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ADJOINING INVERSES TO COMMUTATIVE BANACH ALGEBRAS

BY

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ABSTRACT. Let A be a commutative unital Banach algebra. Suppose $G \subset A$ is such that $\|a\| \le \|ga\|$ for all $g \in G$, $a \in A$. Two questions are considered in the paper. Does there exist a superalgebra B of A in which every $g \in G$ is invertible? Can one always have also $\|g^{-1}\| \le 1$ if $g \in G$? Arens proved that if $G = \{g\}$ then there is an algebra containing g^{-1}, with $\|g^{-1}\| \le 1$. In the paper it is shown that if G is countable B exists, but if G is uncountable, this is not necessarily so. The answer to the second question is negative even if G consists of only two elements.

1. Introduction. Given a commutative unital Banach algebra and a set of elements, when is there a norm-preserving extension of the algebra ("superalgebra") in which every element of the set is invertible and, preferably, has small norm? In other words, when can we adjoin the inverses of a set of elements to the algebra? The best known result in this direction is Shilov's [6] classical theorem about the possibility of adjoining inverses of a single element; other partial solutions were given by R. Arens ([1], [2]), C. E. Rickart [5], R. Arens and K. Hoffman [4]. In this note we shall investigate the problem above and will answer some of the very natural questions, raised by R. Arens [2]. Some problems in a similar vein were discussed by R. Arens in [3] as well.

Throughout this note A will denote a commutative unital Banach algebra, i.e. $1 \in A$, $\|1\| = 1$. Denote by $G(A)$ the set of those elements in A which are not topological zero divisors, i.e.

$$G(A) = \{g\colon g \in A,\ \inf\{\|ag\|\colon \|a\| = 1\} > 0\}.$$

Shilov [6] proved in 1947 that if A is a singly generated algebra then it can be extended to a larger algebra B containing the inverse of some element $g \in A$ if and only if $g \in G(A)$. This result is also easily proved in the following sharp form (see R. Arens [2]).

Proposition 1.1. *The following two conditions are equivalent.*

(i) $\|a\| \le t_0\|ag\|$ *for all* $a \in A$.

(ii) *There exists a superalgebra* B *of* A *such that* $g^{-1} \in B$ *and* $\|g^{-1}\| \le t_0$.

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If $g_1, g_2, \cdots, g_n \in G(A)$, clearly $g_1 g_2 \cdots g_n \in G(A)$. Therefore, by taking a superalgebra B containing $(g_1 g_2 \cdots g_n)^{-1}$, one can simultaneously adjoin to A the inverses of a finite set in $G(A)$ (see [2]).

Proposition 1.2. *Suppose $g_i \in A$, $\|g_i\| = G_i$ and $\|a\| \le \|ag_i\|$ for all $a \in A$ $(i = 1, 2, \cdots, n)$. Then there exists an algebra B, containing A, such that*

$$g_i^{-1} \in B \quad \text{and} \quad \|g_i^{-1}\| \le \prod_{j=1;\, j\neq i}^{n} G_j, \qquad i = 1, 2, \cdots, n.$$

However, the bound for $\|g_i^{-1}\|$, obtained above, is clearly rather rough. One is likely to expect (see R. Arens [2, §7]) that Proposition 1.1 can be generalised in the following obvious way to construct inverses for some sets of elements of $G(A)$:

Let λ be a positive function on $G(A)$ such that $\lambda(g)\|ag\| \ge \|a\|$ for all $g \in G(A)$, $a \in A$, and let H be a "nice" subset of $G(A)$ (i.e. H is finite or compact or closed, etc.). Then there exists a Banach algebra B containing A such that for all $h \in H$ we have $h^{-1} \in H$, $\|h^{-1}\| \le \lambda(h)$.

On the other hand, as far as I know, nothing is known about the *possibility* of adjoining the inverses of a countable or uncountable set of elements in $G(A)$, no matter with what norms.

