ON DERIVED FUNCTORS OF LIMIT

BY

DANA MAY LATCH

ABSTRACT. If $\mathcal{A}$ is a cocomplete category with enough projectives and $\mathcal{C}$ is a $\lambda$-finite small category, then there is a spectral sequence which shows that the cardinality of $\mathcal{C}$ and colimits over finite initial subcategories $\mathcal{C}'$ of $\mathcal{C}$ are determining factors for computation of derived functors of colimit. Applying a recent result of Mitchell to this spectral sequence we show that if the cardinality of $\mathcal{C}$ is at most $\kappa$, and the flat dimension of $\Delta^k Z$ (constant diagram of type $\mathcal{C}^{\text{op}}$ with value $Z$) is $k$, then the derived functors of $\text{lim}_C : \mathcal{A}^{\mathcal{C}} \to \mathcal{A}^X$ vanish above dimension $n + 1 + k$.

Introduction. The purpose of the paper is to study derived functors of limit. This topic was first considered by Milnor [7], Yeh [17], and Roos [14]. The results of Roos, Noebeling [11], André [1], and Laudal [6] all show that derived functors of colimit can be interpreted as the homology of a simplicial complex. This paper introduces a spectral sequence, which isolates the cardinality of $\mathcal{C}$ and colimits over finitely generated initial subcategories $\mathcal{C}'$ of $\mathcal{C}$ as determining factors for the vanishing of derived functors of colimit (dually limit).

If $\mathcal{A}$ is an abelian category, Stauffer [16] shows that there exists an AB5 category $D(\mathcal{A})$, called the directed completion of $\mathcal{A}$, and an exact, Ext-preserving, projective preserving embedding $J : \mathcal{A} \to D(\mathcal{A})$. $D(\mathcal{A})$ is similar to the cocontinuous extension of $\mathcal{A}$ studied by Hilton [4] and to Grothendieck's category of Pro-objects of $\mathcal{A}$ [3].

If $\mathcal{A}$ is cocomplete, we get a coreflection $U : D(\mathcal{A}) \to \mathcal{A}$ of $J : \mathcal{A} \to D(\mathcal{A})$. These two functors together give rise to a factorization

$$\text{colim}_C : \mathcal{A}^{\mathcal{C}} \to \mathcal{A} \quad \text{into} \quad \mathcal{A}^{\mathcal{C}} \xrightarrow{\text{colim}_C} D(\mathcal{A}) \xrightarrow{U} \mathcal{A}.$$

When $\mathcal{C}$ is a $\lambda$-finite small category and $\mathcal{A}$ a cocomplete category with pro-
jectives, we apply a well-known technique of Grothendieck [2] to the above factorization of \( \text{colim}_C : \mathcal{C} \rightarrow \mathfrak{D} \). This results in a first quadrant spectral sequence

\[
E^2 = (L_* U) \left( L_* \text{colim}_C \right) (f^C(\mathcal{A})) \cong L_* \text{colim}_{C' \in \mathcal{F}(C)} \left( L_* \text{colim}_C \right) (\mathcal{A} | C')
\]

which converges to \( (L_* \text{colim})_C(\mathcal{A}) \), where \( \mathcal{A} \) is a diagram in \( \mathfrak{D} \) of type \( C' \) and \( \mathcal{F}(C) \) the \( \mathcal{I} \)-finite directed ordered set of all finite initial subcategories \( C' \).

Many generalizations of ring theoretic results prove useful in applying the spectral sequence. Using a recent result of Mitchell [10], we show that if \( C \) is a \( \mathcal{I} \)-finite small category of cardinality at most \( \kappa_n \) and

\[
k = \sup \{ \mathfrak{m} | 0 \neq L_{n} \text{colim}_C : \mathcal{A} b^C \rightarrow \mathfrak{A} b \},
\]

then \( R' \text{lim}_{C_{op}} : \mathfrak{A} b_{op}^C \rightarrow \mathfrak{A} b \) vanishes for \( r > n + 1 + k \).

I wish to express my appreciation to Alex Heller for his encouragement and aid. Also, I thank Barry Mitchell for many enlightening conversations about this problem.

