THE LATTICE TRIPLE PACKING OF SPHERES IN EUCLIDEAN SPACE

BY

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ABSTRACT. We say that a lattice $\Lambda$ in $n$-dimensional Euclidean space $E^n$ provides a $k$-fold packing for spheres of radius 1 if, when open spheres of radius 1 are centered at the points of $A$, no point of space lies in more than $k$ spheres. The multiple packing constant $\Delta_k^{(n)}$ is the smallest determinant of any lattice with this property. In the plane, the first three multiple packing constants $\Delta_2^{(2)}$, $\Delta_3^{(2)}$, and $\Delta_4^{(2)}$ are known, due to the work of Blundon, Few, and Heppes. In $E_3$, $\Delta_2^{(3)}$ is known, because of work by Few and Kanagasabapathy, but no other multiple packing constants are known. We show that $\Delta_3^{(3)} \leq \sqrt{6}/27$ and give evidence that $\Delta_3^{(3)} = \sqrt{6}/27$. We show, in fact, that a lattice with determinant $8\sqrt{3}/27$ gives a local minimum of the determinant among lattices providing a 3-fold packing for the unit sphere in $E_3$.

1. Introduction. Let $\Lambda$ be an $n$-dimensional lattice in $n$-dimensional Euclidean space $E^n$, such that, if open spheres of radius 1 are centered at the points of $A$, then no point of space is covered more than $k$ times. That is, for any point $X$ in $E^n$ there do not exist distinct points $L_1, L_2, \ldots, L_{k+1}$ of $\Lambda$ such that $|X - L_1|, \ldots, |X - L_{k+1}| < 1$. Then we say that $\Lambda$ provides a $k$-fold packing for spheres of radius 1. The terms single, double and triple are synonymous with $k$-fold for $k = 1, 2$ and 3.

Let $d(\Lambda)$ denote the determinant of $\Lambda$, and let $\Delta_k^{(n)}$ denote the lower bound of $d(\Lambda)$, taken over all lattices $\Lambda$ that provide a $k$-fold packing for spheres of radius 1. (Thus $\Delta_1^{(n)}$ is the critical determinant of a sphere of radius 2.) It is well known and easy to see (e.g., divide one generator of the lattice by $k$) that $\Delta_k^{(n)} \leq \Delta_1^{(n)}/k$.

It has been shown by Few [1] that $\Delta_2^{(2)} = \frac{1}{2}\Delta_1^{(2)}$, and Heppes [5] showed that $\Delta_2^{(2)} = \Delta_1^{(2)}/k$ if and only if $k \leq 4$.

In [4] Few and Kanagasabapathy determined the exact value of $\Delta_2^{(3)}$, namely $3\sqrt{3}/2$, which is less than $\Delta_2^{(3)}/2 = 2\sqrt{3}/3$. By constructing particular lattices they also showed that $\Delta_2^{(n)} < \Delta_1^{(n)}/2$ for every $n \geq 3$.

Few remarks in [2] that $\Delta_2^{(3)}$ is the only multiple packing constant known exactly in three dimensions or more, and in this note I shall prove that $\Delta_3^{(3)} \leq \frac{8\sqrt{3}}{27}$.

Received by the editors September 29, 1972.
Key words and phrases. Spheres, lattice packing, multiple packing.

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457
$8\sqrt{38}/27 < \Delta_1^{(3)/3} = 4\sqrt{2}/3$ and give evidence suggesting that $\Delta_3^{(3)} = 8\sqrt{38}/27$.

In fact, I prove

**Theorem 1.** A certain lattice $\Lambda_0$ of determinant $d_0 = 8\sqrt{38}/27$ provides a triple packing for the unit sphere $S$. Also $\Lambda_0$ has generators $P, Q, R$ with $|P| = 2/3$.

**Theorem 2.** Any lattice $\Lambda$ having generators $P', Q', R'$ with $|P'| \leq 0.95$ providing a triple packing for $S$ must have determinant $d(\Lambda) \geq d_0$ with equality only when $\Lambda = \Lambda_0$. Hence $\Lambda_0$ gives a local minimum of $d(\Lambda)$ for triple packing of unit spheres.

**Remark.** There is extensive numerical evidence that $d(\Lambda)$ does not fall below $d_0$ for any triple packing with $S$.

2. An economical lattice $\Lambda_0$.

**Theorem 1.** The best lattice triple packing for spheres in $E^3$ has determinant $d(\Lambda) \leq 8\sqrt{38}/27 = \sqrt{2432/729} \approx 1.82649 \ldots$, since indeed the lattice $\Lambda_0$ generated by $P, Q$ and $R$ where $P = (a, 0, 0) = (2/3, 0, 0)$, $Q = (b, b, 0) = (1/3, \sqrt{3}, 0)$ and $R = (g, f, c)$, where $g = 1/3$, $f = (11\sqrt{3})/27$ and $c^2 = 3 - f^2$, provides a triple packing for the unit sphere $S$.

**Proof.** Convention: The letters $\lambda, \mu$ and $\nu$ will denote integers. $S(A, r)$ will be the open sphere of radius $r$ centered at $A$; $S(A)$ will denote $S(A, 1)$; thus $S = S(\text{origin})$. Suppose that the point $X = (x, y, z)$ is covered four times. Translating $X$ by a lattice point, we may suppose that $X \in S$, and replacing $X$ by $-X$ if necessary we may suppose $z > 0$. The three other spheres covering $X$ can be written $S(AP + \nu Q + \nu R) = S(\mu P + \mu Q + \nu R)$ where $(\lambda, \mu, \nu) \neq (0, 0, 0)$. We must have $|\lambda P + \mu Q + \nu R| < 2$ since they must intersect $S$. Therefore

$$
(\lambda a + \mu b + \nu g) + (\mu b + \nu f)^2 + \nu c^2 < 4
$$

and $|\nu| < 2/c$. Since $c > 1$, we have $\nu \in \{-1, 0, 1\}$. Now $\nu$ cannot be $-1$, since otherwise $|X - (\lambda P + \mu Q - R)| \geq |c + z| > 1$. Hence $\nu \in \{0, 1\}$. From $(\ast)$ we also get $|\mu b + \nu f| < 2$. Since $0 \leq \nu \leq 1$ and $0 \leq / \leq b/2$ and $b = \sqrt{3} > 4/3$ this gives $-2 < \mu < 2, \mu \in \{-1, 0, 1\}$. We divide the proof into two parts.