In §2 we shall show that Shilov's theorem *can not be generalised* in its sharp form for *even two* elements (i.e. $|H| = 2$). Another, more complicated but more powerful, example will be given in §4.

§3 contains our main positive result. We shall prove that it is *always possible* to adjoin the inverses of a *countable set* of elements which are not topological zero-divisors.

Another negative result will be proved in §4. We shall show that there exist *uncountable* sets of nontopological divisors of zero such that *no extension* of the algebra can contain the inverses of all elements in the set.

These results show that the complete answer to the question at the beginning of the note will certainly not be trivial.

2. Shilov's theorem can not be generalised in its sharp form.

Theorem 2.1. *Given any N (> 3) there exists an algebra $L = L(N)$, containing elements g_1, g_2 such that*

(i) $\|x\| \le \|g_i x\|$, $i = 1, 2$, *for all* $x \in L$;

(ii) *if an extension M of L contains g_1^{-1} and g_2^{-2} then* $\|g_1^{-1}\| + \|g_2^{-2}\| \ge N - 1$.

Proof. Let $\mathbf{B} = \mathbf{B}(b_0, b_1, b_2; g_1, g_2)$ consist of the following elements:

$$g_1^i g_2^j, \qquad i, j = 0, 1, \cdots, \text{ where } g_1^0, g_2^0 = 1,$$

$$b_0 g_1^k, \quad b_0 g_2^l, \quad b_1 g_1^k g_2^l, \quad b_2 g_1^k g_2^l, \qquad k, l = 0, 1, \cdots.$$

Put

$$\|g_1^i g_2^j\| = N^{i+j},$$

$$\|b_0\| = \|b_0 g_i\| = N - 1, \qquad \|b_0 g_i^k\| = N^{k-1}(N-1) \qquad (1 \le k),$$

$$\|b_i g_i^k g_2^l\| = N^{k+l} \qquad (i = 1, 2).$$

Let $L = L(N)$ be the Banach space with basis $\mathbf{B}$, consisting of the formal sums

$$x = \sum_{b \in \mathbf{B}} \lambda_b b, \qquad \sum_{b \in \mathbf{B}} |\lambda_b| \, \|b\| < \infty, \qquad \lambda_b \in \mathbf{C},$$

with norm $\|x\| = \Sigma \, |\lambda_b| \, \|b\|$.

Define a commutative product in L by the following relations:

$$b_i b_j = 0, \qquad i, j = 0, 1, 2, \qquad b_0 g_1 g_2 = b_1 g_1 + b_2 g_2.$$

It is easily seen that with this product L is a Banach algebra.

Let us show that $\|x\| \le \|x g_2\|$ for all $x \in L$. By considering the effect of multiplication by g_2 on the basis elements, it suffices to prove the following inequality:

$$A = \|(z_0 b_0 g_1^k + z_2 b_2 g_1^{k-1}) g_2\| \ge \|z_0 b_0 g_1^k + z_2 b_2 g_1^{k-1}\| = B.$$

By the definitions of the product and norm we have

$$A = \|z_0 b_1 g_1^k + z_0 b_2 g_1^{k-1} g_2 + z_2 b_2 g_1^{k-1} g_2\| = |z_0| N^k + |z_0 + z_2| N^k,$$

$$B = |z_0| N^{k-1}(N-1) + |z_2| N^{k-1}.$$

Thus $A \ge B$ for all $z_0, z_2 \in \mathbf{C}$ and so $\|x\| \le \|x g_2\|$. Similarly $\|x\| \le \|x g_1\|$.