1. Preliminaries. If \( C \) is a small category, let \( |C| \) denote the set of objects of \( C \) and \( C(p, q) \) the set of morphisms from \( p \) to \( q \). If \( \alpha \) is a morphism of \( C \), then \( d\alpha \) and \( r\alpha \) will denote the domain and range of \( \alpha \), respectively. Let \( \|C\| \) represent the cardinality of the set \( C \). Then \( C \) is said to be an \( n \)-category if \( \|C\| \leq \kappa_n \) for \( n > 0 \), and a finite category if \( \|C\| < \kappa_0 \).

A subcategory \( C' \) of \( C \), denoted by \( C' \subseteq C \), will be called initial if \( \alpha \in C \) with \( r\alpha \in |C'| \) implies \( \alpha \in C' \) (and consequently \( d\alpha \in |C'| \)). It is clear that any initial subcategory is full. Let \( C(p) \) denote the smallest initial subcategory containing \( p \). Then if \( C(p, q) \neq \emptyset \), it is clear that \( C(p, q) \leq C(q) \). Also, \( C' \) initial implies \( C' = \bigcup_{p \in |C'|} C(p') \) and \( C(p') \leq C' \) for every \( p' \in |C'| \).

Definition 1.1. A small category \( C \) is said to be downward finite, \( \mathcal{I} \)-finite, if \( C(p) \) is finite for every \( p \in |C| \).

Let \( \mathcal{F}(C) \) represent the collection of all finitely-generated initial subcategories \( C' \) of \( C \). If \( C \) is \( \mathcal{I} \)-finite, then clearly \( \mathcal{F}(C) \) satisfies the following conditions:

(i) \( \mathcal{F}(C) \) is a directed ordered set under the natural ordering of inclusion of categories, with initial element the empty subcategory \( \emptyset \).

(ii) \( \mathcal{F}(C) \) is \( \mathcal{I} \)-finite, i.e. any finitely-generated initial subcategory has a finite number of initial subcategories.

(iii) If \( C \) is a \( n \)-category, then so is \( \mathcal{F}(C) \), i.e. \( \|C\| \leq \kappa_n \) implies \( \|\mathcal{F}(C)\| \leq \kappa_n \).

(iv) For every \( p \in |C| \), \( C(p) \in \mathcal{F}(C) \).

If \( \mathfrak{D} \) is an abelian category, then \( \mathfrak{D}^C \) will denote the abelian category of all diagrams of type \( C \), i.e. covariant functors \( \mathcal{A} : C \rightarrow \mathfrak{D} \), with \( \mathfrak{D}(\mathcal{A}, \mathcal{B}) \) the abelian group of natural transformations from \( \mathcal{A} \) to \( \mathcal{B} \). In particular, let \( \Delta A : C \rightarrow \mathfrak{D} \)
represent the constant functor with value $A$ and $\Delta^* A : C^{op} \rightarrow \mathcal{A}$ the dual diagram. If $F : \mathcal{A} \rightarrow \mathcal{B}$ is any functor, let $F^C : \mathcal{A}^C \rightarrow \mathcal{B}^C$ denote the canonical functor given by $F^C(\mathcal{A})_p = F(A_p^*)$.

It is well known [8] that if $\mathcal{A}$ is a cocomplete abelian category with enough projectives and/or injectives, then so is $\mathcal{A}^C$. For example, if $\mathcal{A} = \mathcal{A}^b$, the category of abelian groups, then $\mathcal{A}^b^C$ is an AB5 category with enough projectives and injectives.

When $\mathcal{A}$ is complete, there is a functor $\mathbf{W} : \mathcal{A}^C \rightarrow (\mathcal{A}^C)^C$ defined by $(\mathbf{W}A)_C = \text{colim}_C A | C'$ with $(\mathbf{W}A)_C^C = (\mathbf{W}A)_C^C : (\mathbf{W}A)_C^C \rightarrow (\mathbf{W}A)_C^C$ the canonical map of colimits induced by the inclusion $C' \leq C^C$.