**Part 1.** $\nu \geq 0$. Then $\mu \in \{0, 1\}$; in fact if $\mu = -1$, then for $X \in S(\lambda P + \mu Q + \nu R)$ we would have $|X - (\lambda P - Q + \nu R)|^2 \geq (b + y - \nu f)^2 > (16b/27)^2 = 256/243$, since $\nu \in \{0, 1\}$. Also $(\mu, \nu) \neq (1, 1)$, since $|\lambda P + Q + R|^2 > b^2 + c^2 > 4$. Hence $(\mu, \nu) \in \{(0, 0), (1, 1), (0, 0)\}$.

**Type 1 spheres.** Suppose $(\mu, \nu) = (0, 0)$. Then $\lambda P + \mu Q + \nu R = \lambda P$, and $S(\lambda P) \cap S = \emptyset$ if $|\lambda| > 2$, since $|3P| = 3a = 2$. The $S(\lambda P)$ such that $0 < |\lambda| \leq 2$ are called Type 1 spheres.
Type 2 spheres. Suppose \((\mu, \nu) = (1, 0)\). Then \(\lambda P + \mu Q + \nu R = \lambda P + Q\), and 
\(S(\lambda P + Q) \cap S = \emptyset\) if \(\lambda \notin \{0, -1\}\), since then \(|\lambda P + Q|^2 = b^2 + (\lambda a + b)^2 = 3 + |2\lambda/3 + 1/3|^2 > 4\). The \(S(Q - P)\) and \(S(Q)\) are called Type 2 spheres.

Type 3 spheres. Suppose \((\mu, \nu) = (0, 1)\). Then \(\lambda P + \mu Q + \nu R = R + \lambda P\), 
\(S(R + \lambda P) \cap S = \emptyset\) if \(\lambda \notin \{0, -1\}\). To see this, observe that if \(S(R + \lambda P) \cap S \neq \emptyset\) then \((\lambda a + g)^2 + j^2 + c^2 < 4\); since \(j^2 + c^2 = 3\), \(|2\lambda/3 + 1/3| < 1\) and \(\lambda \notin \{0, -1\}\). We call \(S(R)\) and \(S(R - P)\) Type 3 spheres.

It follows from the discussion of Type 2 and Type 3 spheres that \(S \cap S(\lambda P + E) = \emptyset\) if \(\lambda \notin \{-1, 0\}\) where \(E \notin \{R, Q\}\). In particular,

\[
\emptyset = S \cap S(E + P) = S(-P) \cap S(E) = S(-2P) \cap S(E - P),
\emptyset = S \cap S(E + 2P) = S(2P) \cap S(E),
\emptyset = S \cap S(E - 2P) = S(P) \cap S(E - P) = S(2P) \cap S(E),
\emptyset = S \cap S(E - 3P) = S(2P) \cap S(E - P).
\]

From the discussion of Type 1 spheres \(S \cap S(\lambda P) = \emptyset\) for \(|\lambda| > 2\) so that

\[
\emptyset = S \cap S(3P) = S(-P) \cap S(2P) = S(-2P) \cap S(P),
\emptyset = S \cap S(4P) = S(-2P) \cap S(2P).
\]

We now draw a graph \(G\) where edges \(A\) and \(B\) are joined only if we know that \(S(A) \cap S(B) = \emptyset\).

We next observe that \(\emptyset = S(Q + \lambda P) \cap S \cap S(R + \lambda' P + v Q)\). For \(|Q + \lambda P| \geq |Q| = \sqrt{b^2 + b^2} > \sqrt{3}\). Therefore the height (maximal value of the \(z\) coordinate of the closure) of \(S(Q + \lambda P) \cap S\) is less than \(\sqrt{1 - 3/4} = \frac{\sqrt{2}}{2} < c - 1\), since \(c = 1.5 \cdots > 3/2\). Since \(R + \lambda' P\) has \(z\) component \(c\), the above intersection is void. Hence we cannot have \(X\) simultaneously inside a sphere of Type 2 and a sphere of Type 3, so
\[ X \in S(\lambda_1 P) \cap S(\lambda_2 P) \cap S(E + \lambda_3 P) \quad \text{with} \quad 0 < |\lambda_1|, \quad |\lambda_2| \leq 2, \quad -1 \leq \lambda_3 \leq 0, \]
or
\[ X \in S(\lambda_1 P) \cap S(E + \lambda_2 P) \cap S(E + \lambda_3 P) \quad \text{with} \quad 0 < |\lambda_1| \leq 2, \quad -1 \leq \lambda_2, \lambda_3 \leq 0, \]
where \( E \in \mathbb{R}, \ Q \). Both of these contradict the graph \( G \), and Part 1 follows.

**Part 2.** We now suppose that \( y < 0 \). Recall that if \( S \cap S(\lambda P + \mu Q + \nu R) \neq \emptyset \), then \(-1 \leq \mu \leq 1\) and \( 0 \leq \nu \leq 1 \). For those \( S(\lambda P + \mu Q + \nu R) \) containing \( X \) we must have \(-1 \leq \mu \leq 0 \). For suppose that \( \mu = 1 \); since \( X = (x, y, z) \), \( y < 0 \), we would have \( |\lambda P + \mu Q + \nu R - X| \geq |b + v| - y| > 1 \). The spheres \( S(\lambda P + \mu Q + \nu R) \) containing \( X \) other than \( S \) may therefore be divided into four types, as follows:

**Type 1 spheres, when** \((\mu, \nu) = (0, 0)\). As before the only spheres \( S(\lambda P) \) intersecting \( S \) satisfy \( 0 < |\lambda| \leq 2 \), i.e., the Type 1 spheres are \( S(2P), S(P), S(-P) \) and \( S(-2P) \).