Let now M be an algebra, containing L, such that $g_1^{-1}, g_2^{-1} \in M$. As $b_0 g_1 g_2 = b_1 g_1 + b_2 g_2$, we have $b_0 = b_1 g_2^{-1} + b_2 g_1^{-1}$. Therefore

$$N - 1 = \|b_0\| \le \|g_1^{-1}\| + \|g_2^{-1}\|.$$

3. Adjoining the inverses of a sequence of elements. Suppose $g_1, g_2, \cdots \in A$ are such that $\|a\| \le \|g_i a\|$ for all i and $a \in A$. Theorem 11.2 implies the existence of an extension of A containing all inverses g_i^{-1} only if $\Pi_1^\infty \|g_i\| < \infty$. However, it is easy to see that we can adjoin all the inverses g_i^{-1} for any sequence $\{g_i\}$ if there are functions $\phi(M)$, $\psi(M, N)$ with the following property. If g_1, g_2 are such that $\|a\| \leq \|g_i a\|$ for all $a \in A$, $i = 1, 2$, and $\|g_1\| \leq M$, $\|g_2\| \leq N$, then there is an extension B of A such that $g_1^{-1} \in B$, $g_2^{-1} \in B$ and

$$\|g_1^{-1}\| \leq \phi(M), \qquad \|g_2^{-1}\| \leq \psi(M, N).$$

We shall show that $\phi(M) = M$ and $\psi(M, N) = N$ are such functions, i.e. for $n = 2$ the bounds in Proposition 1.2 on $\|g_1^{-1}\|$ and $\|g_2^{-1}\|$ can be interchanged.

Theorem 3.1. *Suppose $x, y \in A$ are such that*

$$\|a\| \leq \|xa\|, \qquad \|a\| \leq \|ya\| \quad \textit{for all } a \in A. \tag{1}$$

Then A can be isometrically embedded into a Banach algebra B, in which x and y have inverses satisfying the inequalities

$$\|x^{-1}\| \leq \|x\|, \qquad \|y^{-1}\| \leq \|y\|.$$

Proof. For the sake of simplicity introduce the notations $\|x\| = X$, $\|y\| = Y$. Conditions (1) imply $1 \leq X, Y$. Consider the commutative algebra $A(x^{-1}, y^{-1})$, obtained from A by adjoining two elements, x^{-1}, y^{-1}, with the relations $xx^{-1} = 1$, $yy^{-1} = 1$. Put

$$U = \left\{ \sum_{i,j=0}^{n} c_{ij} x^{-i} y^{-j} : \sum_{i,j=0}^{n} \|c_{ij}\| x^i y^j \leq 1,\ n = 0, 1, \cdots \right\}.$$

U is an absolutely convex subset of $A(x^{-1}, y^{-1})$ and it is closed under multiplication.

For $w \in A(x^{-1}, y^{-1})$ define $\|w\|' = \inf\{c : c > 0, w \in cU\}$. Let B be the completion of the algebra $A(x^{-1}, y^{-1})/\{w : \|w\|' = 0\}$, with norm $\|\cdot\|'$. Clearly $\|x^{-1}\|' \leq \|x\|$, $\|y^{-1}\|' \leq \|y\|$. To complete the proof we have to show only that on A the norm $\|\cdot\|'$ coincides with the original norm. Since $a \in A$ belongs to the ideal generated by $xx^{-1} - 1$ and $yy^{-1} - 1$ if and only if $ax^n y^n = \sum_{i,j=0}^{n} c_{ij} x^{n-i} y^{n-j}$, for some integer n and some $c_{ij} \in A$, this is a consequence of the following lemma.

Lemma 3.2. *If*

$$ax^n y^n = \sum_{i,j=0}^{n} c_{ij} x^{n-i} y^{n-j} \qquad (a, c_{ij} \in A) \tag{2}$$

then

$$\|a\| \leq \sum_{i,j=0}^{n} \|c_{ij}\| X^i Y^j.$$

Proof. Put $C_{ij} = \|c_{ij}\|$, $C = (C_{ij})$, $i, j = 0, 1, \cdots, n$.

Define the scalar product of two matrices, $U = (U_{ij})$ and $V = (V_{ij})$ as $UV = \sum_{i,j} U_{ij} V_{ij}$, where the summation goes over the common set of indices. Denote by $\mathfrak{A}$ the set of positive $(n+1)\times(n+1)$ matrices M, such that $\|a\| \leq CM$. We have to show that the matrix $P = (P_{ij}) = (X^i Y^j)$, $i, j = 0, 1, \cdots, n$, belongs to $\mathfrak{A}$.