**Lemma 1.2.** If $\mathcal{A}$ is a cocomplete abelian category and $C$ is a $\lambda$-finite small category, then

\begin{equation}
\mathcal{A}^C \xrightarrow{\mathbf{W}} (\mathcal{A}^C)^C \xrightarrow{\text{colim}_C} \mathcal{A}^C \xrightarrow{\text{colim}_C} (\mathcal{A}^C)^C
\end{equation}

commutes up to an isomorphism.

This follows easily from the definitions.

Furthermore, when $\mathcal{A}$ cocomplete, there are two associated functors between $\mathcal{A}$ and $\mathcal{A}^C$ for each $p \in |C|$. The first is the canonical evaluation functor $\text{ev}_p : \mathcal{A}^C \rightarrow \mathcal{A}$ defined by $\text{ev}_p(\mathcal{A}) = A_p$, where $\mathcal{A} \in \mathcal{A}^C$. It is exact since exactness in $\mathcal{A}^C$ is "pointwise". The second functor is $E_p : \mathcal{A} \rightarrow \mathcal{A}^C$ which is constructed in the following way. For each $X \in \mathcal{A}$ and $q \in |C|$, let $(E_p X)_q = \prod_{p \rightarrow q} X$, and let $(E_p X)(\beta) : (E_p X)_q \rightarrow (E_p X)_{q'}$, $\beta : q \rightarrow q'$ in $C$, be the canonical morphism such that $(E_p X)(\beta)i_a = i_{\beta.a}$, $i_a : X \rightarrow \prod_{p \rightarrow q} X$ being the natural inclusion into the coproduct. Similarly, for each morphism $f : X \rightarrow Y$ in $\mathcal{A}$, there is a natural transformation $(E_p f) : (E_p X) \rightarrow (E_p Y)$ defined by $(E_p f)i_a = i_a \cdot f$.

**Proposition 1.3.** If $\mathcal{A}$ is cocomplete and abelian, then

(i) $E_p : \mathcal{A} \rightarrow \mathcal{A}^C$ is the coadjoint of $\text{ev}_p : \mathcal{A}^C \rightarrow \mathcal{A}$.

(ii) $E_p : \mathcal{A}^C \rightarrow \mathcal{A}$ is right exact and also preserves projectives (since $\text{ev}_p : \mathcal{A}^C \rightarrow \mathcal{A}$ is exact).

(iii) When $\mathcal{A}$ has enough projectives $\mathcal{A}^C$ has enough canonical projectives of the form $\prod_{q \in |C|} E_q P_q$, $P_q$ projective in $\mathcal{A}$. If $A \in \mathcal{A}^C$ and for each $q \in |C|$, $P_q \rightarrow A_q$ is an epimorphism with $P_q$ projective, then $\prod_{q \in |C|} E_q P_q \rightarrow A$ is an epimorphism in $\mathcal{A}^C$. 
2. \( D(\mathcal{A}) \) and the spectral sequence.

**Theorem 2.1.** Associated with any abelian category \( \mathcal{A} \) there is an AB5 category \( D(\mathcal{A}) \) (called the directed completion of \( \mathcal{A} \)), and a natural embedding \( J : \mathcal{A} \to D(\mathcal{A}) \) such that \( J : \mathcal{A} \to D(\mathcal{A}) \) is exact, full, projective-preserving and \( \text{Ext} \)-preserving (i.e. \( \text{Ext}^n(J(A), J(B)) \cong \text{Ext}^n(A, B) \)). Furthermore, \( J : \mathcal{A} \to D(\mathcal{A}) \) and \( D(\mathcal{A}) \) together satisfy the following universal extension property:

(i) If \( \mathcal{B} \) is any cocomplete abelian category and \( F : \mathcal{A} \to \mathcal{B} \) is right exact, then there exists a unique cocontinuous (i.e. colimit-preserving) functor \( G : D(\mathcal{A}) \to \mathcal{B} \) such that

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{J} & D(\mathcal{A}) \\
\downarrow{F} & & \downarrow{G} \\
\mathcal{B} & \xrightarrow{id} & \mathcal{B}
\end{array}
\]

commutes up to isomorphism.