**Type 2 spheres, when** \((\mu, \nu) = (-1, 0)\). If \( S(\lambda P + \mu Q + \nu R) \) is to intersect \( S \) we must have \( 4 > |\lambda P + \mu Q + \nu R|^2 = (\lambda a - b)^2 + b^2 = (2\lambda/3 - 1/3)^2 + 3 \). Hence \( 0 < \lambda \leq 1 \), i.e., the Type 2 spheres are \( S(-Q) \) and \( S(-P) \).

**Type 3 spheres, when** \((\mu, \nu) = (0, 1)\). As in Part 1, \(-1 \leq \lambda \leq 0 \) if \( S(\lambda P + \mu Q + \nu R) \) intersects \( S \), i.e., the Type 3 spheres are \( S(R) \) and \( S(-Q) \).

**Type 4 spheres, when** \((\mu, \nu) = (-1, 1)\). If \( S(\lambda P + \mu Q + \nu R) \) intersects \( S \), then \( (\lambda a + \mu - b)^2 + (\nu - b)^2 + c^2 < 4, 4\lambda^2/9 < 4 - (\mu - b)^2 - c^2 = 4/9, \lambda^2 < 1, \lambda = 0 \), and \( S(R - Q) \) is the only sphere of Type 4.

As we did in Part 1, we deduce several new disjoint pairs of spheres. From the discussion of Type 4 spheres, \( S \cap S(-Q + R + LP) = \emptyset \) if \( \lambda \neq 0 \), so we have \( S(\lambda P) \cap S(-Q + R) = \emptyset \) if \( \lambda \neq 0 \). From the Type 2 spheres we have
\[
\emptyset = S \cap S(-Q + 2P) = S(\emptyset) \cap S(-Q + P) = S(-2P) \cap S(-Q),
\emptyset = S \cap S(-Q + 3P) = S(-2P) \cap S(-Q + P),
\emptyset = S \cap S(-Q - P) = S(\emptyset) \cap S(-Q) = S(2P) \cap S(-Q + P),
\emptyset = S \cap S(-Q - 2P) = S(2P) \cap S(-Q).
\]
If we combine these with some of the disjoint pairs that we already know from Part 1 and draw a graph, \( G' \), in which \( A \) is joined to \( B \) only if we know \( S(A) \cap S(B) = \emptyset \), we obtain the following graph.

In addition to these disjoint spheres, we observe that, for any \( \lambda \) and \( \lambda' \), \( S(-Q + \lambda P) \cap S(R + \lambda' P) \cap S = \emptyset \), since \( |-Q + \lambda P| \geq |Q| \), so that the height of \( S(-Q + P) \) is not greater than the height of \( S \cap S(Q) \), which is less than \( \frac{1}{2} < c - \frac{1}{4} \), and \( c \) is the height of \( R + \lambda' P \).

Also, for any \( \lambda, \emptyset = S \cap S(Q + \lambda P) \cap S(R) = S \cap S(-Q - \lambda P) \cap S(-R) = S(R) \cap S(R - Q - \lambda P) \cap S \).

In particular,
\[
S(R) \cap S(R - Q) \cap S = \emptyset.
\]
The following enumeration of possibilities shows that $X$ cannot be contained in the necessary spheres, and Theorem 1 follows:

Clearly Types 2 and 3 cannot both occur by the fourth paragraph above, and two spheres of Type 1 cannot occur with anything else by the graph $G'$. Again by the graph $G'$, if two spheres of Type 2 (or two spheres of Type 3) occur, the remaining sphere cannot have Type 1. Hence the only remaining possibilities are

\[(2) \quad X \in S(\lambda_1 P) \cap S(\lambda_2 P - Q) \cap S(R - Q), \quad 0 < |\lambda_1| \leq 2, \ 0 \leq \lambda_2 \leq 1,\]

\[(3) \quad X \in S(\lambda_1 P) \cap S(\lambda_2 P + R) \cap S(R - Q), \quad 0 < |\lambda_1| \leq 2, \ -1 \leq \lambda_2 \leq 0,\]

\[(4) \quad X \in S(P - Q) \cap S(-Q) \cap S(R - Q), \quad \text{or}\]

\[(5) \quad X \in S(R) \cap S(-P + R) \cap S(R - Q).\]

Now (2) and (3) contradict $G'$, and (1) excludes (5). To eliminate (4) we observe that, since $S(P) \cap S \cap S(R) \cap S(Q) = \emptyset$, we must have $S(P - Q) \cap S(-Q) \cap S(R - Q) \cap S = \emptyset$.

3. The lattice $\Lambda_0$ is locally optimal.

**Remark 1.** An arbitrary lattice $\Lambda$ in $E_3$ has a basis $P, Q, R$ where $|P| \leq |Q| \leq |R|$ are the successive minima of the unit sphere, $P = (a, 0, 0), Q = (b, b, 0), R = (g, f, c), a, b, c > 0, 0 \leq b \leq a/2, 0 \leq f \leq b/2, \text{ and } -a/2 < g < a/2.$ Such a basis is said to be reduced in the sense of Gauss or simply reduced. For a proof, see [6, p. 163 et seq., "Seeber's inequality"].

**Remark 2.** If $\Lambda$ has a reduced basis $P, Q, R$ with $P = (a, 0, 0), Q = (b, b, 0), R = (g, f, c)$ and if $d(\Lambda) < d_0$, then $b^2 \leq b_m^2$, where $b_m^2 = a^2/6 + (2/3)(a^4/16 + 3a_0^2/a^2).

**Proof.** Using $|R^2| = g^2 + f^2 + c^2 \geq |Q|^2 = b^2 + b^2$, and the other inequalities of reduction, we have $a_0^2 \geq d^2(\Lambda) = a^2 b^2 c^2 \geq a^2 b^2 (b^2 + b^2 - g^2 - f^2) \geq$
Putting \( t = b^2 \), we get \( 3a^2 t^2 - a^4 t - 4d_0^2 \leq 0 \). Hence \( b^2 \) must lie between the roots \( a^2/6 - (2/3)\sqrt{a^4/16 + 3d_0^2/a^2} \) and \( a^2/6 + (2/3)\sqrt{a^4/16 + 3d_0^2/a^2} \) of the quadratic.