As a simple example, take $n = 2$. By repeatedly making use of (1) and finally applying (2) we obtain

$$
\begin{aligned}
\|a\| &\leq \|a - c_{00}\| + C_{00} \leq \|ax - c_{00}x - c_{10}\| + C_{00} + C_{10} \\
&\leq \|axy - c_{00}xy - c_{10}y - c_{01}x - c_{11}\| + C_{00} + C_{10} + C_{01}X + C_{11} \\
&\leq \|axy^2 - c_{00}xy^2 - c_{10}y^2 - c_{01}xy - c_{11}y - c_{02}x - c_{12}\| \\
&\quad + C_{00} + C_{10} + C_{01}X + C_{11} + C_{02}X + C_{12} \\
&\leq \|ax^2y^2 - c_{00}x^2y^2 - c_{10}xy^2 - c_{01}x^2y - c_{11}xy - c_{02}x^2 \\
&\qquad\qquad - c_{12}x - c_{20}y^2 - c_{21}y - c_{22}\| \\
&\quad + C_{00} + C_{10} + C_{01}X + C_{11} + C_{02}X + C_{12} + C_{20}Y^2 + C_{21}Y + C_{22} \\
&= C_{00} + C_{01}X + C_{02}X + C_{10} + C_{11} + C_{12} + C_{20}Y^2 + C_{21}Y + C_{22}.
\end{aligned}
$$

In other words, the following matrix belongs to $\mathfrak{A}$:

$$
\begin{pmatrix}
1 & X & X \\
1 & 1 & 1 \\
Y^2 & Y & 1
\end{pmatrix}
$$

We shall show now that in general $\mathfrak{B} \subset \mathfrak{A}$, where $\mathfrak{B}$ is the set of matrices obtained in the following way.

Put 1 into the upper left corner. Divide the remaining part of the $(n+1)\times(n+1)$ matrix into $1 \times k$ and $l \times 1$ rectangles $(1 \leq k, l \leq n)$ in such a way that the $1 \times k$ rectangles have their first element in the first column of the matrix and the $l \times 1$ rectangles have an element in the first row. Then fill the $1 \times k$ rectangles from the right by numbers $1, Y, \cdots, Y^{k-1}$ and fill the $l \times 1$ rectangles, starting from below, with the elements $1, X, \cdots, X^{l-1}$. One of the possibilities for $n = 7$ is shown by the figure.

1	1	1	1	X^3	X^5	X^6	X^6
Y^3	Y^2	Y	1	X^2	X^4	X^5	X^5
Y^3	Y^2	Y	1	X	X^3	X^4	X^4
Y^3	Y^2	Y	1	1	X^2	X^3	X^3
Y^4	Y^3	Y^2	Y	1	X	X^2	X^2
Y^4	Y^3	Y^2	Y	1	1	X	X
Y^5	Y^4	Y^3	Y^2	Y	1	1	1
Y^7	Y^6	Y^5	Y^4	Y^3	Y^2	Y	1

Figure

Let $B \in \mathfrak{B}$. Clearly there is a unique sequence $\{R_t\}_1^{2n+1}$, where R_t is a $(k_t + 1) \times (l_t + 1)$ matrix and is the union of t of these $1 \times l$ and $k \times 1$ rectangles. (In the figure $k_1 = k_2 = k_3 = k_4 = 0$, $l_1 = 0$, $l_2 = 1$, $l_3 = 2$, $l_4 = 3$, $k_5 = 1$, $l_5 = 3$, $k_6 = 2$, $l_6 = 3$, etc.) As $R_{2n+1} = B$, to show $B \in \mathfrak{A}$ it suffices to prove the following inequality:

$$\|a\| \leq CR_t + \left\| a - \sum_{i=0}^{k_t} \sum_{j=0}^{l_t} c_{ij} x^{k_t - i} y^{l_t - j} \right\|.$$