(ii) If \( \mathcal{B} \) is AB5 and \( F : \mathcal{A} \to \mathcal{B} \) is exact, then \( G : D(\mathcal{A}) \to \mathcal{B} \) is cocontinuous and exact.

For the details of the proof see Stauffer [16].

In particular, when \( \mathcal{A} \) itself is cocomplete there exists a unique cocontinuous (and consequently right exact) functor \( U : D(\mathcal{A}) \to \mathcal{A} \) such that \( U \cdot J \cong \text{id}_\mathcal{A} : \mathcal{A} \to \mathcal{A} \). Thus \( \mathcal{A} \) can be considered as a coreflective subcategory of \( D(\mathcal{A}) \). The next proposition follows easily from the facts that \( U : D(\mathcal{A}) \to \mathcal{A} \) is cocontinuous and \( U \cdot J \cong \text{id}_\mathcal{A} \).

**Proposition 2.2.** If \( \mathcal{C} \) is any small category and \( \mathcal{A} \) is cocomplete and abelian, then \( U(\text{colim}_\mathcal{C} J^{\mathcal{A}}(A)) \cong \text{colim}_\mathcal{C}(A) \) for all \( A \in \mathcal{A}^\mathcal{C} \).

By Proposition 2.2, \( \text{colim}_\mathcal{C} : \mathcal{A}^\mathcal{C} \to \mathcal{A} \) is factored into \( \text{colim}_\mathcal{C} : \mathcal{A}^\mathcal{C} \to D(\mathcal{A}) \) and \( U : D(\mathcal{A}) \to \mathcal{A} \). This factorization, for \( \mathcal{C} \) a \( \downarrow \)-finite small category and a cocomplete abelian category with enough projectives, will yield the spectral sequence which is the major tool of this paper. As a first step, we prove a series of lemmas to show that \( J^\mathcal{C} : \mathcal{A}^\mathcal{C} \to D(\mathcal{A})^\mathcal{C} \) preserve canonical projectives.

For the remainder of this section, \( \mathcal{A} \) will be assumed to be a cocomplete abelian category with enough projectives.

**Lemma 2.3.** For every \( p \in |\mathcal{C}| \), the following diagram commutes.

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{J} & D(\mathcal{A}) \\
\downarrow{E_p} & & \downarrow{E_p} \\
\mathcal{A}^\mathcal{C} & \xrightarrow{J^\mathcal{C}} & (D(\mathcal{A}))^\mathcal{C}
\end{array}
\]
Proof. It suffices to show that, for each \( q \in |C| \), \( J^C(E_p X)_q = E_p (j(X))_q \).

By definition, \( J^C(E_p X)_q = j(E_p X)_q = j(\coprod_{p \in q} X)_q \). Since \( C \) is \( \downarrow \)-finite, \( (E_p X)_q = \coprod_{p \in q} X \) is a finite coproduct. \( J : \mathfrak{A} \to D(\mathfrak{A}) \) additive insures that \( j(\coprod_{p \in q} X) = \coprod_{p \in q} j(X) = E_p (j(X))_q \), and the lemma follows.

Using \( \downarrow \)-finiteness of \( C \), a proof similar to the above yields the next lemma.

**Lemma 2.4.** Let \( C \) be any \( \downarrow \)-finite small category, \( \{ X_p \}_{p \in |C|} \) any collection of objects in \( \mathfrak{A} \). Then

\[
J^C \left( \coprod_{p \in |C|} E_p X_p \right) = \coprod_{p \in C} E_p (j(X)_p). 
\]

**Corollary 2.5.** \( J^C : \mathfrak{A}^C \to D(\mathfrak{A})^C \) preserves canonical projectives.

**Proof.** That \( J^C : \mathfrak{A}^C \to D(\mathfrak{A})^C \) preserves projectives follows immediately from Lemma 2.4, the definition of a canonical projective (1.3) and the fact that both \( E_p : \mathfrak{A} \to \mathfrak{A}^C \) and \( j : \mathfrak{A} \to D(\mathfrak{A}) \) preserve projectives.