Let \( p^+ \) denote \( \max\{0, p\} \).

**Theorem 2.** If \((\Lambda, S)\) is a triple packing and \( P = (a, 0, 0), Q = (b, b, 0) \) and \( R = (g, f, c) \) gives a basis for \( \Lambda \) reduced in the sense of Gauss, and \( a \leq 1 \), then

\[
d(\Lambda) = abc > a^2 \sqrt{4 - (a + b)^2 - ((b^2 - 2ab - b^2)/(2b))^2} = f_1(a, b, b)
\]

when \( 0 \leq g \leq b \), and

\[
d(\Lambda) = abc > ab \sqrt{14 - (a + b)^2 - ((b^2 - 2ab - b^2)/(2b))^2} = f_2(a, b, b)
\]

when \(-a/2 \leq g \leq 0\) and when \( b \leq g \leq a/2 \). Furthermore \( f_1(a, b, b) \geq f_2(a, b, b) \), so that in fact

\[
d(\Lambda) \geq f_2(a, b, b)
\]

in all cases. Also, if \( d(\Lambda) \leq d_0 \) and \( 2/3 \leq a \leq 0.9508 \), we have

\[
f_2(a, b, b) \geq \min\{p(a), d_2^2 + 1/100\},
\]

where \( p(a) = 2a^6 - 11a^4 + 12a^2 \), \( p(2/3) = d_0^2 \), and

\[
p(a) > d_2^2 \text{ for } 2/3 < a \leq 0.9508.
\]

Hence \( d(\Lambda) \geq d_0 \) for \( 2/3 \leq a \leq 0.9508 \), and with equality only if \( a = 2/3 \).

**Proof.** Suppose that \((\Lambda, S)\) gives a triple packing and that \( P, Q, R \) form a reduced basis of \( \Lambda \). From reduction, we have

\[
|P| \leq |Q| \leq |R|, \quad 0 \leq b \leq a/2, \quad 0 \leq f \leq b/2, \quad \text{and} \quad |g| \leq a/2.
\]

We also have \( a \geq 2/3 \), since otherwise the point \((\frac{1}{2})P\) would be covered by \( S(-P), S, S(P) \) and \( S(2P) \). Observe that the center of the parallelogram with vertices \( P, Q, Q + P \) and the origin will be covered by the four spheres \( S, S(P), S(Q) \), and \( S(Q + P) \) unless one of the diagonals \( |Q + P|, |Q - P| \) is at least 2. Since \( |Q + P| \geq |Q - P| \) by (11), it follows that \( 4 \leq |Q + P|^2 = b^2 + (a + b)^2 \); hence

\[
b^2 \geq 4 - (a + b)^2 \quad \text{and} \quad a \geq 2/3.
\]

**Case 1.** In this case we assume

\[
0 \leq g \leq b.
\]

A consequence of (13) is that \(|R - Q - P|\) is not less than \(|R - Q + P|\). They
cannot both be less than 2, since then the center of the parallelogram having vertices $R, Q, Q + P, R + P$ would be covered four times. Hence we have

\[(14) \ |R - Q - P| \geq 2.\]

Another consequence of (13) is that $|R + P|$ is not less than $|R - P|$. Considering the parallelogram with vertices $P, R, R + P$ and the origin shows that

\[(15) \ |R + P| > 2.\]

With a view to proving (6) we imagine $a, b, b$ to be fixed and find the point $R = (g, f, c)$ having least nonnegative $c$ such that (13), (14) and (15) hold and also

\[(16) \ 0 < f \leq b/2.\]

We are in fact looking for the lowest point $X = (x, y, z)$ inside the rectangular prism given by

\[(17) \ Q < x < ¿, \ 0 < y < b/2, \ z > 0,\]

subject to the additional constraint

\[(18) \ |X - P - Q| \geq 2, \ |X + P| \geq 2.\]

The problem is somewhat simplified by the fact that the centers $P + Q, -P$ of the spheres lies on the plane $z = 0$, outside the prism.

If the right-hand side of (6) is zero, there is nothing to prove. Let us suppose, therefore, that it is positive. We shall show that the point $X^*$ that lies on the intersection of the boundary of $S(−P, 2)$ and $S(P + Q, 2)$ and the plane $x = h$ is the lowest point satisfying (17) and (18). We start by finding $X^*$. Let $X^* = (x^*, y^*, z^*)$. Let $\pi$ be the radical plane of $S(−P, 2)$ and $S(P + Q, 2)$ (the plane obtained by subtracting the equations of the two spheres). Then $\pi$ passes through $(\frac{b}{2})Q = (b/2, b/2, 0)$, which is halfway between the center of the two spheres, and has the equation $y - b/2 = (−(2a + b)/b)(x - b/2)$. Putting $x^* = b$, we obtain $y^* = b/2 − (2ab + b^2)/(2b)$. We must show that $0 \leq y^*$ so that (17) is satisfied. By (12) we have $b^2 > 4 - (a + b)^2$ since $b < a/2$; hence $2by^* = b^2 − (2ab + b^2) > 0$ and (17) follows. We see that $z^* = \sqrt{4 - (a + b)^2} - ((b^2 - 2ab - b^2)/(2b))^2 > 0$ by a previous assumption.
The first step in showing that \( X^* \) is optimal is to show that the bottom of the prism is covered by \( S(-P, 2) \) and \( S(P + Q, 2) \). This means that there is no \( X \) satisfying (17) and (18) with \( z = 0 \). Let \( A^* = (x^*, y^*, 0) \), \( E = (0, b/2, 0) \), \( F = (b, b/2, 0) \) and \( G = (b, 0, 0) \). Then \( 2 = |P + Q - X^*| > |P + Q - A^*| \geq |P + Q - F| \), and also \( |P + Q - (\frac{\sqrt{2}}{2})Q| < 2 \), since the spheres \( S(-P, 2) \) and \( S(P + Q, 2) \) intersect and \( (\frac{\sqrt{2}}{2})Q \) is halfway between their centers. Hence the triangle with vertices \((\frac{\sqrt{2}}{2})Q, A^*, F\) lies in the interior of \( S(P + Q, 2) \).