As this inequality clearly holds for $t = 1$, by symmetry it suffices to show that if it holds for t and R_{t+1} is a $(k_t + 2) \times (l_t + 1)$ matrix then it also holds for $t + 1$. By making use of (1), we obtain

$$\begin{aligned}
\|a\| &\leq CR_t + \left\| a - \sum_{i=0}^{k_t} \sum_{j=0}^{l_t} c_{ij} x^{k_t - i} y^{l_t - j} \right\| \\
&\leq CR_t + \left\| \left(a - \sum_{i=0}^{k_t} \sum_{j=0}^{l_t} c_{ij} x^{k_t - i} y^{l_t - j} \right) x - \sum_{j=0}^{l_t} c_{k_t+1,j} y^{l_t - j} \right\| \\
&\quad + \left\| \sum_{j=0}^{l_t} c_{k_t+1,j} y^{l_t - j} \right\| \\
&= CR_{t+1} + \left\| a - \sum_{i=0}^{k_t+1} \sum_{j=0}^{l_t+1} c_{ij} x^{k_t+1-i} y^{l_t+1-j} \right\|.
\end{aligned}$$

Thus $B \in \mathfrak{A}$.

To show the matrix $P = (X^i Y^j)_0^n$ belongs to $\mathfrak{A}$ it suffices to show that $\mathfrak{B}$ (or, even more, the convex hull of $\mathfrak{B}$) contains a matrix $M = (M_{ij})_0^n$ such that $M_{ij} \leq P_{ij}$ for all i and j. In its turn, as $1 \leq X, Y$, this will follow if we show that the set of permissible rectangles, whose each element is not greater than the appropriate entry in P, covers the $(n + 1) \times (n + 1)$ square. This clearly holds since, given i and j, either $Y^j \leq X^i$ or $X^i \leq Y^j$, so either we can put $Y^j, Y^{j-1}, \cdots, Y, 1$ into the ith row or $X^i, X^{i-1}, \cdots, X, 1$ into the jth column.

The proof is complete.

Remark 3.3. It is easily seen that the proof of Lemma 3.2 can be generalized from squares to arbitrary dimensional cubes to give the following result.

Let $x_1, \cdots, x_m \in A$ be such that $\|b\| \leq \|bx_k\|$ for all $b \in A$, and $k = 1, \cdots, m$. Suppose

$$ax_1^n \cdots x_m^n = \sum_{\substack{0 \leq i_j \leq n \\ 0 \leq j \leq m}} c_{i_1 \cdots i_m} x_1^{n-i_1} \cdots x_m^{n-i_m},$$

where $a \in A$ and $c_{i_1 \cdots i_m} \in A$. Then

$$\|a\| \leq \sum \|c_{i_1 \cdots i_m}\| \tilde{X}_1^{i_1} \cdots \tilde{X}_m^{i_m}$$

where $\tilde{X}_i = \|x_i\|^{2^i - 1}$, $i = 1, \cdots, m$.

This result implies immediately that it is possible to adjoin the inverses of a sequence of elements in $G(A)$.

Theorem 3.4. *Let* $g_1, g_2, \cdots \in A$ *be such that* $\|a\| \leq \|g_i a\|$ *for all* $a \in A$. *Put*

$$G_n = \|g_n\|^{2^n - 1}, \qquad n = 1, 2, \cdots.$$

Then there is an extension A_∞ *of* A *such that*

$$g_i^{-1} \in A_\infty \quad \text{and} \quad \|g_i^{-1}\| \leq G_i, \qquad i = 1, 2, \cdots.$$

Proof. Define, as usual (see [2] and Theorem 3.1), the *maximal algebra norm* $\|\cdot\|'$ on $A(g_1^{-1}, g_2^{-1}, \cdots)$ under the conditions $\|g_n^{-1}\|' \leq G_n$, $n = 1, \cdots$. Let A_∞ be the completion of this algebra. In order to complete the proof, we have to show only that on A the norm $\|\cdot\|'$ coincides with the original norm. This is an immediate consequence of Remark 3.3.