**Theorem 2.6** (Spectral sequence). If \( C \) is a \( \downarrow \)-finite ordered set, \( \mathfrak{A} \) is cocomplete with projectives, and \( \Lambda \in \mathfrak{A}^C \), then there is a first quadrant spectral sequence

\[
E^2_{pq} = (L_p U) \left( L_q \colim_C (J^C(\Lambda)) \right)
\]

converging to \( (L_{p+q} \colim_C (J^C(\Lambda))) \).

**Proof.** Both \( \colim_C : D(\mathfrak{A})^C \to D(\mathfrak{A}) \) and \( J^C : \mathfrak{A}^C \to D(\mathfrak{A})^C \) (by Corollary 2.5) preserve projectives. Hence, the hypotheses of the "Grothendieck Two Functor Theorem" [2] are satisfied since \( U \circ \colim_C J^C \simeq \colim_C : \mathfrak{A}^C \to \mathfrak{A} \). \( U : D(\mathfrak{A}) \to \mathfrak{A} \) is right exact and \( \colim_C J^C : \mathfrak{A}^C \to D(\mathfrak{A}) \) preserves projectives. Applying this theorem of Grothendieck yields a spectral sequence with \( E^2_{pq} = (L_p U)(L_q \colim_C J^C(\Lambda))(\Lambda) \) converging to \( (L_{p+q} \colim_C (J^C(\Lambda))) \). But since \( J^C : \mathfrak{A}^C \to D(\mathfrak{A})^C \) is both exact and projective-preserving,

\[
(L_p \colim_C J^C(\Lambda)) \simeq (L_p \colim_C (J^C(\Lambda))
\]

giving the required form.

Also, \( D(\mathfrak{A}), AB5, \) and \( J^C : \mathfrak{A}^C \to D(\mathfrak{A})^C \) exact yield the next corollary.

**Corollary 2.7.** If \( \Lambda \) is a \( \downarrow \)-finite directed ordered set, and \( \Lambda \in \mathfrak{A}^A \), then

\( (L_p U)(\colim_{\Lambda} J^A(\Lambda)) \simeq (L_p \colim_{\Lambda} (J^A(\Lambda))) \) for every \( p > 0 \).
Recall that \( W : \mathcal{C} \rightarrow \mathcal{F}(\mathcal{C}) \) is the functor defined by \( (W\mathcal{A})_{\mathcal{C}'} = \text{colim}_{\mathcal{C}'} \mathcal{A} \mid \mathcal{C} \), where \( \mathcal{F}(\mathcal{C}) \) is the \( \downarrow \)-finite directed ordered set consisting of all finitely-generated initial subcategories \( \mathcal{C}' \).

**Lemma 2.8.** If \( \mathcal{C} \) is a \( \downarrow \)-finite small category, then

\[
\left( L_p \text{ colim}_{\mathcal{C}} f^C \right)(\mathcal{A}) \cong \text{colim}_{\mathcal{F}(\mathcal{C})} \left( L_p W \right)(\mathcal{A})
\]

for every \( \mathcal{A} \in \mathcal{C} \).

**Proof.** Since \( J : \mathcal{A} \rightarrow D(\mathcal{A}) \) is exact, it commutes with finite colimits, and therefore \( J((W\mathcal{A})_{\mathcal{C}'}) \cong W(f^C(\mathcal{A}))_{\mathcal{C}'} \), for every \( \mathcal{C}' \in \mathcal{F}(\mathcal{C}) \). But by Lemma 1.2, 
\[
\text{colim}_{\mathcal{C}} f^C(\mathcal{A}) \cong \text{colim}_{\mathcal{F}(\mathcal{C})} W(f^C(\mathcal{A})),
\]

and thus \( \text{colim}_{\mathcal{C}} f^C(\mathcal{A}) \cong \text{colim}_{\mathcal{C}' \in \mathcal{F}(\mathcal{C})} J((W\mathcal{A})_{\mathcal{C}'}) \)

\( \cong \text{colim}_{\mathcal{F}(\mathcal{C})} J(\mathcal{F}(\mathcal{C}))(\mathcal{A}) \). Since \( \mathcal{F}(\mathcal{C}) \) is a \( \downarrow \)-finite directed ordered set and \( D(\mathcal{A}) \) is AB5, \( \text{colim}_{\mathcal{F}(\mathcal{C})} f^C(\mathcal{A}) : \mathcal{F}(\mathcal{C}) \rightarrow D(\mathcal{A}) \) is exact and therefore commutes with homology. Consequently, \( (L_\ast \text{colim}_{\mathcal{C}} f^C(\mathcal{A})) \cong \text{colim}_{\mathcal{F}(\mathcal{C})} J(f^C(\mathcal{A})) \).