Similarly, \( 2 = |-P - X^*| > |-P - A^*| > |-P - G| > |P| \) and \( 2 > |P - (\frac{\sqrt{2}}{2})Q| \geq |P - E| \) so that the convex pentagon with vertices \( G, A^*, (\frac{\sqrt{2}}{2})Q, E \) and the origin lies in the interior of \( S(-P, 2) \). Hence the bottom of the prism is covered.

We now let \( X_1 = (x_1, y_1, z_1) \) be a lowest point satisfying (17) and (18). We know that \( X_1 \) exists, because the set of solutions is nonempty and closed. The point \( X_1 \) must be on the boundary of \( S(-P, 2) \) or \( S(P + Q, 2) \) since otherwise it could be lowered and still satisfy (18).

Let us suppose first that \( X_1 \) is on the boundary of \( S(-P, 2) \). We shall deduce that \( X_1 \) is on the boundary of \( S(P + Q, 2) \). Suppose not. Then \((x_1, y_1)\) must be the point satisfying (17) that is farthest from \(-P\), namely \((b, b/2)\). But then the point \( X_1 = (0, b/2, (a - b + a^2 - b^2/4)^{1/2}) \) is easily seen to be inside \( S(P + Q, 2) \), contrary to (18).

Suppose that \( X_1 \) is not on the boundary of \( S(-P, 2) \). Then \( |X_1 - P - Q| = 2 \), and \( X_1 = (0, 0, \sqrt{4 - (a + b)^2 - b^2/4}) \) lies inside \( S(-P, 2) \) contrary to (18).

Hence \( X_1 \) lies on the arc of the intersection of \( S(-P, 2) \) and \( S(P + Q, 2) \) with \( z \geq 0 \). The highest point of the arc is the point directly above \( (\frac{\sqrt{2}}{2})Q \), which is on the boundary of the prism, and the lowest point in the prism is \( X^* \), where the arc cuts the \( x = b \) plane. Hence \( X_1 = X^* \) and (6) follows.

Case 2. Assume

\( (19) \ - a/2 \leq g \leq 0. \)

Since the center of the parallelogram with vertices \( R, Q - P, R + P, Q \) must not be covered four times, we know that one of its two diagonals \( |R - Q|, |R - Q + 2P| \) must be at least 2. Assuming Gauss reduction, we always have \( |R - Q + 2P| \geq |R - Q| \), since the vectors \( R - Q + 2P \) and \( R - Q \) differ only in the first component, and \( |g - b + 2a| \geq 2a - |g| - b \geq a \geq |g - b| \). Hence \( |R - Q + 2P| \geq 2. \)

As in Case 1, one of \( |R + P|, |R - P| \) must be at least 2, and from (19) we know that \( |R - P| \geq |R + P| \) so that \( |R - P| \geq 2. \)

In a manner similar to Case 1, we are looking for the lowest point \( X = (x, y, z) \) inside the rectangular prism given by

\( (20) \ - a/2 \leq x \leq 0, \ 0 \leq y \leq b/2, \ z \geq 0, \)

such that

\( (21) \ |X - P| \geq 2, \) and \( |X - Q + 2P| \geq 2. \)
The procedure is the same as in Case 1. We may suppose that the right-hand side of (7) is positive. Let \( X^* = (x^*, y^*, z^*) \) be the point on the two spheres and the plane \( x = -a/2 \), with \( z^* \geq 0 \).

The radical plane \( \pi \) of the two spheres passes through \( \frac{1}{2}(Q - P) \). The equation of \( \pi \) is \( 2(b - 3a)x + 2by = b^2 + (b - 3a)(b - a) \). Putting \( x = x^* = -a/2 \), \( \pi \) cuts the two spheres in \( y^* = \frac{(b^2 + b^2 - 3ab)}{(2b)} \). We must show that \( 0 < y < b/2 \). Now \( b^2 > 4 - (a + b)^2 \) for a triple packing; hence \( 2by^* = b^2 + b^2 - 3ab > 4 - a^2 - 5ab \). On the other hand \( b^2 < ab/2 \leq 3ab \), \( 2by^* = b^2 + b^2 - 3ab \leq b^2 \), \( y^* \leq b/2 \). We see that \( z^* = \) \( \frac{\sqrt{4 - (3a/2)^2} - (b^2 + b^2 - 3ab)/(2b)}{2} > 0 \) by assumption.

We now show that the bottom of the prism is covered by the two spheres \( S(P, 2) \) and \( S(Q - 2P, 2) \). Let \( A^* = (x^*, y^*, 0) \), \( E = (-a/2, b/2, 0) \), \( F = (0, b/2, 0) \) and \( G = (-a/2, 0, 0) \). Then \( 2 > |Q - 2P - A| > |Q - 2P - A^*| \geq |Q - 2P - E| \), and also \( |Q - 2P - \frac{1}{2}(Q - P)| < 2 \) since the spheres \( S(P, 2) \) and \( S(Q - P, 2) \) intersect and \( \frac{1}{2}(Q - P) \) is halfway between their centers, hence the triangle with vertices \( \frac{1}{2}(Q - P), E, A^* \) lies in the interior of \( S(Q - 2P, 2) \) and \( S(Q - P, 2) \). Similarly \( 2 > |P - \frac{1}{2}(Q - P)| > |P - F| \geq |P| \) and \( 2 > |P - A^*| \geq |P - G| \), so that the convex pentagon with vertices \( G, A^*, \frac{1}{2}(Q - P), F \), and the origin lies in the interior of \( S(P, 2) \). Hence the bottom of the prism is covered.

We now let \( X_1 = (x_1, y_1, z_1) \) be a lowest point satisfying (20) and (21); \( X_1 \) exists because the set of points satisfying (20) and (21) is closed and nonempty. Since \( z_1 > 0 \), \( X_1 \) must be on the boundary of one of the spheres. We suppose first that \( |X_1 - Q| = 2 \), and \( |X_1 - Q + 2P| > 2 \). Then \( x_1, y_1 \) must be as far from \((a, 0)\) as possible still satisfying (20). Hence \( X_1 = (-a/2, b/2, \sqrt{4 - b^2, 4 - (3a/2)^2}) \), and calculation shows that \( |X_1 - Q + 2P| < 2 \), which is a contradiction.

