ converges in B.

To show the other part, observe that $\{Y_n\}$ is a Schauder decomposition of X implies that α-reflexivity coincides with k-reflexivity. An application of [10, Theorem 4.7] completes the proof.

The following theorem characterizes the α-reflexivity of X_s.

Theorem 7.3. *Let* $X = B\text{-}\Sigma_n X_n$. *Then* X_s *is* α*-reflexive if and only if* (X_n) *forms a Schauder decomposition of* X *and* X_n *is reflexive for each* n.

Proof. If X_s is α-reflexive, $J(X) = A(\alpha\text{-}X_s)$. Since $A(\alpha\text{-}X_s)$ has a Schauder decomposition $\{A(X_s)^*_k, p^*_k\}$, it follows that $A(X_s)^*_k = p^*_k(J(X))$ and $\{p^*_k(J(X)), p^*_k\}$ forms a Schauder decomposition for $J(X)$. But for any k, $p^*_k(J(X)) = J(Y_k)$. For, if $f = \Sigma_k f_k \in A(X_s)$, $x = \{x_k\} \in X$,

$$p^*_k(J(x))f = J(x)p_k(f) = f_k(x) = f_k(y_k)$$
$$= J(y_k)f_k = J(y_k)f$$

so that $p^*_k(J(x)) = J(y_k)$, $y_k = \{x_j \delta_{k,j}\}$, whence $p^*_k(J(X)) \subseteq J(Y_k)$. Since $p^*_k(J(y_k)) = J(y_k)$, the other inclusion is clear and our assertion is proved. It follows then that $A(X_s)^*_k = J(Y_k)$ for every k, whence Y_k is reflexive for every k, and $J(X) = \bigoplus \Sigma_k J(Y_k)$, that is, $X = \bigoplus \Sigma_k Y_k$.

The converse follows from [10, Theorem 4.7] and the fact that if $X = \bigoplus \Sigma_k Y_k$, then α-reflexivity coincides with k-reflexivity. This completes the proof.

Corollary 7.4. *Let $X = B\text{-}\Sigma_n X_n$. If X_s is γ-complete and α-reflexive, then X_s is γ-reflexive.*

Let us remark that the converse to the above corollary is false. One may take $X = (m)$. Here m_s is γ-reflexive but not α-reflexive. However, we have the following theorem.

Theorem 7.5 (cf. [8]). *Let $X = B\text{-}\Sigma_n X_n$. The following are equivalent:*

(1) *X is reflexive.*

(2) *X_s is α-reflexive, (X_n) is boundedly complete and (X_n) is shrinking for $(X, \| \|)^*$.*

(3) *X_n is reflexive for each n, (X_n) is boundedly complete and (X_n) is shrinking for $(X, \| \|)$.*

Proof. X is reflexive if and only if X_s is γ-reflexive and saturated [2, Theorem 3.7]. Recall that X_s is saturated if $A(X_s) = (X, \| \|)^*$. The equivalence of (1) and (3) now follows from Theorem 6.4 and Theorem 5.8. That (2) implies (1) is clear since $X = \bigoplus \Sigma_n X_n$ in this case, α-reflexivity is identical with k-reflexivity and the conclusion follows from [8, Theorem 2]. It remains to show that (1) implies (2).

The reflexivity of X immediately implies the reflexivity of Y_n, hence that of X_n, for each n. As observed before, X_s is γ-reflexive and saturated so that (X_n) is boundedly complete and (X_n) is shrinking for $(X, \| \|)^*$. To complete the proof, we must show that X_s is α-reflexive. Since X is reflexive, X_n is reflexive for each n and $A(X_s) = (X, \| \|)^*$, we have

$$J(X) = (X, \| \|)^{**} = A(X_s)^* = \sum_k{}^* \, p_k^*(A(X_s)^*) = \sum_k{}^* \, J(Y_k)$$

where Σ^* denotes convergence in the weak-* topology of $(X, \| \|)^{**}$. This means that (Y_k) is a weak Schauder decomposition, and hence [7, I.20] a Schauder decomposition of X in the norm topology. Hence X_s is α-reflexive by Theorem 7.3 completing the proof.

Corollary 7.6. *If $X = B\text{-}\Sigma_n X_n$, where (X_n) does not form a Schauder decomposition of X, then X is not norm-reflexive. In particular, for any sequence (X_n) of B-spaces, $m\text{-}\Sigma_n X_n$ is not reflexive.*

8.0 Use as counterexamples. We conclude this paper with a remark on the usefulness of the spaces $X = B\text{-}\Sigma_n X_n$ as counterexamples. Day(1) gives examples of reflexive B-spaces not isomorphic to uniformly convex spaces. Here one takes $B =$

(1) Bull. Amer. Math. Soc. 47 (1941), 313–317.

l^p, $1 < p < \infty$, and the spaces X_n are finite dimensional spaces. If we take $B = l$ and $X_n = c_0$ for each n (all over $\mathbf{R}$), we obtain [10, Remark after Theorem 5.3] an example of a B-space with a boundedly complete Schauder decomposition but not isometrically isomorphic to a dual space (cf. [8]). Incidentally, this example also illustrates Theorem 7.2 since l_s is known to be α-reflexive. If one takes $B = l$ and $X_n = m$ for each n, one obtains an example of a nonseparable B-space with a Schauder decomposition. Finally, by taking $B = m$, $X_n = c_0$ for each n, one obtains [6, p. 113] an example of a B-space with a *countable determing set* (that is a countable set $D = \{f_n\} \subset \{f : |f|_1^* \leq 1\}$ such that $\|x\| = \sup\{|f(x)| : f \in D\}$, $\| \|$ and $| |_1^*$ being the norms in the space and its conjugate respectively) which is neither separable nor the dual of a separable space. For other counterexamples concerning two-norm spaces, the reader is referred to [6].

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