Combining Lemma 2.8, Theorem 2.6, and Corollary 2.7 yields several equivalent forms for the spectral sequence.

**Theorem 2.9.** If \( \mathcal{A} \) is cocomplete with enough projectives, \( \mathcal{C} \) a \( \downarrow \)-finite small category, and \( \mathcal{A} \in \mathcal{C} \), then there is a first quadrant spectral sequence

\[
E^{pq}_2 \cong \left( L_p U \right) \left( L_q \text{ colim}_{\mathcal{C}} f^C(\mathcal{A}) \right) \cong \left( L_p U \left( \text{colim}_{\mathcal{F}(\mathcal{C})} \left( L_q W(\mathcal{A}) \right) \right) \right) \cong \left( L_p \text{ colim}_{\mathcal{F}(\mathcal{C})} \left( L_q W(\mathcal{A}) \right) \right)
\]

converging to \( (L_{p+q} \text{ colim}_{\mathcal{C}} f^C)(\mathcal{A}) \).

Thus from the factorization of \( \text{colim}_{\mathcal{C}} : \mathcal{A} \rightarrow \mathcal{A} \) into \( \text{colim}_{\mathcal{C}} f^C : \mathcal{A} \rightarrow D(\mathcal{A}) \) and \( U : D(\mathcal{A}) \rightarrow \mathcal{A} \), we get a spectral sequence which involves derived functors of colimit over a directed ordered set, namely, \( \mathcal{F}(\mathcal{C}) \).

3. Applications. In this section, we apply a recent result of Mitchell [10] to the spectral sequence. This shows the cardinality of \( \mathcal{C} \) is related to the vanishing of higher derived functors of \( \text{colim}_{\mathcal{C}} : \mathcal{A} \rightarrow \mathcal{A} \), \( \mathcal{C} \) an AB4 category and \( \mathcal{C} \) a \( \downarrow \)-finite small category. The method will employ generalizations of dimension theory for rings developed by Mitchell in Rings with several objects [9].

If \( \mathcal{A} \in \mathcal{C} \), then the homological (projective) dimension of \( \mathcal{A} \), denoted \( \text{hd}_\mathcal{C} \mathcal{A} \), is defined to be \( \sup \{ k \mid \text{Ext}^k_{\mathcal{C}}(\mathcal{A},-) \neq 0 \} \) or equivalently, to be the smallest integer for which there is a projective resolution

\[
0 \rightarrow P_n \rightarrow \cdots \rightarrow P_0 \rightarrow \mathcal{A} \rightarrow 0
\]
when \( \mathcal{G} \) is cocomplete with projectives.

**Proposition 3.1.** \( \text{hd}_C\mathcal{A} = \sup|k| 0 \neq R^k \lim_C \colon \mathcal{A}bC \to \mathcal{A} \).

**Proof.** Let \( \Delta \colon \mathcal{A}b \to \mathcal{A}bC \) be the full exact embedding which assigns to each \( G \in \mathcal{A}b \) the constant diagram \( \Delta G \). By definition, \( \Delta \colon \mathcal{A}b \to \mathcal{A}bC \) is the coadjoint of \( \lim_C \colon \mathcal{A}bC \to \mathcal{A}b \) and therefore \( \mathcal{A}bC(\Delta Z, \mathcal{A}) \simeq \mathcal{A}(Z, \lim_C \mathcal{A}) \simeq \lim_C \mathcal{A} \). Taking derived functors gives the result.