Suppose now that \( |X_1 - P| > 2 \), so that \( |X_1 - Q + 2P| = 2 \). Then \( x_1, y_1 \) must be as far as possible from \((b - 2a, b)\) and still satisfy (20). Now \( -2a \leq b - 2a \leq -3a < a/2 \). Hence \( X_1 = (0, 0, \sqrt{4 - (b - 2a)^2 - b^2}) \). Hence \( |X_1 - P|^2 = a^2 + 4 - (b - 2a)^2 - b^2 < a^2 + 4 - 9a^2/4 < 4 \), which is a contradiction.
Hence $X_1$ must be on the boundary of both spheres. In a manner similar to Case 1, $X_1$ lies on a circular arc whose highest point $\frac{1}{2}(Q-P)$ is on the boundary of the prism and whose lowest point inside the prism is $X^*$. Hence $X_1 = X^*$ and (7) is proved for this case.

Case 3. Assume

(22) $b \leq g \leq a/2$.

The vectors $R-Q+P, R-Q-P$ differ only in the first component and, by (22), $|g-b+a| \geq |g-b-a|$. Since the center of the parallelogram with vertices $P, R-Q, R-Q+P$ and the origin must not be covered four times, we conclude that $|R-Q+P| \geq 2$. On the other hand, as in Case 1, $|R+P| \geq 2$.

We are seeking the lowest point $X_1$ of the prism

(23) $b \leq x \leq a/2, 0 \leq y \leq b/2, z \geq 0$,

such that

(24) $|X+P| \geq 2, |X-Q+P| \geq 2$.

If the right-hand side of (7) is zero, there is nothing to prove. Let us suppose that it is positive. Let $X$ be the point on the two spheres and the plane $x = a/2$. To find $X = (x, y, z)$ we solve

$$9a^2/4 + y^2 + z^2 = 4, (3a/2 - b)^2 + (y - b)^2 + z^2 = 4,$$

and obtain $x^* = a/2, y^* = (b^2 + b^2 - 3ab)/(2b), z^* = \sqrt{4 - 9a^2/4 - (y^*)^2}$.

This is the same $y^*$ that appears in Case 2, so $0 \leq y^* \leq b/2$ and $X^*$ satisfies (23) and (24).

Now let $X_1$ be a lowest point satisfying (23) and (24), and we shall show that $X_1 = X^*$.

We must show that the bottom of the prism is covered. Since $x = a/2$ maximizes the horizontal distance from both $-P$ and $Q-P$, it is sufficient to show that the line segment $\{(a/2, t, 0): 0 \leq t \leq b/2\}$ is covered. The spheres $S(-P, 2)$ and $S(Q-P, 2)$ intersect the plane $x = a/2$ in circles which intersect at $(a/2, y^*, z^*)$. Since $z^* > 0$, the segment is covered.

For a lowest point $X_1$ we must have $x_1 = a/2$. Since the spheres intersect
LATTICE PACKING OF SPHERES IN EUCLIDEAN SPACE

the $x = a/2$ plane in circles whose centers have $y$ components 0 and $b$ respectively, it is clear that $X_1 = X^*$ and inequality (7) holds. This finishes Case 3.

Hence $d(\Lambda) \geq \min \{f_1, f_2\}$. We observe immediately that $f_1 \geq f_2$. This would follow if $\psi(b) \geq 0$ where

$$\psi(b) = 9a^2/4 + (b/2 - (3a - b)b)/(2b) - b^2 - (b/2 - (b/2)b + 2a)/b)^2.$$ 

Now $\psi(0) = 9a^2/4 - a^2 = (5/4)a^2 > 0$ and $\psi(a/2) = 0$. Differentiating, we have

$$\psi'(b) = -[(5a)/(2b)][(b^2 + 3b^2 - ab)] < 0$$

since $b^2 \geq a^2 - b^2 \geq (3/4)a^2$.

We now prove

$$(9) \quad f^2 \geq \min \{d_0^2 + 1/100, 2a^6 - 11a^4 + 12a^2\}$$

under the hypothesis that $d(\Lambda) \leq d_0$ and $2/3 \leq a \leq 0.9508$. Write $F(a, b, b) = f^2 = a^2b^2/4 - 9a^2/4 - (b/2 + b(b - 3a)/(2b))^2$ and put $t = b^2$. Clearly $\partial^2 F / \partial t^2 = -a^2/2 < 0$. Hence $F(a, b, b) \geq \min \{\phi(a, b), \psi(a, b)\}$, where

$$\phi(a, b) = F(a, b, \sqrt{4(a + b)^2}), \quad \psi(a, b) = F(a, b, b_m),$$

and $\psi(a, b) = F(a, b, b_m)$, where $b_m$ was defined in Remark 2. Now

$$\phi(a, b) = a^2/4 - (a + b)^2[(4 - 9a^2/4 - (b/2 + b(b - 3a)/(2b))^2 - (b/2 - (b/2)b + 2a)/b)^2$$

and calculation shows that $\partial^2 \phi / \partial b^2 = -8a^2 / -8a^4 < 0$. Since $\phi(a, 0) = \phi(a, a/2) = p(a)$ where $p(x) = 2x^6 - 11x^4 + 12x^2$, we have $\phi \geq p(a)$.

To complete the proof of (9), we will show that $\psi(a, b) \geq d_0 + 1/100$. Now

$$\psi(a, b) = F(a, b, b_m) = a^2b^2/4 - 9a^2/4 - (b_m/2 + b(b - 3a)/(2b_m))^2$$

and calculation shows that $\partial^2 \psi / \partial b^2 = -a^2b_m^2 + 3a^2b_m^2 - 9a^3b + 9a^4/4$. We shall show that $\partial^2 \psi / \partial b^2 < 0$, so that $\psi(a, b)$ is a concave function of $b$.