4. **Adjoining the inverses of an uncountable set of elements.** To conclude our general investigations, let us show that, apart from the bounds on the norms of the inverses, Theorem 3.4 is best possible in a sense; namely, if $S \subset G(A)$ is an uncountable set, then A does not necessarily have an extension in which every element of S has an inverse.

Theorem 4.1. *There exists an algebra* A, *containing a set* G, $|G| = \aleph_1$, *such that* $\|g\| = 2$ *and* $\|a\| \leq \|ag\|$ *for all* $g \in G$, $a \in A$, *but in no extension of* A *is every element of* G *invertible.*

Proof. Let G, $|G| = \aleph_1$, be a set of variables. Let $\tilde{P}$ be the completion of the algebra of complex polynomials in the variables g, $g \in G$, where the norm is defined as

$$\left\| \sum c_{g_1 \cdots g_j k_1, \cdots k_j} g_1^{k_1} \cdots g_j^{k_j} \right\| = \sum |c_{g_1 \cdots g_j k_1 \cdots k_j}| 2^{k_1 + \cdots + k_j}.$$

Suppose that to every finite subset $\{g_1, g_2, \cdots, g_n\}$ of G there belongs a Banach space $B = B(g_1, \cdots, g_n)$ with the following properties. If the subalgebra of $\tilde{P}$, generated by $g_1, g_2, \cdots, g_n$, is denoted by P, the multiplication on P can be extended to an algebra multiplication on $P + B$, where the norm on $P + B$ is defined as the sum of the norms. This multiplication is such that if $p \in P$ and $b, b' \in B$ then $pb \in B$, $bb' = 0$ and for all $x \in P + B$, $1 \leq i \leq n$, we have $\|x\| \leq$

$\|g_i x\|$. We also require that if in some extension of $P + B$ all the inverses g_i^{-1}, $1 \le i \le n$, exist, then

$$\sum_{i=1}^{n} \|g_i^{-1}\| \geq 2^n - 1. \tag{3}$$

Then put

$$\widetilde{A} = \widetilde{P} + \sum_{\{g_i\}} \sum_{\{g_{\alpha_j}, \beta_j\}} \left(\prod_1^m g_{\alpha_j}^{\beta_j} \right) B(g_1, \cdots, g_n),$$

where $(\Pi_1^m g_{\alpha_j}^{\beta_i})B$ denotes the Banach space consisting of the elements $x \,\Pi_1^m g_{\alpha_j}^{\beta_j}$: $x \in B$, with norm $\|x \,\Pi_1^m g_{\alpha_j}^{\beta_j}\| = 2^{\Sigma \beta_j} \|x\|$; the first summation goes over all finite nonempty subsets $\{g_i\}$ of G and the second over all finite subsets $\{g_{\alpha_j}\}_1^m$ of $G - \{g_i\}$ and all m-tuples $\{\beta_j\}$ of natural numbers. Norm the vector space $\widetilde{A}$ by taking the sum of the norms and denote by A its completion. Clearly A becomes a Banach algebra if we extend the multiplications on the subalgebras $P + B(g_i, \cdots, g_n)$ in the obvious way and by putting $bb' = 0$ if $b \in B(g_1, \cdots, g_n)$, $b' \in B(g_1', \cdots, g_m')$.

By construction $\|g\| = 2$ and $\|a\| \le \|ga\|$ for all $g \in G$ and $a \in A$. Furthermore, (3) implies that in any extension B of A the inequality $\|g^{-1}\| \le (2^n - 1)/n$ holds for at most n elements g in G. Consequently B *can contain the inverses of only a countable number of elements in* G.

Thus, to complete the proof, we have to construct only the Banach algebras $P + B(g_1, g_2, \cdots, g_n)$ with the properties above.

Let the following set be a basis of the Banach space $B = B(g_1, g_2, \cdots, g_n)$: $\{b \,\Pi_1^n g_i^{k_i} : \Pi_1^n k_i = 0,\ k_i \ge 0;\ b_j \,\Pi_1^n g_i^{k_i} : j = 1, 2, \cdots, n,\ k_i \ge 0\}$.