If \( \mathcal{G} \) is any cocomplete category and \( C \) is any small category, there exists a covariant additive cocontinuous (colimit-preserving) bifunctor \( \otimes_C \colon \mathcal{A}bC^{\text{op}} \times \mathcal{C} \to \mathcal{G} \) (whose value on the pair \( M, F \) is denoted by \( M \otimes_C F \)), such that for every \( M \in \mathcal{A}bC^{\text{op}}, F \in \mathcal{C} \), and \( X \in \mathcal{G} \), \( \mathcal{A}bC^{\text{op}}(M, \mathcal{A}(F, X)) \simeq \mathcal{A}(M \otimes_C F, X) \) (where \( \mathcal{A}(F, X) \colon C_{\text{op}} \to \mathcal{G} \) is given by \( \mathcal{A}(F, X)_p = \mathcal{A}(F_p, X) \)). Define \( \text{Tor}_*(M, F) = H_p(P \otimes_C F) \), where \( P \) is a projective resolution for \( M \). From [12], we know that when \( \mathcal{G} \) is AB4 and when \( M \) has free values (for example \( M = \Delta^*Z \)), \( \text{Tor}_*(M, \_ ) \) is the sequence of left satellites (left derived functors when \( \mathcal{C} \) has enough projectives) of \( M \otimes_\_ : C_{\text{op}} \to C_{\text{op}} \).

**Lemma 3.2.** If \( \mathcal{G} \) is AB4, then \( \text{Tor}_*(\Delta^*Z, \_ ) : C_{\text{op}} \to \mathcal{G} \) and \( \text{L}_* \text{colim}_C : \mathcal{C} \to \mathcal{G} \) are isomorphic.

**Proof.** If \( F \in \mathcal{C} \) and \( X \in \mathcal{G} \), then by definitions of \( \text{colim}_C : \mathcal{C} \to \mathcal{G} \) and \( \text{lim}_{C_{\text{op}}} : \mathcal{A}bC^{\text{op}} \to \mathcal{A}b \),

\[
\mathcal{A}(\Delta^*Z \otimes_C F, X) \simeq \mathcal{A}bC^{\text{op}}(\Delta^*Z, \mathcal{A}(F, X)) \simeq \mathcal{A}b\left(Z, \lim_{C_{\text{op}}} \mathcal{A}(F, X)\right)
\simeq \lim_{C_{\text{op}}} \mathcal{A}(F, X) \simeq \mathcal{A}\left(\text{colim}_C F, X\right).
\]

By Yoneda, this composite natural equivalence must come from a natural equivalence. Hence

\[
\Delta^*Z \otimes_C F \simeq \text{colim}_C F \quad \text{and} \quad \Delta^*Z \otimes_{C_{\text{op}}} \simeq \text{colim}_C : \mathcal{C} \to \mathcal{G}.
\]

Since \( \mathcal{G} \) is AB4, \( \text{L}_* \text{colim}_C \simeq \text{Tor}_*(\Delta^*Z, \_ ) : \mathcal{C} \to \mathcal{G} \).

If \( \mathcal{G} = \mathcal{A}b \), we say the **weak (or flat) dimension of** \( M \in \mathcal{A}bC^{\text{op}} \), denoted \( \text{wd}_C M \), is the \( \sup|k|0 \neq \text{Tor}_k \rangle C(M, \_ ) : \mathcal{A}bC \to \mathcal{A}b \). Thus by Lemma 3.2, \( \text{wd}_C \Delta^*Z = \sup|k|0 \neq \text{L}_k \text{colim}_C : \mathcal{A}bC \to \mathcal{A}b \). Now when \( \mathcal{G} \) is AB5, we can use flat resolutions of \( M \) to compute \( \text{Tor}(M, F) \). This yields the second part of the following (see [9]).

**Corollary 3.3.** (i) If \( \mathcal{G} \) is AB4 and \( \text{hd}_C \Delta^*Z = r \), then \( 0 = \text{L}_k \text{colim}_C : \mathcal{C} \to \mathcal{G} \) for every \( k > r \).