We digress for a moment to show that $b \leq b_m(a)$, where $b_m = \sqrt[4]{4 - b_m^2} - a$. To see this, recall that for triple packing we must have $b^2 \geq 4 - (a + b)^2$, whereas $b^2 \leq b_m^2$, since $d(\Lambda) \leq d_0$. The juxtaposition $b_m^2 \geq 4 - (a + b)^2$ yields $b \geq b_m$.

Putting $b = a/2$ we see that $b_m \leq a/2$. It is conceivable that $b_m$ is negative, even though $b$ never is. For what follows, it is useful to know that $b_m \geq -a/2$. To see this, note that

$$b_m^2 = a^2/6 + (2/3)\sqrt{a^4/16 + 3d_0^2/a^2} < 1/6 + (2/3)\sqrt{1/16 + 27} < 3.64,$$

since $2/3 \leq a \leq 1$ and $3 < a^2 < 4$, that $4 - b_m^2 > 4a^2/4$, $b_m + a = \sqrt{4 - b_m^2} > a/2$, and $b_m > -a/2$.

We now return to showing that $\psi(a, b)$ is a concave function of $b$ for $b_m \leq b \leq a/2$. Since $\partial^2 \psi / \partial b^2 = 6a^6b - 9a^5b \leq 3a^3 - 9a^3 = -6a^3 < 0$, it is enough to show that $f(a) = (1/a^2)\partial^2 \psi / \partial b^2 |_{b = b_m} < 0$.

Since $f(a)$ is a rather complicated function of the single variable $a$, we shall simply find an upper bound for its derivative as a function of $a$ and use a computer to evaluate it on a fine grid.
Let $F(x, y, z) = -z + 3y^2 - 9xy + 9x^2/4$ so that $F(a, b_m, b_m^2) = f(a)$. Then

$$|f'(a)| \leq \left| \frac{\partial F}{\partial x} \right|_0 + \left| \frac{\partial F}{\partial y} \right|_0 + \left| \frac{\partial F}{\partial z} \right|_0 \left| \frac{d}{da} b_m^2 \right|,$$

where the subscript 0 indicates that the partial derivatives are evaluated at $(x, y, z) = (a, b_m, b_m^2)$.

Then

$$\left| \frac{\partial F}{\partial x} \right|_0 = \left| -9b_m + 9a/2 \right| \leq 9a \leq 9, \quad \left| \frac{\partial F}{\partial y} \right|_0 \leq \left| 6b_m \right| + 9a \leq 12, \quad \text{and} \quad \left| \frac{\partial F}{\partial z} \right|_0 = 1.$$

Let $u = a^2$, $g(u) = b_m^2 = u/6 + (2/3)\sqrt{u^2/16 + 3d_0^2}/u$. Then

$$g'(u) = 1/6 + (1/3)(u/8 - 3d_0^2/u^2)/\sqrt{u^2/16 + 3d_0^2/u},$$

and

$$|g'(u)| \leq 1/6 + (1/3)(1/8 + 3 \times 4/(4/9)^2)/\sqrt{(4/9)^2/16 + 9} = 1/6 + (1/3)(1/8 + 243/4)/\sqrt{9} < 7.$$ 

Therefore $|dg(a^2)/da| = |2ag'(a^2)| < 14$. That is, $|db_m^2/da| < 14$.

Finally, $b_m = \sqrt{4 - b_m^2} - a$, $db_m/da = 1 - (1/2)(db_m^2/da)/\sqrt{4 - b_m^2}$ and

$$|db_m/da| \leq 1 + 7\left[ 1 - \frac{b_m^2}{4} \right]^{1/2} < 1 + 7[0.36]^{1/2} < 13,$$

since $b_m^2 < 3.64$. Hence

$$|f'(a)| \leq 9 + 12 \times 13 + 1 \times 14 = 179.$$ 

Using a computer we verified that (allowing for roundoff error) $f(a_i) < -0.2$ at the points $2/3 = a_0 < a_1 < \cdots < a_n = 1$, where $n = 500$, and $|a_{i-1} - a_i| < 1/1200$ for $1 \leq i \leq n$.

Let $a$ be an arbitrary number in the interval $[2/3, 1]$. Then $a \in [a_{i-1}, a_i]$ for some $i$, and therefore

$$f(a) = f(a_{i-1}) + \int_{a_{i-1}}^a f'(t) \, dt \leq -0.2 + \frac{179}{1200} < -0.05.$$ 

Hence $\psi(a, b)$ is a concave function of $b$ as claimed, so $\psi(a, b) \geq \min \{\psi(a, b_m), \psi(a, a/2)\}$.

The functions $\psi(a, b_m)$ and $\psi(a, a/2)$ are also rather complicated functions of the single variable $a$, and they are both above $d_0^2 + 1/100$. We shall simply find an upper bound for their derivatives as functions of $a^2$ or $a$ and use a computer to evaluate them on a fine mesh.

Let $u = a^2$ and $f(u) = \psi(a, a/2)$. Then $f(u) = u g(u) \Omega(u, g(u)) - 25u^3/64$ where $g(u) = b_m^2 = u/6 + (2/3)\sqrt{u^2/16 + 3d_0^2}/u$, and $\Omega(u, v) = 4 - (9/4)v - v^2/4 + (5/8)u$. Then

$$f'(u) = g(u) \Omega(u, g(u)) + g'(u)\Omega(u, g(u)) + u g'(u) \partial \Omega/\partial u + g'(u) \partial \Omega/\partial v - 75u^2/64.$$ 

To estimate $|f'(u)|$, we must estimate $g$, $\Omega$, and their derivatives. We have $|g(u)| \leq 3.64 < 4$ from before and $|g(u)| \geq u/3$, trivially. Hence $|\Omega(u, g(u))| \leq$
4 + 13u/8 + |g(u)/4| \leq 4 + 13/8 + 1 \leq 7.  We also have \(|g'(u)| \leq 7\) from before.  On
the other hand \(|\partial \Omega/\partial u| = |13/8| < 2\) and \(|\partial \Omega/\partial v| = \frac{1}{4}\).  Putting the estimates together,
\(|f'(u)| < 4 \times 7 \times 7 \times 7 + 4 \times 2 + 5 + 2 = 92.