If π is a permutation of the indices $1, 2, \cdots, n$ (i.e. $\pi \in P_n$) and $a \in P + B$, let $\pi(a) \in P + B$ be the element obtained from a by permuting the indices according to π.

Define the product on $P + B$ by the relations:

$$\pi(a)\pi(a') = \pi(aa'), \qquad a, a' \in P + B,\ \pi \in P_n,$$

$$\prod_1^n g_i^{k_i} \beta \prod_1^n g_i^{k_i'} = \beta \prod_1^n g_i^{k_i + k_i'}, \qquad \beta \in \{b, b_1, \cdots, b_n\},$$

$$\beta \prod_1^n g_i^{k_i} \beta' \prod_1^n g_i^{k_i'} = 0, \qquad \beta, \beta' \in \{b, b_1, \cdots, b_n\},$$

and

$$b g_1 g_2 \cdots g_n = \sum_{i=1}^{n} b_i g_1 \cdots g_{i-1} g_{i+1} \cdots g_n. \tag{4}$$

By using the distributivity and commutativity of the product these relations define a multiplication on $P + B(g_1, \cdots, g_n)$.

Similarly, define the norm to be invariant under permutations. Furthermore, let it be the sum of the norms on the permutations of the subspaces spanned by the vectors

$$b\prod_1^m g_i^{k_i},\ b_i\left(\prod_1^m g_j^{k_j}\right)\Big/ g_i, \qquad i = 1, 2, \cdots, m,$$

where $(k_1, k_2, \cdots, k_m)$ is a fixed set of natural numbers (i.e. $k_i \geq 1$) and $0 \leq m \leq n$.

Given a sequence of complex numbers, $\mu, \mu_1, \mu_2, \cdots, \mu_m$, arrange the indices in such a way that $|\mu + \mu_i|$ increases, $i = 1, 2, \cdots, m$. Then define

$$\|x\| = \left\|\mu b\prod_1^m g_j + \mu_i \sum_{i=1}^m b_i\left(\prod_1^m g_j\right)\Big/ g_i\right\| \tag{5}$$

$$= \sum_{i=1}^m |\mu + \mu_i| 2^{i-1} + |\mu|(2^n - 2^m).$$

In other words

$$\|x\| = \max_\pi \sum_{i=1}^m |\mu + \mu_{\pi(i)}| 2^{i-1} + |\mu|(2^n - 2^m) \tag{6}$$

where the maximum is taken over all permutations of the index set $\{1, 2, \cdots, m\}$.

This definition of the norm is clearly compatible with the relation (4).

In general, if $k_i \geq 1$, $i = 1, 2, \cdots, m$, define

$$\left\|\mu b\prod_1^m g_i^{k_i} + \sum_{i=1}^m \mu_i b_i\left(\prod_1^m g_j^{k_j}\right) g_i\right\|$$

$$= 2^{\sum_1^m k_i - m}\left\|\mu b\prod_1^m g_j + \sum_{i=1}^m \mu_i b_i\left(\prod_1^m g_j\right)\Big/ g_i\right\|.$$

It is easy to show that $P + B(g_i, \cdots, g_n)$ is a Banach algebra with the given multiplication and norm, and $\|a\| \leq \|g_i a\|$ for all $a \in P + B(g_1, \cdots, g_n)$. For both of these assertions follow at once if we show that for the element x in (5) we have $\|x\| \leq \|x g_i\| \leq 2\|x\|$ and this readily follows from the form (6) of the norm.

Finally inequality (3) is a consequence of the relation (4) since $\|b\| = 2^n - 1$, $\|b_i\| = 1$ and if g_i^{-1} exists for $i = 1, 2, \cdots, n$, we have $b = \Sigma_1^n b_i g_i^{-1}$, implying $2^n - 1 = \|b\| \leq \Sigma_1^n \|g_i^{-1}\|$.

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