(ii) If \( \mathcal{G} \) is AB5 and \( \text{wd}_C \Delta^*Z = r \), then \( 0 = \text{L}_k \text{colim}_C : \mathcal{C} \to \mathcal{G} \) for every \( k > r \).
Using other generalizations of ring theoretic results of Osofsky [13], Mitchell [10] proves the next result.

**Theorem 3.4.** Let $\kappa_n$ be the smallest cardinal number of a cofinal subset of the directed (upward) ordered set $\Lambda (-1 \leq n \leq \infty)$. Then $\text{hd}_{\Lambda}^{\text{op}} \Lambda^* Z = n + 1$.

This and the above corollary immediately imply that $L_p \colim_A: \mathfrak{A}^\Lambda \to \mathfrak{A}$ vanish for $p$ above $n + 1$ whenever $\mathfrak{A}$ is AB4, e.g. $\mathfrak{A} = \mathfrak{A}_b$.

Using these preliminary results, we now consider the spectral sequence.

**Theorem 3.5.** Suppose $\mathfrak{A}$ is an AB4 category with projectives, and $C$ is a small $\downarrow$-finite $\Pi$-category with $\text{wd}_C \Lambda^* Z = k$. Then $L_r \colim_C: \mathfrak{A}^C \to \mathfrak{A}$ vanishes whenever $r > n + 1 + k$.

**Proof.** By Theorem 2.9, there exists a first quadrant spectral sequence

$$E_{pq}^2 = (L_p U) \left( L_q \colim_C \mathcal{F} C(A) \right) \left( \mathcal{F} C(A) \right) \cong \left( L_p \colim_C \mathcal{F} C(A) \right) \left( L_q W(A) \right) C^A$$

converging to $(L_p + q \colim_C C(A)) A$ for every $A \in \mathfrak{A}^C$. We first hold $p$ constant.

Since $\mathfrak{A}(C)$ is AB5, Corollary 3.3(ii) and $\text{wd}_C \Lambda^* Z = k$ insure that $(L_p U)$$\left( L_q \colim C \mathcal{F} C(A) \right)$ is zero for $q > k$. Next, let $q$ be held constant. $C$ an $\Pi$-category implies $\mathfrak{A}(C)$, the directed set of all finite initial subcategories, is also a $\Pi$-category, i.e. $\| \mathfrak{A}(C) \| < \kappa_n$. Therefore, by Proposition 3.4, $\text{hd}_{\mathfrak{A}(C)^{\text{op}}} \Lambda^* Z = n + 1$ and $(L_p \colim_C \mathcal{F} C(A)) \left( L_q W(A) \right) C^A = 0$ for $p > n + 1$. Combining these together yields $(L_r \colim_C \mathcal{F} C(A)) \left( L_q W(A) \right) C^A = 0$ for $r > n + 1 + k$.

The dual statement is the following.

**Theorem 3.6.** If $\mathfrak{A}$ is an AB4 category with injectives and $C$ is a $\downarrow$-finite small $\Pi$-category with $\text{wd}_C \Lambda^* Z = k$, then $R^r \lim_{C^{\text{op}}} \mathfrak{A}^{C^{\text{op}}} \to \mathfrak{A}$ is zero for $r > n + 1 + k$.

In the case when $\mathfrak{A} = \mathfrak{A}_b$, the following corollary holds.

**Corollary 3.7.** If $C$ is a $\downarrow$-finite small $\Pi$-category with $\text{wd}_C \Lambda^* Z = k$, then $\text{hd}_{\mathfrak{A}_b^{\text{op}}} \Lambda^* Z \leq n + 1 + k$.

This follows from Lemma 3.1.

Lastly, putting Corollary 3.3 and Corollary 3.7 together, we can drop the hypothesis of Corollary 3.7 that $\mathfrak{A}$ have enough projectives.

**Theorem 3.9.** If $\mathfrak{A}$ is an AB4 category and $C$ is a $\downarrow$-finite small $\Pi$-category with $\text{wd}_C \Lambda^* Z = k$, then $L_r \colim_C: \mathfrak{A}^C \to \mathfrak{A}$ vanishes for $r > n + 1 + k$.
BIBLIOGRAPHY