We will show that \(f(u) \geq d_0^2 + 1/100\) for \(4/9 \leq u \leq 1\).  Let us suppose that a
computing machine has verified that \(f(u) \geq d_0^2 + 1/50 + \epsilon\) for \(4/9 = u_0 < u_1 < \cdots < u_n = 1\), where
\(|u_{i-1} - u_i| < (50 \times 92)^{-1}\).  It then follows that \(f(u) \geq d_0^2 + \epsilon\) for \(4/9 \leq u \leq 1\).  For \(u \in [u_{i-1}, u_i]\) for some \(i\), and
\[
|f(u)| \geq |f(u_{i-1})| - \left| \int_{u_{i-1}}^u f'(t) \, dt \right|
\]
\[
\geq d_0^2 + 1/50 + \epsilon - 92|u_{i-1} - u_i| \geq d_0 + \epsilon.
\]

A computing machine was programmed to find the minimum value of \(f(u_i)\) for
\(1 \leq i \leq 8251\), where \(u_i = 4/9 + (i - 1)/14850\), and the answer was 3.51822.

Had there been no roundoff error, we could say that \(f(u_i) \geq d_0^2 + 1/6\).  It is
certainly safe to say that \(f(u) \geq d_0^2 + 1/100\) for 2/3 < \(u < 1\).

We shall use the same method for \(\psi(a, b^m)\).  Let us rename \(f(a) = \psi(a, b^m) = a^2 b^2 (-9/4)a^2 - b^2/4 - b_m (b_m - 3a)/2 - a^2 b^2 (b_m - 3a)/4\).  It is unfortunate
that \(\psi(a, b^m)\) cannot be written simply as a function of \(a^2\); all our functions are
now functions of \(a\).  Let \(g_1(a) = b^2 = g(a^2)\), and let \(b(a) = b^m\).  Put \(\Phi(x, y, z) = 4
- (9/4)x^2 - y/4 - z(z - 3x)/2\) and \(\Theta(x, y, z) = -x^2 z^2 (z - 3x)/4\).  Then
\[
f(a) = a^2 g_1(a) \Phi(a, g_1(a), b(a)) - \Theta(a, b(a)),
\]
and
\[
f'(a) = 2a g_1(a) \Phi(a, g_1(a), b(a)) + a^2 g_1'(a) \Phi(a, g_1(a), b(a))
\]
\[+ a^2 g_1(a) \partial \Phi/\partial x + (\partial \Phi/\partial y) g_1'(a) + (\partial \Phi/\partial z) b'(a)] - \partial \Theta/\partial x - (\partial \Theta/\partial z) b'(a).
\]

Using some of the estimates from before and making some new ones, we see that
\(|g_1(a)| = |g(a^2)| < 4, |b(a)| = |\sqrt{4 - b^2 - a}| \leq \frac{1}{2}, |\Phi| \leq 4 + 9/4 + 1 + b^2(2)/2 +
3|b(a)|/2 \leq 1/8 + 29/4 < 8, |g_1'(a)| < 14, \partial \Phi/\partial x = 9x/2 + 3z/2, |\partial \Phi/\partial x| \leq 9/2 +
(3/2) \times \frac{1}{2} < 6, |\partial \Phi/\partial z| = \frac{1}{4}, |\partial \Phi/\partial z| = |z + 3x/2| \leq \frac{1}{2} + 3/2 = 2, and |b'(a)| < 13.

By calculation, \(\partial \Phi/\partial x = xz^2 - 2x + 18x^2)/2\).  Taking the maximum of the positive and negative parts, \(\partial \Phi/\partial x \leq b^2 + 2\times 18a^2, \partial b_m^2 \leq 9/2 |x| < 13\).

Similarly, \(\partial \Phi/\partial z = xz^2 (x^2 - 2x + 9x^2 + 3x^2)/2, \partial \Phi/\partial x \leq (b_m^2) \times \max(1/8 + 29/4 + 3x/2) + 3 + 3 \times 13 = 430\).  To show that \(f(a) \geq d_0^2 + \epsilon\), therefore, it is enough to show that \(f(a_i) \geq d_0^2 + 1/50 + \epsilon\)
for \(2/3 < a_0 < \cdots < a_n = 1\) where \(|a_{i-1} - a_i| < 1/25,000\).

A computing machine was programmed to find the minimum value of \(f(a_i)\) for
\(1 \leq i \leq 100,001\) where \(a_i = 2/3 + i/300,000\), and the answer was 3.40344 . . . .
Had there been no roundoff error we could say that \( f(a) \geq d_0^2 + 1/15 \). It is certainly safe to say that \( f(a) \geq d_0 + 1/50 + 1/100 \) so that \( \psi(a, h_m) = f(a) \geq d_0^2 + 1/100 \) for \( 2/3 \leq a \leq 1 \). Therefore \( \psi(a, h) \geq \min \{ \psi(a, a/2), \psi(a, h_m) \} \geq d_0^2 + 1/100 \) and \( f_2^2 \geq \min \{ \psi(a, b), \phi(a, b) \} \geq \min \{ d_0^2 + 1/100, p(a) \} \) as claimed. Thus (9) is proved.

We now prove (10). Let \( f(t) = 2t^3 - 11t^2 + 12t - d_0^2 \). Then \( f'(t) = 6t^2 - 22t + 12 \) and \( f''(t) = 12t - 22 < 0 \) for \( 0 < t < 1 \). Hence \( f(t) \) is a concave function and has at most two zeroes in the range \([0, 1]\). In fact, \( f(4/9) = 0 \), and \( f(\alpha^2) = 0 \) where \( \alpha^2 = 0.90402... \) and \( f(t) > 0 \) for \( 4/9 < t < \alpha^2 \). Since \( f(a^2) = p(a) - d_0^2 \), we conclude that \( p(a) > d_0^2 \) for \( 2/3 < a < \alpha = 0.950802... \) and (10) is proved.

It now follows from (8), (9) and (10) that \( d(A) \geq d_0 \) for \( 2/3 \leq a \leq 0.9508 \) with equality only when \( a = 2/3 \).

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